RSK CORRESPONDENCE WITH KING TABLEAUX AND SEMISTANDARD OSCILLATING TABLEAUX

MASATO KOBAYASHI*

ABSTRACT. In 2025, the author and Tomoo Matsumura found a bijective RSK correspondence of type C for King tableaux with Berele, Lee and Sundaram's ideas all together. We briefly explain this progress and mention a conjecture on the generating function of semistandard oscillating tableaux.

CONTENTS

1. INTRODUCTION

In 2025, the author and Tomoo Matsumura (International Christian University) found a bijective RSK correspondence of type C for King tableaux with Berele, Lee and Sundaram's ideas altogether. In this note, we briefly explain this progress and mention a conjecture on the generating function of semistandard oscillating tableaux.

In combinatorics of Young tableaux, there is a series of important results which we often call an RS (Robinson-Schensted) or RSK (Robinson-Schensted-Knuth) correspondence; there are many variants such as a "dual" RSK, though. As shown in the table below, Type A RS, RSK for Young tableaux and type C RS correspondences for King tableaux have been already established. However, we have been missing ??? part since initiation of the theory on King tableaux in 1976. Here, our progress fills this gap.

Date: July 29, 2025.

Key words and phrases. Berele insertion, Cauchy identity, King tableaux, RSK correspondence, Semistandard oscillating tableaux.

*Department of Engineering, Kanagawa University.

This is based on the author's talk at Kyoto RIMS, Recent Developments on Representation theory, Lie theory and Related Areas organized by Nagatoshi Sasano, 1:40-2:30pm, on June 24, 2025.

	RS	RSK
type A	Robinson-Schensted	Knuth
type C	Berele	???

Now we wish to mention three particular authors.

- Berele (1986) [1]
- Sundaram (1986) [4]
- Lee (2025) [3]

Berele found a type C RS correspondence with its Q-symbol *oscillating tableaux*. Sundaram was almost close to a type C RSK-correspondence. However, at that time of her writing, there was no precise formulation of an appropriate Q-symbol for this purpose. Instead, she discussed certain triples including King, oscillating and Littlewood-Richardson tableaux. After about 40 years, Lee then introduced a notion of *semistandard oscillating tableaux* in context of Kashiwara's crystal theory.

2. TYPE C RSK CORRESPONDENCE

Following [2], here we review basic terms and main ideas.

Consider barred numbers $\overline{1}, \dots, \overline{k}$ and let $[\overline{k}] = \{1, \overline{1}, \dots, k, \overline{k}\}$. Define the *symplectic order*

$$1 < \overline{1} < 2 < \overline{2} < \cdots < k < \overline{k}.$$

Definition 2.1. A *King tableau* (KT) of shape λ is a filling of the Young diagram of λ on the alphabet $[\overline{k}]$ with all of the following.

(1) Entries are weakly increasing along rows.
(2) Entries are strictly increasing down columns.
(3) All entries in row i are $\geq i$ (symplectic condition).

Denote $\mathrm{KT}_k(\lambda)$ by the set of King tableaux of entries $[\overline{k}]$ of shape λ.

Now we are going to define *Berele insertion* $T \leftarrow x$ for a KT T and $x \in [\overline{k}]$.

Define $T \leftarrow x$ with a series of row-insertions of type A except the following case: If row i contains at least one $\overline{i}$, and we row-insert i to this row, then, instead of adding i and bumping $\overline{i}$ to the $(i+1)$-st row,

(1) replace the first $\overline{i}$ to i,
(2) replace the first i to empty box,

$$\text{row } i\text{:}\ \boxed{i}\boxed{i}\boxed{\cdots}\boxed{i}\boxed{\overline{i}}\boxed{\overline{i}}\boxed{\cdots} \rightarrow \boxed{}\boxed{i}\boxed{\cdots}\boxed{i}\boxed{i}\boxed{\overline{i}}\boxed{\cdots}$$

and play jeu de taquin (as an SSYT of entries $[\overline{k}]$) with this empty box. As Berele showed [1], the resulting tableau is always King.

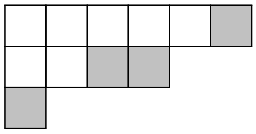
Definition 2.2. A *Knuth array* w of length n is a two-line array

$$w = \begin{pmatrix} u_1 & \cdots & u_n \\ v_1 & \cdots & v_n \end{pmatrix}$$

such that $u_j \in [l], v_j \in [k]$ with the following lexicographic conditions.

(1) $u_j \leq u_{j+1}$.
(2) $u_j = u_{j+1} \Longrightarrow v_j \leq v_{j+1}$.

Denote the set of such arrays by $\mathbf{A}_k^l$.

A skew shape is a *horizontal strip* if it contains at most one box at each column. For example, lightgray boxes indicate such. 

Definition 2.3. A *k-semistandard oscillating tableau* (k-SSOT) of the final shape λ is a finite sequence $S = (S^1, S'^2, S^2, S'^3, \ldots, S'^k, S^k)$ of partitions such that

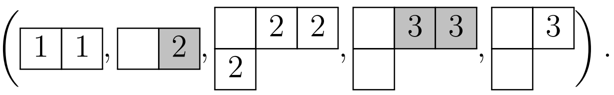
(1) $S^i \supseteq S'^{i+1}$ $(1 \leq i \leq k-1)$ and $S'^i \subseteq S^i$ $(2 \leq i \leq k)$. Moreover, $S^i \backslash S'^{i+1}$, $S^i \backslash S'^i$ are horizontal strips.
(2) $S^k = \lambda$.

Denote by $\mathrm{SSOT}_k(\lambda)$ the set of all such SSOTs.

This is a technical definition. More intuitively, this is a sequence of partitions starting at $S'^1 = \emptyset$ and ending at λ such that at $S^i \supseteq S'^{i+1}$, as the first part of a step i, we delete several boxes so that those form a horizontal strip and at $S'^i \subseteq S^i$, as the second part of the step i, we add several boxes similarly. It is a good idea to think that we either add or delete one box at one substep. At an addition step, we add boxes from *left* and at a deletion step, we delete them from *right*. Each partition in such S must have at most k rows since at each step $S'^i \subseteq S^i$, we add at most one box to columns.

Remark 2.4. We borrowed this idea from Lee [3]. However, this definition is slightly different from his. He allows it to end with deleting boxes at the final step. The purpose of this modification is to construct the bijective RSK correspondence with Berele insertion.

There is a convenient way to compactly express an SSOT S within one multiset-valued tableau. Every time a box is added or deleted at a substep of S, we record its *step number* into a tableau (of possibly larger than the final shape) at the same position. For example, we can encode the above S as



All together, we identify S with

$$\begin{array}{|c|c|c|}\hline 1 & 12233 & 23 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

Definition 2.5. From

$$w = \begin{pmatrix} u_1 & \cdots & u_n \\ v_1 & \cdots & v_n \end{pmatrix} \in \mathbf{A}_{\overline{k}}^{l},$$

we construct a pair of tableaux of the same shape

$$(P(w), Q(w)) \in \mathrm{KT}_k(\lambda) \times \mathrm{SSOT}_{ln}(\lambda)$$

for some λ ($\ell(\lambda) \leq k, l$) as follows. Start with $(P_0(w), Q_0(w)) = (\emptyset, \emptyset)$. For $j \geq 1$, let

$$P_j(w) = P_{j-1}(w) \leftarrow v_j \quad \text{(Berele insertion)}$$

and $Q_j(w)$ be the multiset-valued tableau with placing u_j into the box which we added or deleted at $P_{j-1}(w) \leftarrow v_j$. Finally, set $(P_C(w), Q_C(w)) = (P_n(w), Q_n(w))$. In fact, $Q_C(w)$ is an SSOT [2, Lemma 4.13].

Example 2.6. Let $k = 2, l = 5, n = 11$,

$$w = \begin{pmatrix} 1 & 1 & 1 & 2 & 3 & 3 & 4 & 4 & 4 & 5 & 5 \\ \overline{1} & 2 & \overline{2} & 2 & 1 & \overline{1} & 1 & 1 & \overline{1} & 1 & \overline{2} \end{pmatrix}.$$

We find that

$$\begin{array}{|c|}\hline \overline{1} \\ \hline\end{array} \to \begin{array}{|c|c|}\hline \overline{1} & 2 \\ \hline\end{array} \to \begin{array}{|c|c|c|}\hline \overline{1} & 2 & \overline{2} \\ \hline\end{array} \to \begin{array}{|c|c|c|}\hline \overline{1} & 2 & 2 \\ \hline \overline{2} & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \to \begin{array}{|c|c|}\hline 2 & 2 \\ \hline \overline{2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

$$\to \begin{array}{|c|c|}\hline \overline{1} & 2 \\ \hline\end{array} \to \begin{array}{|c|}\hline 2 \\ \hline\end{array} \to \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 1 & \overline{1} \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \to \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 1 & \overline{2} \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}.$$

Thus,

$$(P(w), Q(w)) = \left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}, \begin{array}{|c|c|c|}\hline 1 & 14455 & 13 \\ \hline 234 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \right).$$

Theorem 2.7 (RSK correspondence of type C [2, Theorem 4.17])**.**

$$\Phi = \Phi_C : \mathbf{A}_{\overline{k}}^{l} \to \bigcup_{\ell(\lambda)\le k,l} \mathrm{KT}_k(\lambda) \times \mathrm{SSOT}_l(\lambda), \quad w \mapsto (P_C(w), Q_C(w))$$

is a bijection.

The key idea of the proof is the *standardization* of an array to use Berele's correspondence.

Now we generalize this in the unbounded case. Let

$$[\infty] = \{1, 2, \dots\}, \quad [\overline{\infty}] = \{1, \overline{1}, 2, \overline{2}, \dots\}$$

and

$$\mathbf{A}_{\overline{\infty}}^{\infty} = \left\{ w = \begin{pmatrix} u_1 & \cdots & u_n \\ v_1 & \cdots & v_n \end{pmatrix} \Big| u_j \in [\infty], v_j \in [\overline{\infty}], n \ge 0 \right\}.$$

Recall from Theorem 2.7 that for all $k, l \ge 1$,

$$\Phi = \Phi_C = \Phi_{\overline{k}}^{l} : \mathbf{A}_{\overline{k}}^{l} \to \bigcup_{\ell(\lambda)\le k,l} \mathrm{KT}_k(\lambda) \times \mathrm{SSOT}_l(\lambda), \quad w \mapsto (P_C(w), Q_C(w))$$

is a bijection.

Given $w = \begin{pmatrix} u_1 & \cdots & u_n \\ v_1 & \cdots & v_n \end{pmatrix} \in \mathbf{A}_{\overline{\infty}}^{\infty}$, define

$$\Phi = \Phi_{\overline{\infty}}^{\infty} : \mathbf{A}_{\overline{\infty}}^{\infty} \to \bigcup_{\lambda} \mathrm{KT}(\lambda) \times \mathrm{SSOT}(\lambda)$$

as follows. Take $l = u_n$ and let k be the minimal integer such that $v_j \le \overline{k}$ for all j.

Then we can think w as $w \in \mathbf{A}_{\overline{k}}^{l}$ so that $\Phi(w) := \Phi_{\overline{k}}^{l}(w)$ makes sense.

Theorem 2.8.

$$\Phi : \mathbf{A}_{\overline{\infty}}^{\infty} \to \bigcup_{\lambda} \mathrm{KT}(\lambda) \times \mathrm{SSOT}(\lambda)$$

is a bijection.

Proof. First, we show that Φ is surjective. Suppose $(T, S) \in \bigcup_\lambda \mathrm{KT}(\lambda) \times \mathrm{SSOT}(\lambda)$. Then

$$(T, S) \in \mathrm{KT}_k(\lambda) \times \mathrm{SSOT}_l(\lambda)$$

for some k, l and a partition λ. We know that $\Phi_{\overline{k}}^{l}$ is surjective. Thus, $\Phi_{\overline{k}}^{l}(w) = (T, S)$ for some $w \in \mathbf{A}_{\overline{k}}^{l} (\subseteq \mathbf{A}_{\overline{\infty}}^{\infty})$. Therefore, $\Phi(w) = \Phi_{\overline{k}}^{l}(w) = (T, S)$.

Next, let us see that Φ is injective. Suppose $\Phi(w) = \Phi(w')$, say this is (T, S). Let l be the maximum of all entries in S and k be the minimal integer such that $T(i, j) \le \overline{k}$ for all i, j. The assumption $\Phi(w) = \Phi(w')$ now implies $\Phi_{\overline{k}}^{l}(w) = \Phi_{\overline{k}}^{l}(w')$ and hence $w = w'$ by injectivity of $\Phi_{\overline{k}}^{l}$. □

This bijection derives an expansion of Cauchy identity with two generating functions as we now explain. Let $\mathbf{x}^{\pm} = (x_1, x_1^{-1}, \dots)$ and $\mathbf{y} = (y_1, \dots)$ be two sets of infinitely many variables. For any partition λ, define *symplectic Schur function*

$$\mathrm{sp}_\lambda(\mathbf{x}^{\pm}) = \sum_{T \in \mathrm{KT}(\lambda)} \mathbf{x}^T, \quad \mathbf{x}^T = \prod_{i,j} x_{T(i,j)}$$

and the *SSOT function*

$$\mathrm{ss}_\lambda(\mathbf{y}) = \sum_{S \in \mathrm{SSOT}(\lambda)} \mathbf{y}^S.$$

Corollary 2.9.

$$\prod_{i,j=1}^{\infty} (1 - x_j y_i)^{-1} (1 - x_j^{-1} y_i)^{-1} = \sum_{\lambda} \mathrm{sp}_\lambda(\mathbf{x}^{\pm}) \mathrm{ss}_\lambda(\mathbf{y}).$$

where the sum takes over all partitions λ.

It would be also nice if we could understand this in terms of *the ring of symmetric functions* and *the representation theory of the symplectic groups*. We take another opportunity to discuss such ideas.

3. Conjecture

Conjecture 3.1. For a partition λ, we have

$$\mathrm{ss}_\lambda(\mathbf{y}) = \left(\prod_{i<j} (1 - y_i y_j)^{-1} \right) s_\lambda(\mathbf{y}).$$

Recall that this product is *Littlewood generating function*:

$$\prod_{i<j} (1 - y_i y_j)^{-1} = \sum_{\beta' \text{ even}} s_\beta(\mathbf{y}).$$

Here, s_β is the Schur function and a partition is even if all its rows have even number of boxes and β' means the conjugate of a partition β. The product of two Schur functions $s_\beta s_\lambda$ reminds us *Littlewood-Richardson rule*. We thus expect that

$$\mathrm{ss}_\lambda(\mathbf{y}) = \sum_{\beta, \mu} c^{\mu}_{\beta\lambda} s_\mu(\mathbf{y})$$

where the sum takes over all partitions β, μ such that β' is even, $\mu \supseteq \lambda$, $|\mu| - |\lambda|$ $(= |\beta|)$ is even, say $2r$, μ is obtained by adding $2r$ boxes to λ, at most r new boxes in each row and $c^{\mu}_{\beta\lambda}$ the LR coefficient. This gives an expansion of $\mathrm{ss}_\lambda(\mathbf{y})$ as an *infinite* sum of Schur functions with positive integer coefficients. It should be possible to discuss properties of such a generating function. At least, we proved that it is symmetric in [2] with constructing Bender-Knuth-like involution.

Acknowledgment.
The author would like to thank Tomoo Matsumura for fruitful discussions on this topic. This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

References

[1] A. Berele, A Schensted-type correspondence for the symplectic group, J. Combin. Theory Ser. A, 43 (1986), 320-328.
[2] M.Kobayashi-T. Matsumura, RSK correspondence for King tableaux with Berele insertion, `math arXiv 2506.06951`.
[3] S. J. Lee, Crystal Structure on King Tableaux and Semistandard Oscillating Tableaux. Transf. Groups 30 (2025), 823-853.
[4] S. Sundaram, On the combinatorics of representations of $Sp(2n, \mathbf{C})$, Massachusetts Institute of Technology, ph. D thesis, 1986.

Masato Kobayashi, Department of Engineering, Kanagawa University, Rokkakubashi, Yokohama, Japan.
Email address: `masato210@gmail.com`