NORM BOUNDS FOR THE SPECTRAL PROJECTIONS OF THE HEISENBERG SUB-LAPLACIAN

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Contents

Abstract. This paper concerns the boundedness of the generalized spectral projections $\mathcal{P}_\lambda$ associated with the sub-laplacian on the Heisenberg group, viewed as operators

$$\mathcal{P}_\lambda : L^1_t L^p_z \longrightarrow L^\infty_t L^q_z.$$

We formulate a conjecture describing the range of exponents p, q for which these projections are bounded, and we provide several motivations and partial results in support of this conjecture.

1. Introduction

This article is a survey of the lecture I delivered at the RIMS conference "Recent Developments in Representation Theory, Lie Theory and Related Areas." It is a pleasure to express my deep appreciation to RIMS for its support, and to the organizing committee of the meeting, in particular to Nagatoshi Sasano. His efforts and generous hospitality contributed greatly to making the conference a most fruitful and enjoyable event.

In Section 2 we establish the connection between the spectral theory of the Laplacian and the restriction theory of the Fourier transform in $\mathbb{R}^d$. This section also contains a brief introduction to the Fourier restriction problem.

In Sections 3 and 4 we turn to the Heisenberg group setting, discussing some old and new results on the mapping properties of the spectral projections of the sub-laplacian, and formulating a conjecture analogous to the one that arises in the Euclidean restriction theory.

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2. RESTRICTION OPERATOR AND SPECTRAL DECOMPOSITION OF THE LAPLACIAN

If f is a Schwartz function on $\mathbb{R}^d$, the Fourier inversion formula in polar coordinates reads

$$\begin{aligned} f(x) &= \int_{\mathbb{R}^d} \hat{f}(\xi)\, e^{2\pi i x\cdot\xi}\, d\xi \\ &= \int_0^\infty \left(\int_{S^{d-1}} \hat{f}(r\omega)\, e^{2\pi i r\,\omega\cdot x}\, d\sigma(\omega) \right) r^{d-1}\, dr, \end{aligned} \tag{2.1}$$

where $d\sigma$ is the surface measure on the unit sphere S^{d-1}, and

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i x\cdot\xi}\, dx$$

is the Fourier transform of f.

Applying the Laplacian Δ in the x-variable to the inner integral in (2.1), we obtain

$$\Delta \int_{S^{d-1}} \hat{f}(r\omega)\, e^{2\pi i r\,\omega\cdot x}\, d\sigma(\omega) = -4\pi^2 r^2 \int_{S^{d-1}} \hat{f}(r\omega)\, e^{2\pi i r\,\omega\cdot x}\, d\sigma(\omega).$$

In light of this, (2.1) may be viewed as a spectral decomposition of f with respect to Δ. Setting $\lambda = 2\pi r$, we can rewrite (2.1) as

$$\begin{aligned} f(x) &= \frac{1}{(2\pi)^d} \int_0^\infty \left(\int_{S^{d-1}} \hat{f}\left(\tfrac{\lambda\omega}{2\pi}\right) e^{i\lambda\,\omega\cdot x}\, d\sigma(\omega) \right) \lambda^{d-1}\, d\lambda \\ &= \int_0^\infty \mathcal{P}_\lambda f(x)\, d\lambda, \end{aligned}$$

where

$$\mathcal{P}_\lambda f(x) = \frac{\lambda^{d-1}}{(2\pi)^d} \int_{S^{d-1}} \hat{f}\left(\tfrac{\lambda\omega}{2\pi}\right) e^{i\lambda\,\omega\cdot x}\, d\sigma(\omega)$$

is the λ-spectral component of f with respect to Δ. Following R. Strichartz [Str], we call the operators $\mathcal{P}_\lambda$ the generalized spectral projections of Δ.

The key observation linking the spectral theory of the Laplacian with the restriction theory of the Fourier transform is that

$$\mathcal{P}_{2\pi} f(x) = \int_{S^{d-1}} \hat{f}(\omega)\, e^{2\pi i\,\omega\cdot x}\, d\sigma(\omega) = S(\hat{f})(x), \tag{2.2}$$

where S is the extension operator from the unit sphere, defined for a locally integrable function g on S^{d-1} by

$$Sg(x) = \int_{S^{d-1}} g(\omega)\, e^{2\pi i\,\omega\cdot x}\, d\sigma(\omega).$$

The adjoint of S is the restriction operator R, which maps a Schwartz function f on $\mathbb{R}^d$ to the restriction $\hat{f}|_{S^{d-1}}$.

In the 1960s, E. Stein observed that if p is sufficiently close to 1, then

$$\|\hat{g}\|_{L^2(S^{d-1})} \lesssim \|g\|_{L^p(\mathbb{R}^d)} \tag{2.3}$$

for all Schwartz functions g on $\mathbb{R}^d$. If $f \in L^p(\mathbb{R}^d)$, we can use (2.3) to define the restriction of $\hat{f}$ to S^{d-1} (in the L^2 sense) as follows.

Since the Schwartz space $\mathcal{S}$ is dense in $L^p(\mathbb{R}^d)$, there exists a sequence $\{f_n\} \subset \mathcal{S}$ converging to f in L^p. The Fourier transforms $\hat{f}_n$ are Schwartz functions and hence continuous, so taking their restrictions $\hat{f}_n|_{S^{d-1}}$ makes sense even though S^{d-1} has Lebesgue measure zero. From (2.3), the sequence $\{\hat{f}_n\}$ is Cauchy in $L^2(S^{d-1})$, and thus $\{\hat{f}_n|_{S^{d-1}}\}$ converges in $L^2(S^{d-1})$. We then define

$$Rf := \lim_{n\to\infty} \hat{f}_n|_{S^{d-1}} \quad \text{in } L^2(S^{d-1}),$$

and call Rf the restriction of $\hat{f}$ to the sphere.

Some years later, Stein and P. Tomas independently proved the following theorem.[1]

Theorem 2.1 (P. Tomas, E. Stein). *The estimate*

$$\|Rf\|_{L^2(S^{d-1})} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

holds for all $1 \le p \le \frac{2(d+1)}{d+3}$.

The range of exponents in this theorem is sharp, as follows from an example due to A. Knapp.

The proof of the Stein–Tomas theorem is based on the observation that the boundedness of $R : L^p(\mathbb{R}^d) \to L^2(S^{d-1})$ is equivalent to the boundedness of the composition

$$S \circ R : L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d).$$

It is easy to see that

$$S(Rf)(x) = \int_{S^{d-1}} \hat{f}(\omega)\, e^{2\pi i\, \omega\cdot x}\, d\sigma(\omega) = f * \hat{\sigma}(x),$$

where $\hat{\sigma}$ is the Fourier transform of surface measure on S^{d-1}.

Comparing this with (2.2), we see that

$$\mathcal{P}_{2\pi} f(x) = f * \hat{\sigma}(x),$$

so the Stein–Tomas theorem can be rephrased as the statement that

$$\mathcal{P}_{2\pi} : L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d) \quad \text{for } 1 \le p \le \frac{2(d+1)}{d+3}.$$

2.1. **The restriction conjecture.**

From classical stationary phase estimates for $\hat{\sigma}$, one deduces that

$$\|\hat{f}\|_{L^q(S^{d-1})} \lesssim \|f\|_{L^p(\mathbb{R}^d)} \tag{2.4}$$

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[1] Tomas first proved in [To] that the estimate holds for $1 \le p < \frac{2(d+1)}{d+3}$. Stein then reached the endpoint $p = \frac{2(d+1)}{d+3}$ by combining Tomas's argument with complex interpolation; see [St].

fails whenever $p \geq \frac{2d}{d+1}$. On the other hand, Knapp's example shows that (2.4) cannot hold if $q < \frac{d-1}{d+1}\, p'$. In the mid-1970s, Stein conjectured that (2.4) should hold precisely in the complementary range

$$1 \leq p < \frac{2d}{d+1}, \qquad q \geq \frac{d-1}{d+1}\, p'.$$

For $d = 2$, Stein together with C. Fefferman [Fef], and independently A. Zygmund [Z], proved that this conjecture is indeed true. For $d \geq 3$, however, the restriction conjecture remains open.

3. The Heisenberg group

As noted in (2.2), restriction estimates for the Fourier transform on $\mathbb{R}^d$ can be interpreted as mapping properties of the (generalized) spectral projections of the Laplacian. This point of view underlies many works in which, in settings without a natural global Fourier transform, one studies analogues of restriction estimates via spectral projections of distinguished operators. This perspective was essentially introduced by Strichartz in [Str].

As far as we know, the first systematic investigation in this direction is due to C. Sogge, who in his PhD thesis [So1] (see also [So2]) studied the mapping properties of the spectral projections of the Laplace–Beltrami operator on spheres. Later, together with A. Seeger, he obtained analogous results for the Laplace–Beltrami operator on compact Riemannian manifolds, proving an analogue of the Tomas–Stein theorem in that setting [SeeSo].

Subsequently, V. Casarino derived similar bounds for the joint spectral projections of the Laplace–Beltrami operator and the sub-laplacian on complex spheres in [Ca1, Ca2]. In this connection see also [CCi], where the bounds for the complex sphere obtained in [Ca2] are transferred to the reduced Heisenberg group. Related analyses have been carried out on quaternionic and octonionic spheres; see [CCi3, BCCi, Ca3].

The first restriction-type estimates in a genuinely sub-Riemannian setting are due to D. Müller, who in [Mu] established an inequality for the spectral projections of the sub-laplacian on the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}_n$. For simplicity we state this result on $\mathbb{H}_1$, which we simply denote by $\mathbb{H}$.

We now fix notation (for more details, see [Th]). Topologically, $\mathbb{H} \cong \mathbb{R}^3$. We use coordinates (x, y, t) and define the left-invariant vector fields

$$X = \partial_x - \frac{y}{2}\,\partial_t, \qquad Y = \partial_y + \frac{x}{2}\,\partial_t, \qquad T = \partial_t.$$

Then $[X, Y] = T$, and all other commutators among X, Y, T vanish. Equipped with this Lie bracket, $\mathbb{R}^3$ carries the structure of the Heisenberg Lie algebra $\mathfrak{h}$. We denote by $\mathbb{H}$ the connected, simply connected Lie group with Lie algebra $\mathfrak{h}$, and refer to it as the (reduced) Heisenberg group.

The sub-laplacian on $\mathbb{H}$ is the second-order differential operator

$$L = -X^2 - Y^2 = -\partial_x^2 - \partial_y^2 + (x\partial_y - y\partial_x)\,\partial_t - \frac{1}{4}(x^2 + y^2)\,\partial_t^2.$$

By Hörmander's theorem, L is hypoelliptic. Moreover, L admits a unique positive self-adjoint extension on $L^2(\mathbb{H})$.

Let

$$f(x,y,t) = \int_0^\infty \mathcal{P}_\lambda f(x,y,t)\, d\lambda$$

be the spectral decomposition of a Schwartz function f with respect to L, so that

$$L\,\mathcal{P}_\lambda f = \lambda\,\mathcal{P}_\lambda f.$$

Our goal is to study the mapping properties of the operators $\mathcal{P}_\lambda$ between suitable Lebesgue spaces.

As mentioned above, the first result on the Heisenberg group is due to Müller [Mu], who proved a Heisenberg analogue of the Tomas–Stein theorem. His result is formulated in terms of mixed Lebesgue norms: for a measurable function f on $\mathbb{H} \cong \mathbb{R}^3$, we set

$$\|f\|_{L^r_t L^p_{x,y}} = \left(\int_{\mathbb{R}^2} \left(\int_{-\infty}^{\infty} |f(t,x,y)|^r\, dt \right)^{p/r} dx\, dy \right)^{1/p},$$

with the usual modifications when $r = \infty$ or $p = \infty$.

Theorem 3.1 (D. Müller)**.** *For every* $1 \le p < 2$ *and every Schwartz function* f *on* $\mathbb{H}$*, the estimate*

$$\|\mathcal{P}_\lambda f\|_{L^\infty_t L^{p'}_{x,y}} \lesssim_\lambda \|f\|_{L^1_t L^p_{x,y}} \tag{3.1}$$

holds, where the implicit constant may depend on λ *but not on* f*.*

Remark 3.2. Note that, in general, estimates of the form

$$\|\mathcal{P}_\lambda f\|_{L^s_t L^q_{x,y}} \lesssim_\lambda \|f\|_{L^r_t L^p_{x,y}}$$

fail whenever $(r,s) \neq (1,\infty)$. In particular, Theorem 3.1 is sharp with respect to the time exponents.

4. IMPROVING ON MÜLLER'S BOUNDS

Some years ago, together with V. Casarino, we obtained bounds that improve on (3.1). Since the estimates (3.1), by interpolation, imply a family of estimates of the same type with the exponent p' on the left replaced by any larger exponent, "improving" Müller's result means obtaining estimates in which the exponent p' is replaced by a strictly smaller exponent q, that is

$$\|\mathcal{P}_\lambda f\|_{L^\infty_t L^q_{x,y}} \lesssim_\lambda \|f\|_{L^1_t L^p_{x,y}}, \qquad q < p'. \tag{4.1}$$

By duality, it is easy to see that (4.1) holds if and only if

$$\|\mathcal{P}_\lambda f\|_{L^\infty_t L^{p'}_{x,y}} \lesssim_\lambda \|f\|_{L^1_t L^{q'}_{x,y}}$$

for exponents satisfying $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

Using sharp estimates from [KR], we showed in [CCi1, CCi2] that the exponent on the left-hand side of (4.1) can in fact be reduced as far as $q = 2$.

Theorem 4.1. *For every $1 \le p < 2$ and every Schwartz function f on $\mathbb{H}$,*

$$\|\mathcal{P}_\lambda f\|_{L^\infty_t L^2_{x,y}} \lesssim_\lambda \|f\|_{L^1_t L^p_{x,y}}.$$

Analogous results hold on more general groups, such as the $(2n+1)$-dimensional Heisenberg groups $\mathbb{H}_n$ and, more generally, groups of Heisenberg type.

Furthermore, by localizing the bounds, one shows that $\mathcal{P}_\lambda$ is bounded

$$\mathcal{P}_\lambda : L^p_{x,y} L^1_t \longrightarrow L^q_{x,y} L^\infty_t$$

for

$$1 \le p \le \frac{6}{5}, \quad q > \frac{4}{3}$$

on $\mathbb{H}$, and for

$$1 \le p \le 2\frac{2n+1}{2n+3}, \quad q > \frac{4n}{2n+1}$$

on $\mathbb{H}_n$.

By invariance under (twisted) translations there can be no estimates of the form (4.1) with $q < p$ (see [H]). Moreover, analyzing the action of $\mathcal{P}_\lambda$ on radial functions, one shows that there are no such estimates when $q \le \tilde{p} := \frac{4n}{2n+1}$.

On $\mathbb{H}$ we proved that the bounds hold in the full complementary range of exponents [CCi4].

Theorem 4.2. *The estimate*

$$\|\mathcal{P}_\lambda f\|_{L^\infty_t L^q_{x,y}} \lesssim \lambda^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^1_t L^p_{x,y}}$$

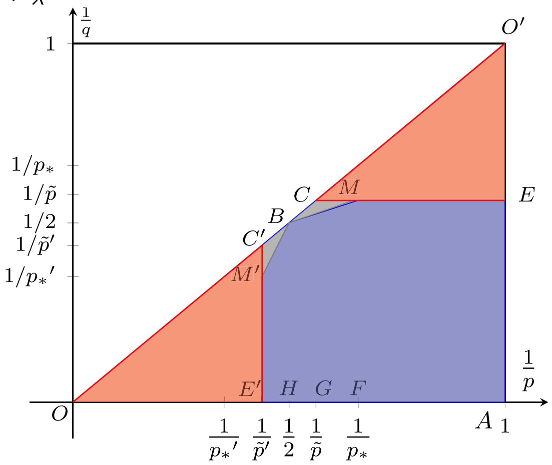
holds if and only if $(1/p, 1/q)$ lies in the interior of the pentagon with vertices

$$(1,0),\ (1,\tfrac{3}{4}),\ (\tfrac{3}{4},\tfrac{3}{4}),\ (\tfrac{1}{4},\tfrac{1}{4}),\ (\tfrac{1}{4},0),$$

in the interior of the sides from $(1,0)$ to $(1,\frac{3}{4})$ and from $(\frac{1}{4},0)$ to $(1,0)$, and at the point $(1,0)$ itself.

We conjecture that the same picture should hold in higher dimensions.

To clarify what is known and what we expect on $\mathbb{H}_n$, we include the Riesz diagram for $\mathcal{P}_\lambda$:



In this picture $\tilde{p} = \frac{4n}{2n+1}$ and $p_* = 2\,\frac{2n+1}{2n+3}$.

We know that the estimates (4.1) hold in the blue region, fail in the two closed red triangles, and we conjecture that they should hold in the grey "bowtie" region. In the case of $\mathbb{H}$ (i.e., $n = 1$), we know that the estimates are indeed valid also in the bowtie.

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