LOWER BOUNDS FOR SPHERICAL HARMONICS ON THE OCTONIONIC SPHERE

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ABSTRACT. We prove L^q bounds, with $q \geqslant 2$, for some bigraded spherical harmonics on the octonionic sphere Σ^{15} in $\mathbb{R}^{16}$. These estimates imply bounds from below for the (L^p, L^2) norm of octonionic harmonic projectors for certain values of $p \in [1, 2]$.

1. Introduction

Given a rank one non-compact Riemannian symmetric space X, one has $X = G/K$, where G is a simply connected non-compact Lie group with finite center, K is a maximal compact subgroup of G, and, if $G = KAN$ is an Iwasawa decomposition of G, then $\dim A = 1$. Denoting by M the centraliser of A in K, it is well-known that the Furstenberg boundary K/M of G/K is not a symmetric space, in general. Nevertheless, since (K, M) turns out to be a Gelfand pair (that is, the convolution algebra of the functions on K which are invariant on either side by M is commutative), it is possible to develop some harmonic analysis on K/M.

If $\mathbb{F}$ is one of the fields $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, we set $d = \dim_{\mathbb{R}} \mathbb{F}$. The case where G is the group of all $\mathbb{F}$-linear transformations g on $\mathbb{F}^{n+1}$ (seen as a right vector space over $\mathbb{F}$), which preserve the quadratic form

$$Q(x_1, \dots, x_n, x_{n+1}) := |x_1|^2 + \dots + |x_n|^2 - |x_{n+1}|^2$$

and are such that $\det g = 1$ if $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, was first studied by Kostant and Johnson and Wallach [17, 16]; in the real case, some results had been previously obtained by Takahashi and Vilenkin [21, 23]. If $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, then G and K/M coincide with $SO(n, 1)$, $SU(n, 1)$, $Sp(n, 1)$, and with $\mathrm{SO}(n)/\mathrm{SO}(n-1)$, $\mathrm{SU}(n)/\mathrm{SU}(n-1)$, $(\mathrm{Sp}(n) \times \mathrm{Sp}(1))/(\mathrm{Sp}(n-1) \times \mathrm{Sp}(1))$, respectively. Moreover, K/M may be identified with Σ^{dn-1}, the unit sphere in $\mathbb{R}^{dn}$.

The study of the octonionic case, where $G = F_{4(-20)}$, was initiated in [15]. In this exceptional setting, the maximal compact subgroup is $K = \mathrm{Spin}(9)$, the centraliser is $M = \mathrm{Spin}(7)$, and the quotient space $F_{4(-20)}/\mathrm{Spin}(9)$ can be realized as the unit ball in $\mathbb{R}^{16}$. Its Furstenberg boundary K/M can be identified with Σ^{15}, the unit sphere in $\mathbb{R}^{16}$. In this context, it has been proved in [15, 22] that the unitary irreducible M-spherical

Key words and phrases. Octonionic sphere, Cayley numbers, bigraded spherical harmonics, harmonic projection operators, Spin(9)-irreducible decomposition.

This work originates from the seminar held by the second named author at RIMS workshop "Recent Development on Representation Theory, Lie theory and Related Areas", June 2025. The second author would like to thank the organizers of this workshop for the kind invitation. The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) .

representations of K are parametrized by pairs of integers (ℓ, ℓ') with $\ell \geqslant \ell' \geqslant 0$, and that

$$L^2(K/M) = L^2(\Sigma^{15}) = \bigoplus_{\ell \geqslant \ell' \geqslant 0} \mathcal{H}^{\ell\ell'}. \tag{1.1}$$

For a characterization of the subspaces $\mathcal{H}^{\ell\ell'}$ in the octonionic framework, see [22, Théorème 1] or [15, Theorem 3.1]. We just note that the indices (ℓ, ℓ') in (1.1) correspond to $(p, q) = (\ell - \ell', \ell')$ in [22, Théorème 1], and to $(m, l) = (\ell + \ell', \ell - \ell')$ in [15, Theorem 3.1].

A similar decomposition also holds for $L^2(\Sigma^{dn-1})$, $d = 1, 2, 4$ (see [17, 16]), that is, the space of square integrable functions on Σ^{dn-1} turns out to be the direct sum of certain subspaces (still denoted, with a slight abuse of language, by the symbol $\mathcal{H}^{\ell\ell'}$), which consist of real, complex or quaternionic spherical harmonics. We are interested in the mapping properties of the projection operators $\pi_{\ell\ell'}$ mapping $L^2(K/M) = L^2(\Sigma^{dn-1})$ onto the subspaces $\mathcal{H}^{\ell\ell'}$. Recently, in [10], we formulated a conjecture on the growth of the norm $\|\pi_{\ell\ell'}\|_{(L^p, L^2)}$, in terms of powers of ℓ, ℓ' and $\ell - \ell'$. Our conjecture has been proved in the real case by Sogge in [19], in the complex framework by the second author [7, 8]; partial results are known both in the quaternionic and in the octonionic case [9, 5, 10].

In this note we present some considerations and some improvements in the exceptional framework. Since in the case of Σ^{15} (which is called, sometimes, Cayley sphere) the homogeneous dimension of N is 22, the dimension of the imaginary part of the octonions is $d_{\mathfrak{z}} = 7$ and $d_{\mathfrak{v}} := Q - 2d_{\mathfrak{z}} = 8$, in the octonionic context the conjecture formulated in [10] takes the following form.

Conjecture 1.1. *Let $1 \leqslant p \leqslant 2$. Then for all $0 \leqslant \ell' \leqslant \ell$, $\ell, \ell' \in \mathbb{N}$, the following estimate holds*

$$||\pi_{\ell\ell'} f||_{L^2(\Sigma^{15})} \leqslant C_p (1+\ell)^{\alpha(1/p)} (1+\ell')^{\beta(1/p)} (1+\ell-\ell')^{\gamma(1/p)} \, ||f||_{L^p(\Sigma^{15})} \,, \tag{1.2}$$

where

$$\alpha\Big(\frac{1}{p}\Big) = 4\Big(\frac{1}{p} - \frac{1}{2}\Big) \quad \textit{for all } 1 \leqslant p \leqslant 2\,,$$

$$\beta\Big(\frac{1}{p}\Big) := \begin{cases} 4\Big(\frac{1}{p} - \frac{1}{2}\Big) - \frac{1}{2} & \textit{if } 1 \leqslant p \leqslant \tilde{p} = \frac{18}{11} \\ \frac{1}{2}\big(\frac{1}{2} - \frac{1}{p}\big) & \textit{if } \tilde{p} \leqslant p \leqslant 2\,, \end{cases}$$

$$\gamma\Big(\frac{1}{p}\Big) := \begin{cases} 7\big(\frac{1}{p} - \frac{1}{2}\big) - \frac{1}{2} & \textit{if } 1 \leqslant p \leqslant 8/5 \\ 3\big(\frac{1}{p} - \frac{1}{2}\big) & \textit{if } 8/5 \leqslant p \leqslant 2. \end{cases}$$

The behavior of α, β, γ is illustrated in Figure 1.

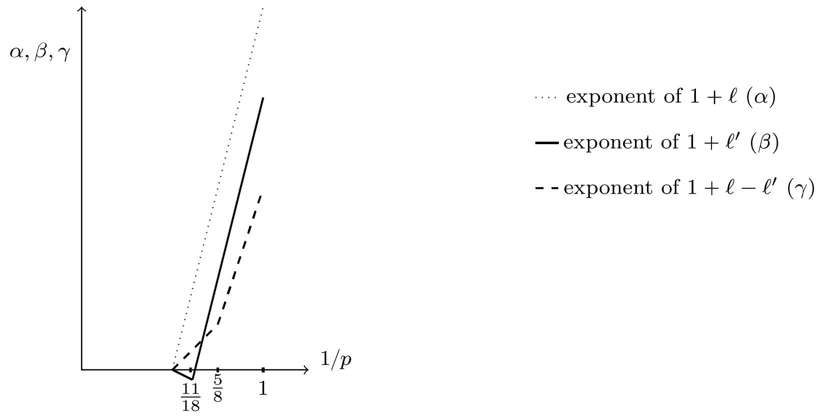

Figure 1. α, β, γ in the estimate (1.2).

A heuristic discussion of the critical exponents $p = 8/5$ and $\tilde{p} = 18/11$ that appear in (1.2), as well as the numerology underlying the conjectured bounds for $\|\pi_{\ell\ell'}\|_{(p,2)}$, can be found in [10]. We do not prove (1.2) here; rather, we collect lower bounds for $\|\pi_{\ell\ell'}\|_{(p,2)}$. Since the adjoint operator $\pi^*_{\ell\ell'} : \mathcal{H}^{\ell\ell'} \to L^q(\Sigma^{15})$ is the natural inclusion (with $1/p + 1/q = 1$), we have

$$\|\pi_{\ell\ell'}\|_{(p,2)} \geqslant \frac{\|Y_{\ell\ell'}\|_q}{\|Y_{\ell\ell'}\|_2}, \qquad q \geqslant 2,\ Y_{\ell\ell'} \in \mathcal{H}^{\ell\ell'}.$$

Thus, to establish lower bounds for $\|\pi_{\ell\ell'}\|_{(p,2)}$, we are led to study the L^q-norms of the bigraded octonionic spherical harmonics $Y_{\ell\ell'} \in \mathcal{H}^{\ell\ell'}$ with $q \geqslant 2$.

It is worth noting that the exceptional sphere Σ^{15} has several notable geometric properties. We refer to [18] for a thorough study of the Riemannian geometry of the octonionic Hopf fibration $\Sigma^{15} \to \Sigma^8$ (see also [3]). We recall from [18] that, in connection with the classical problem of vector fields on spheres, Σ^{15} is the lowest-dimensional sphere admitting more than seven linearly independent vector fields [14]. Moreover, Σ^{15} is the unique sphere admitting three homogeneous Einstein metrics [24]. For additional geometric properties of Σ^{15} and further references, see [18, 3].

Several applications are nowadays known for multiparameter estimates like (1.2); for a general discussion we refer to the Introduction of [8, 9]. Among the most recent results, we just recall [13], where the authors prove higher order Poincaré– Sobolev and Hardy–Sobolev–Maz'ya inequalities on quaternionic hyperbolic spaces and on the Cayley hyperbolic plane using the Helgason–Fourier analysis on symmetric spaces (see also [12]). Moreover, very recently, in the real case, estimates like (1.2) have proved to be useful to study boundary regularity of harmonic function [6].

Notation. For two non-negative quantities A and B, we write $A \lesssim B$, or equivalently $B \gtrsim A$, if $A \leqslant CB$ for some C, and $A \simeq B$ means that $A \lesssim B$ and $B \lesssim A$. By $\mathbb{N}$ we mean $\{0, 1, \dots\}$.

Finally, in the following, the symbol $I_{\mathbb{S}}$ will denote the set of indices

$$\{(\ell, \ell') \in \mathbb{N} \times \mathbb{N} : 0 \leqslant \ell' \leqslant \ell\}.$$

2. The octonionic setting

2.1. The octonionic sphere.

2.1. The octonionic sphere. The symbol $\mathbb{O}$ will denote the noncommutative and non-associative (but alternative) division algebra of octonions, usually described as

$$\mathbb{O} = \Big\{ x = \sum_{j=0}^{7} x_j e_j,\ x_j \in \mathbb{R} \Big\}.$$

This is an eight-dimensional vector space over $\mathbb{R}$, with the standard basis $\{e_0, e_1, \dots, e_7\}$. To define a product between two octonions, one usually defines multiplication between elements of the standard basis, by means of the so called Cayley table (see, for instance, [2, p.150]).

A conjugation $\eta \mapsto \bar{\eta}$ is defined as follows. If $\eta = \sum_{j=0}^{7} a_j e_j$, with $a_j \in \mathbb{R}$, then the conjugate $\bar{\eta}$ of η is given by

$$\bar{\eta} = a_0 e_0 - \sum_{j=1}^{7} a_j e_j.$$

The norm of η is then $\|\eta\|^2 = \eta\,\bar{\eta} = \sum_{j=0}^{7} a_j^2$.

The symbol $\mathbb{S}$ will sometimes denote the unit sphere in the 2-dimensional (over $\mathbb{O}$) vector space $\mathbb{O}^2$. The north pole of $\mathbb{S}$ will be $\mathbf{1} = (1, \underline{0})$, with $\underline{0} \in \mathbb{R}^{15}$.

2.2. Coordinates and measure. A point $\eta \in \mathbb{S}$ will be written in spherical coordinates as

$$\eta = (u, v), \ \text{ with } u = z_1 \cos\theta \text{ and } v = z_2 \sin\theta,\ 0 \leqslant \theta \leqslant \frac{\pi}{2}. \tag{2.1}$$

Here

$$z_1 = \begin{cases} \cos\lambda_1 \\ \sin\lambda_1 \cos\lambda_2 \\ \vdots \\ \sin\lambda_1 \sin\lambda_2 \dots \sin\lambda_6 \cos\sigma \\ \sin\lambda_1 \sin\lambda_2 \dots \sin\lambda_6 \sin\sigma, \end{cases} \qquad z_2 = \begin{cases} \cos\mu_1 \\ \sin\mu_1 \cos\mu_2 \\ \vdots \\ \sin\mu_1 \sin\mu_2 \dots \sin\mu_6 \cos\tau \\ \sin\mu_1 \sin\mu_2 \dots \sin\mu_6 \sin\tau, \end{cases} \tag{2.2}$$

with $0 \leqslant \lambda_j, \mu_j \leqslant \pi$ for $j = 1, \dots, 6$, and $0 \leqslant \sigma, \tau \leqslant 2\pi$ (see [22]).

Then, up to some constant, the invariant measure $d\sigma = d\sigma_{\mathbb{S}}$ on $\mathbb{S}$ is given by

$$\sin^7\theta \cos^7\theta \, d\theta \, dz_1 \, dz_2 \,.$$

Here

$$dz_1 = (\sin\lambda_1)^6 (\sin\lambda_2)^5 \dots (\sin\lambda_5)^2 (\sin\lambda_6) d\lambda_1 \dots d\lambda_6 d\sigma$$

and

$$dz_2 = (\sin\mu_1)^6 (\sin\mu_2)^5 \dots (\sin\mu_5)^2 (\sin\mu_6) d\mu_1 \dots d\mu_6 d\tau.$$

From now on, we shall write φ instead of λ_1.

2.3. **Zonal spherical harmonics.** We recall from [10] how one can obtain an explicit expression for the zonal functions in this framework. Starting from [17] and [15, Theorem 3.1], we write the zonal function $\widetilde{\mathbb{Z}}_{kj}$, satisfying $\widetilde{\mathbb{Z}}_{kj}(\mathbf{1}) = 1$, as

$$\begin{aligned}\widetilde{\mathbb{Z}}_{kj}(\theta,\varphi) = (\cos\varphi)^j\, F_1\left(-\frac{j}{2}, \frac{-j+1}{2}; \frac{7}{2}; -\tan^2\varphi\right) \\ \times (\cos\theta)^k\, F_1\left(\frac{j-k}{2}, \frac{-k-j-6}{2}; 4; -\tan^2\theta\right),\end{aligned} \tag{2.3}$$

where $k \geqslant j \geqslant 0$, $k-j$ even, $\varphi \in [0,\pi]$, $\theta \in [0, \frac{\pi}{2}]$. We take the opportunity to correct a misprint in [10, (2.3)], where $(\cos\theta)^j$ should be replaced by $(\cos\theta)^k$.

We take care of the expression in φ in (2.3) by applying some formulas concerning the hypergeometric function (more precisely, (15.3.4), (15.3.22), (15.4.5) from [1]). One obtains

$$(\cos\varphi)^j F_1\left(-\frac{j}{2}, \frac{-j+1}{2}; \frac{7}{2}; -\tan^2\varphi\right) = \frac{C_j^{(3)}(\cos\varphi)}{C_j^{(3)}(\mathbf{1})}.$$

Here $C_j^{(3)}$ denotes a Gegenbauer polynomial; the well-known relationship between Jacobi and Gegenbauer polynomials (denoted by $P_n^{(\alpha,\beta)}$) leads to

$$\frac{P_j^{(\frac{5}{2},\frac{5}{2})}(\cos\varphi)}{P_j^{(\frac{5}{2},\frac{5}{2})}(\mathbf{1})} = (\cos\varphi)^j F_1\left(-\frac{j}{2}, \frac{-j+1}{2}; \frac{7}{2}; -\tan^2\varphi\right).$$

The expression in θ in (2.3) is then treated by means of [20, (4.3.2)], which yields

$$F_1\left(\frac{j-k}{2}, \frac{-k-j-6}{2}; 4; -\tan^2\theta\right) = (\cos^2\theta)^{\frac{j-k}{2}} \frac{P_{(k-j)/2}^{(3,j+3)}(\cos(2\theta))}{P_{(k-j)/2}^{(3,j+3)}(\mathbf{1})},$$

whence

$$(\cos\theta)^k F_1\left(\frac{j-k}{2}, \frac{-k-j-6}{2}; 4; -\tan^2\theta\right) = (\cos\theta)^j \frac{P_{(k-j)/2}^{(3,j+3)}(\cos(2\theta))}{P_{(k-j)/2}^{(3,j+3)}(\mathbf{1})}.$$

As a consequence,

$$\widetilde{\mathbb{Z}}_{kj}(\theta,\varphi) = \frac{P_j^{(\frac{5}{2},\frac{5}{2})}(\cos\varphi)}{P_j^{(\frac{5}{2},\frac{5}{2})}(\mathbf{1})} (\cos\theta)^j \frac{P_{(k-j)/2}^{(3,j+3)}(\cos(2\theta))}{P_{(k-j)/2}^{(3,j+3)}(\mathbf{1})},$$

with $k \geqslant j \geqslant 0$, $k-j$ even, $\varphi \in [0,\pi]$, $\theta \in [0, \frac{\pi}{2}]$.

Now, we set $\ell' = (k-j)/2$ and $\ell = (k+j)/2$, so that

$$\widetilde{\mathbb{Z}}_{\ell+\ell',\ell-\ell'}(\theta,\varphi) = \frac{P_{\ell-\ell'}^{(\frac{5}{2},\frac{5}{2})}(\cos\varphi)}{P_{\ell-\ell'}^{(\frac{5}{2},\frac{5}{2})}(\mathbf{1})} (\cos\theta)^{\ell-\ell'} \frac{P_{\ell'}^{(3,\ell-\ell'+3)}(\cos(2\theta))}{P_{\ell'}^{(3,\ell-\ell'+3)}(\mathbf{1})},$$

with $\ell \geqslant \ell' \geqslant 0$.

Now we slightly change notation and then choose a different normalization, in analogy with the real, complex and quaternionic frameworks. The zonal function we shall deal with from now on is

$$\mathbb{Z}_{\ell\ell'}(\theta,\varphi) = d_{\ell\ell'} \frac{P_{\ell-\ell'}^{(\frac{5}{2},\frac{5}{2})}(\cos\varphi)}{P_{\ell-\ell'}^{(\frac{5}{2},\frac{5}{2})}(\mathbf{1})} (\cos\theta)^{\ell-\ell'} \frac{P_{\ell'}^{(3,\ell-\ell'+3)}(\cos(2\theta))}{P_{\ell'}^{(3,\ell-\ell'+3)}(\mathbf{1})}, \tag{2.4}$$

with $\ell \geqslant \ell' \geqslant 0$, for some positive constant $d_{\ell\ell'}$ (which will be identified later).

2.4. **Decomposition of $L^2(\mathbb{S})$.** As recalled in the Introduction, in [17, 15, 22] the authors studied the Spin(9)-irreducible decomposition of the space of square integrable functions on $\mathbb{S}$, proving (1.1). For $(\ell,\ell') \in I_{\mathbb{S}}$, $\mathcal{H}^{\ell,\ell'}$ is the finite-dimensional subspace of spherical harmonics of bidegree (ℓ,ℓ'), spanned by elements from the cyclic action of Spin(9) on zonal harmonics $\mathbb{Z}_{\ell\ell'}$ defined in (2.4). It is quite easy to prove that such functions only depend on $|u| = \cos\theta$ and $\Re u = \cos\theta\cos\varphi$.

Moreover, the constant $d_{\ell\ell'}$ appearing in (2.4) is the dimension of $\mathcal{H}^{\ell,\ell'}$. By using the fact that $\|\mathbb{Z}_{\ell\ell'}\|^2_{L^2(\mathbb{S})} \simeq \dim \mathcal{H}^{\ell\ell'}$ we have

$$d_{\ell\ell'} \simeq (1+\ell')^3(\ell+\ell')^4(1+\ell-\ell')^6\,. \tag{2.5}$$

2.5. **Differential operators.** We start by observing that the special orthogonal groups $\mathrm{SO}(1+j)$ with $1 \leqslant j \leqslant 15$ can be naturally identified with a sequence of nested subgroups of SO(16). Each of these groups acts on $\mathbb{S}$ by rotations. The symbol Δ_j will denote the second-order differential operator on $\mathbb{S}$ which corresponds through this action to the Casimir operator on $\mathrm{SO}(1+j)$. The operators Δ_j for $j = 1,\dots,15$ commute pairwise. Moreover, $\Delta := \Delta_{15}$ and Δ_7 turn out to be the Laplace–Beltrami operators on $\mathbb{S}$ and Σ^7, respectively. We then define

$$\mathcal{L} := \Delta - \Delta_7.$$

$\mathcal{L}$ is not elliptic at any point of the submanifold $\Sigma^7 \times \{0\}$; anyway, $\mathcal{L}$ is hypoelliptic and satisfies subelliptic estimates (see [11] for more details on this kind of operators, sometimes called "ultraspherical Grushin operators").

In the coordinate system introduced in Subsection 2.2, the Laplace–Beltrami operator Δ acting on $\mathbb{S}$ takes the form

$$\Delta := \frac{\partial^2}{\partial\theta^2} + 14\cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\cos^2\theta}\Delta_7 + \frac{1}{\sin^2\theta}\Omega.$$

Here Δ_7 is the Laplacian on Σ^7 with respect to the variables $(\varphi,\lambda_2,\dots,\lambda_6,\sigma)$ (with the symbol φ replacing λ_1), while Ω is the Laplacian on Σ^7 with respect to the variables $(\mu_1,\dots,\mu_6,\tau)$. One has, in particular,

$$\Delta_{\Sigma^7} = \frac{\partial^2}{\partial\varphi^2} + 6\cot\varphi\frac{\partial}{\partial\varphi} + \frac{1}{\sin^2\varphi}\Delta_6.$$

We shall use the following facts:

$$\Big(\frac{\partial^2}{\partial\theta^2} + 14\cot\theta\frac{\partial}{\partial\theta} + (\ell+\ell')(\ell+\ell'+14) - \frac{(\ell-\ell')(\ell-\ell'+6)}{\cos^2\theta}\Big)(\cos\theta)^{\ell-\ell'}P_{\ell'}^{(3,\ell-\ell'+3)}(\cos(2\theta)) = 0; \tag{2.6}$$

$$\Big(\frac{\partial^2}{\partial\varphi^2} + 6\cot\varphi\frac{\partial}{\partial\varphi}\Big)P_{\ell-\ell'}^{(\frac{5}{2},\frac{5}{2})}(\cos\varphi) = -(\ell-\ell')(\ell-\ell'+6)P_{\ell-\ell'}^{(\frac{5}{2},\frac{5}{2})}(\cos\varphi). \tag{2.7}$$

Formula (2.6) may be found, for instance, in [22] (see the last displayed formula at page 552, where $p = \ell - \ell'$ and $m = \ell + \ell'$); (2.7) follows from the fact that $P_{\ell-\ell'}^{(\frac{5}{2},\frac{5}{2})}(\cos\varphi)$ is a spherical harmonic of degree $\ell - \ell'$ on Σ^7.

It turns out that the subspace $\mathcal{H}^{\ell,\ell'}$ is an eigenspace for Δ (with eigenvalues $\mu_{\ell\ell'} = -(\ell+\ell')(\ell+\ell'+14)$) and for $\mathcal{L}$ (with eigenvalue $\mu_{\ell\ell'} - \nu_{\ell\ell'} = -4(\ell\ell' + 2\ell + 5\ell')$). Here $\ell \geqslant \ell' \geqslant 0$ and the first non zero eigenvalue of $\mathcal{L}$ is -8.

For other systems of commuting operators acting on $\mathbb{S}$ see [15] or [3].

Remark 2.1. In the following sections, we shall bound the L^q norm, $q \geqslant 2$, of some octonionic spherical harmonics $Y_{\ell\ell'} \in \mathcal{H}^{\ell\ell'}$. Our bounds rely on some classical asymptotic estimates holding when ℓ' or $\ell-\ell'$ are large enough; thus, also the estimates for the norm of $\pi_{\ell\ell'}$ hold under the assumption that ℓ' or $\ell-\ell'$ are sufficiently large. When ℓ' or $\ell-\ell'$ are bounded, the estimates are easier, and one can proceed, for instance, as in [9, Proposition 5.1].

We also refer to [9, Corollary 5.4], where the case of the highest weight vector, which is not treated in this paper, is considered. In fact, it is worth pointing out that in this note we just consider $\|\pi_{\ell\ell'}\|_{p,2}$ for $1 \leqslant p \leqslant 18/11$.

3. Some preliminary results

In the following, we shall need the Mehler–Heine formula for the disk polynomials, which may be found, for instance, in [4, p. 10].

Proposition 3.1. *Fix $n \in \mathbb{N}$. Let $j, k \in \mathbb{N}$, $j \leqslant k$. Then*

$$\lim_{\substack{j\to+\infty\\ k\to+\infty}} \Big(\cos(\frac{\theta}{\sqrt{jk}})\Big)^{k-j} \frac{P_j^{(2n-3,k-j)}\big(\cos(\frac{2\theta}{\sqrt{jk}})\big)}{P_j^{(2n-3,k-j)}(1)} = \Gamma(2n-2)\frac{J_{2n-3}(2\theta)}{\theta^{2n-3}};$$

here the symbol J_{2n-3} denotes the Bessel function of the first kind of order $2n-3$. This limit holds uniformly in every compact interval.

For $q \geqslant 2$ we set

$$\mathcal{I}_q := \left(\int_0^{\pi/2} \left| \frac{P_{\ell'}^{(3,\ell-\ell'+3)}(\cos(2\theta))}{P_{\ell'}^{(3,\ell-\ell'+3)}(\mathbf{1})} (\cos\theta)^{\ell-\ell'} \right|^q (\sin\theta\,\cos\theta)^7\, d\theta\right)^{1/q}$$

Proposition 3.2. *For all $q \geqslant 2$ and for all $(\ell,\ell') \in I_{\mathbb{S}}$ such that ℓ' is sufficiently large, we have*

$$\frac{\mathcal{I}_q}{\mathcal{I}_2} \gtrsim \ell^{4(\frac{1}{2}-\frac{1}{q})}\,(\ell')^{4(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}}. \tag{3.1}$$

Proof. One has

$$\begin{aligned}(\mathcal{I}_q)^q &\gtrsim \int_0^{1/\sqrt{\ell\ell'}} \left| \frac{P_{\ell'}^{(3,\ell-\ell'+3)}(\cos(2\theta))}{P_{\ell'}^{(3,\ell-\ell'+3)}(\mathbf{1})} (\cos\theta)^{\ell-\ell'+7/q} \right|^q \sin^7\theta\, d\theta \\ &\gtrsim \int_0^{1/\sqrt{\ell\ell'}} \left| \frac{P_{\ell'}^{(3,\ell-\ell'+3)}(\cos(2\theta))}{P_{\ell'}^{(3,\ell-\ell'+3)}(\mathbf{1})} (\cos\theta)^{\ell-\ell'+3} \right|^q \sin^7\theta\, d\theta.\end{aligned}$$

Notice that we have used the fact that $\theta \in [0, 1/\sqrt{\ell\ell'}]$. Then

$$
\begin{aligned}
\big(\mathcal{I}_q\big)^q &\gtrsim \int_0^1 \left| \frac{P_{\ell'}^{(3,\ell-\ell'+3)}\,(\cos(2\theta/\sqrt{\ell\ell'}))}{P_{\ell'}^{(3,\ell-\ell'+3)}(\mathbf{1})} \,\big(\cos(\theta/\sqrt{\ell\ell'})\big)^{\ell-\ell'+3} \right|^q \big(\sin(\theta/\sqrt{\ell\ell'})\big)^7 \,\frac{d\theta}{\sqrt{\ell\ell'}} \\
&\simeq \int_0^1 \left| \frac{P_{\ell'}^{(3,\ell-\ell'+3)}\,(\cos(2\theta/\sqrt{\ell\ell'}))}{P_{\ell'}^{(3,\ell-\ell'+3)}(\mathbf{1})} \,\big(\cos(\theta/\sqrt{\ell\ell'})\big)^{\ell-\ell'+3} \right|^q \big(\theta/\sqrt{\ell\ell'}\big)^7 \,\frac{d\theta}{\sqrt{\ell\ell'}} \\
&\simeq (\ell\ell')^{-4} \int_0^1 \left| \frac{P_{\ell'}^{(3,\ell-\ell'+3)}\,(\cos(2\theta/\sqrt{\ell\ell'}))}{P_{\ell'}^{(3,\ell-\ell'+3)}(\mathbf{1})} \,\big(\cos(\theta/\sqrt{\ell\ell'})\big)^{\ell-\ell'+3} \right|^q \theta^7\, d\theta \\
&\simeq (\ell\ell')^{-4} \left\| \frac{P_{\ell'}^{(3,\ell-\ell'+3)}\,(\cos(2\theta/\sqrt{\ell\ell'}))}{P_{\ell'}^{(3,\ell-\ell'+3)}(\mathbf{1})} \,\big(\cos(\theta/\sqrt{\ell\ell'})\big)^{\ell-\ell'+3} \right\|^q_{L^q([0,1];\, \theta^7\, d\theta)} .
\end{aligned}
$$

We now focus on $\mathcal{I}_2$. One first recalls that

$$
\Big(\int_0^\pi |P_{\ell-\ell'}^{(5/2,5/2)}(\cos\varphi)\,|^2 (\sin\varphi)^6 d\varphi \Big)^{1/2} \simeq (1+\ell-\ell')^{-1/2}
$$

in the light of [20, Page 391], and that $\|\mathbb{Z}_{\ell\ell'}\|_{L^2} \simeq \sqrt{d_{\ell\ell'}}$. Thus

$$
\begin{aligned}
\sqrt{d_{\ell\ell'}} &\simeq \|\mathbb{Z}_{\ell\ell'}\|_{L^2} \\
&\simeq d_{\ell\ell'} \binom{5/2+\ell-\ell'}{\ell-\ell'}^{-1} \Big(\int_0^\pi |P_{\ell-\ell'}^{(5/2,5/2)}(\cos\varphi)\,|^2 \sin^6\varphi\, d\varphi \Big)^{1/2} \\
&\quad \left(\int_0^{\pi/2} \left| \frac{P_{\ell'}^{(3,\ell-\ell'+3)}\,(\cos(2\theta))}{P_{\ell'}^{(3,\ell-\ell'+3)}(\mathbf{1})} \right|^2 |\cos\theta|^{2(\ell-\ell')} \sin^7\theta\, \cos^7\theta\, d\theta \right)^{1/2} \\
&\simeq d_{\ell\ell'} (1+\ell-\ell')^{-5/2} (1+\ell-\ell')^{-1/2}\, \mathcal{I}_2,
\end{aligned}
$$

whence

$$
\mathcal{I}_2 \simeq (d_{\ell\ell'})^{-1/2}\, (1+\ell-\ell')^3 \simeq (\ell')^{-3/2} (\ell+\ell')^{-2}
$$

by (2.5). Then finally one has

$$
\begin{aligned}
\frac{\mathcal{I}_q}{\mathcal{I}_2} &\gtrsim (\ell\ell')^{-4/q} \left\| \frac{P_{\ell'}^{(3,\ell-\ell'+3)}\,(\cos(2\theta/\sqrt{\ell\ell'}))}{P_{\ell'}^{(3,\ell-\ell'+3)}(\mathbf{1})} \,\big(\cos(\theta/\sqrt{\ell\ell'})\big)^{\ell-\ell'+3} \right\|_{L^q([0,1];\, \theta^7\, d\theta)} \\
&\qquad \times (\ell')^{3/2} (\ell+\ell')^2 \\
&\simeq \ell^{4(\frac{1}{2}-\frac{1}{q})}\, (\ell')^{4(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}} \\
&\qquad \times \left\| \frac{P_{\ell'}^{(3,\ell-\ell'+3)}\,(\cos(2\theta/\sqrt{\ell\ell'}))}{P_{\ell'}^{(3,\ell-\ell'+3)}(\mathbf{1})} \,\big(\cos(\theta/\sqrt{\ell\ell'})\big)^{\ell-\ell'+3} \right\|_{L^q([0,1];\, \theta^7\, d\theta)} .
\end{aligned}
\tag{3.2}
$$

Then an application of Proposition 3.1, the fact that that for all $j, k \in \mathbb{N}$, $j \leqslant k$,

$$
\sup_{\theta \in [0,\pi/2]} \Big| (\cos\theta)^{k-j}\, \frac{P_j^{(2n-3,k-j)}\,\big(\cos(2\theta)\big)}{P_j^{(2n-3,k-j)}(1)} \Big| \leqslant 1
$$

(see [4, p. 12]), and standard convergence properties yield, for ℓ' sufficiently large, the following estimate

$$\limsup_{\ell'\to+\infty}\left\|\frac{P_{\ell'}^{(3,\ell-\ell'+3)}(\cos(2\theta/\sqrt{\ell\ell'}))}{P_{\ell'}^{(3,\ell-\ell'+3)}(\mathbf{1})}\left(\cos(\theta/\sqrt{\ell\ell'})\right)^{\ell-\ell'+3}\right\|_{L^q([0,1];\,\theta^7\,d\theta)}$$
$$\gtrsim \Gamma(4)\left\|\frac{J_3(2\theta)}{\theta^3}\right\|_{L^q([0,1],\theta^7 d\theta)} > 0.$$

Notice that in the last step we have used standard asymptotics for the Bessel function. This, combined with (3.2), proves (3.1). □

Then, for $q \geqslant 2$ we set

$$\mathcal{J}_q := \left(\int_0^\pi \left|\frac{P_{\ell-\ell'}^{(5/2,5/2)}(\cos\varphi)}{P_{\ell-\ell'}^{(5/2,5/2)}(\mathbf{1})}\right|^q \sin^6\varphi\, d\varphi\right)^{1/q}.$$

For $\ell-\ell'$ sufficiently large, we estimate $\mathcal{J}_q^q$, again by means of [20, Page 391], getting

$$\begin{aligned}\mathcal{J}_q^q &= \int_0^\pi \left|\frac{P_{\ell-\ell'}^{(5/2,5/2)}(\cos\varphi)}{P_{\ell-\ell'}^{(5/2,5/2)}(\mathbf{1})}\right|^q \sin^6\varphi\, d\varphi\\
&\simeq (\ell-\ell')^{-\frac{5}{2}q}\left(\int_0^\pi \left|P_{\ell-\ell'}^{(5/2,5/2)}(\cos\varphi)\right|^q (\sin\varphi)^6\, d\varphi\right)\\
&\simeq \begin{cases}(\ell-\ell')^{-\frac{5}{2}q}(\ell-\ell')^{-\frac{q}{2}} \simeq (\ell-\ell')^{-3q} & \text{if } q<\frac{7}{3},\\ (\ell-\ell')^{-\frac{5}{2}q}(\ell-\ell')^{\frac{5}{2}q-7} \simeq (\ell-\ell')^{-7} & \text{if } q>\frac{7}{3}.\end{cases}\end{aligned}$$

In the critical point $q=\frac{7}{3}$ one obtains

$$\begin{aligned}\mathcal{J}_q^q &\simeq (\ell-\ell')^{-\frac{5}{2}q}(\ell-\ell')^{-\frac{q}{2}}\log(\ell-\ell')\\ &\simeq (\ell-\ell')^{-3q}\log(\ell-\ell').\end{aligned}$$

Notice that again we used [20, p. 391]. By observing that

$$\mathcal{J}_2 \simeq (\ell-\ell')^{-3},$$

one finally obtains, for all $q\geqslant 2$ and for all $(\ell,\ell')\in I_{\mathbb{S}}$ such that $\ell-\ell'$ is sufficiently large,

$$\frac{\mathcal{J}_q}{\mathcal{J}_2} \simeq \begin{cases}1 & \text{if } q<\frac{7}{3};\\ \left(\log(\ell-\ell')\right)^{1/q} & \text{if } q=\frac{7}{3};\\ (\ell-\ell')^{7(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}} & \text{if } q>\frac{7}{3}.\end{cases} \tag{3.3}$$

4. Optimality for $1 \leqslant p \leqslant 8/5$

We are going to show that, when p is close to 1, sharpness can be proved by means of zonal functions.

Proposition 4.1. *For all* $(\ell,\ell')\in I_{\mathbb{S}}$ *such that* ℓ' *and* $\ell-\ell'$ *are sufficiently large, and for all* $q\geqslant 2$ *we have*

$$\frac{||\mathbb{Z}_{\ell\ell'}||_q}{||\mathbb{Z}_{\ell\ell'}||_2}\gtrsim\begin{cases}\ell^{4(\frac12-\frac1q)}\,(\ell')^{4(\frac12-\frac1q)-\frac12} & \text{if } q<\frac73;\\ \ell^{4(\frac12-\frac1q)}\,(\ell')^{4(\frac12-\frac1q)-\frac12}\big(\log(\ell-\ell')\big)^{1/q} & \text{if } q=\frac73;\\ \ell^{4(\frac12-\frac1q)}\,(\ell')^{4(\frac12-\frac1q)-\frac12}\,(\ell-\ell')^{7(\frac12-\frac1q)-\frac12} & \text{if } q>\frac73.\end{cases}$$

Proof. From the definition of $\mathcal{I}_q$ and $\mathcal{J}_q$, and from Proposition 3.2, it follows that

$$\begin{aligned}\frac{\|\mathbb{Z}_{\ell\ell'}\|_q}{\|\mathbb{Z}_{\ell\ell'}\|_2}&\simeq\frac{\mathcal{I}_q}{\mathcal{I}_2}\,\frac{\mathcal{J}_q}{\mathcal{J}_2}\\ &\gtrsim\ell^{4(\frac12-\frac1q)}\,(\ell')^{4(\frac12-\frac1q)-\frac12}\;\times\;\frac{\mathcal{J}_q}{\mathcal{J}_2}.\end{aligned}$$

Then, if $q>\frac73$, (3.3) yields

$$\frac{\|\mathbb{Z}_{\ell\ell'}\|_q}{\|\mathbb{Z}_{\ell\ell'}\|_2}\gtrsim\ell^{4(\frac12-\frac1q)}\,(\ell')^{4(\frac12-\frac1q)-\frac12}\,(\ell-\ell')^{7(\frac12-\frac1q)-\frac12}.$$

The other cases are similar. □

Notice that $q>7/3$ if and only if $p<7/4$ and that $4/7<11/18<5/8$.
Then Proposition 4.1 yields optimality for $\|\pi_{\ell\ell'}\|_{p,2}$ when $p\in(1,8/5)$.

5. Optimality for $8/5\leqslant p\leqslant 18/11$.

In order to prove the sharpness of the bounds in Conjecture 1.1 when $8/5\leqslant p\leqslant 18/11$ one can consider the function $\mathbb{M}_{\ell\ell'}$, which, in the coordinates (2.1), is given by

$$\mathbb{M}_{\ell\ell'}(z_1,z_2,\theta):=q_{\ell-\ell'}(\varphi,\lambda_2,\ldots,\lambda_6,\sigma)(\cos\theta)^{\ell-\ell'}\frac{P_{\ell'}^{(3,\ell-\ell'+3)}(\cos(2\theta))}{P_{\ell'}^{(3,\ell-\ell'+3)}(\mathbf{1})},$$

with $\ell\geqslant\ell'\geqslant 0$; here $q_{\ell-\ell'}$ denotes the highest-weight spherical harmonic of degree $\ell-\ell'$ in Σ^7, that is, the polynomial $(x_{n-1}+ix_n)^{\ell-\ell'}$ (with x_{n-1} and x_n denoting the last two coordinates of z_1 in (2.2)).

A straightforward computation, together with (2.6) and (2.7), shows that

$$\Delta_7\mathbb{M}_{\ell\ell'}=-(\ell-\ell')(\ell-\ell'+6)\mathbb{M}_{\ell\ell'},$$

and that

$$\Delta\mathbb{M}_{\ell\ell'}=-(\ell+\ell')(\ell+\ell'+14)\mathbb{M}_{\ell\ell'}.$$

Thus $\mathbb{M}_{\ell\ell'}$ belongs to $\mathcal{H}^{\ell\ell'}$.

By combining the estimate

$$\frac{\|q_{\ell-\ell'}\|_{L^q(\Sigma^7)}}{\|q_{\ell-\ell'}\|_{L^2(\Sigma^7)}}\simeq(\ell-\ell')^{\frac32-\frac3q},$$

proved in [19, p.55] for all $q\geqslant 2$, with the bound (3.1), one obtains

$$\frac{\|\mathbb{M}_{\ell\ell'}\|_{L^q(\mathbb{S})}}{\|\mathbb{M}_{\ell\ell'}\|_{L^2(\mathbb{S})}}\gtrsim(\ell-\ell')^{\frac32-\frac3q}\,\ell^{4(\frac12-\frac1q)}\,(\ell')^{4(\frac12-\frac1q)-\frac12},\qquad q\geqslant 2. \tag{5.1}$$

A comparison with (1.2) shows that (5.1) yields optimality for $11/18\leqslant 1/p\leqslant 5/8$.

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