Warped tensor products and noncommutative 2d topological quantum field theories

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1 Introduction

The origins of topological quantum field theories can be traced back to the interplay between mathematics and physics, and their modern treatment continues to reflect these interdisciplinary beginnings. In this report we will emphasize an approach inspired by quantum groups to define *noncommutative* topological quantum field theories using deformations. Our main results follow.

Theorem 1.1 (Theorem 3.2)**.** *There is an equivalence between two dimensional topological quantum field theories and commutative Frobenius algebras. This equivalence is as good as it can be, namely it respects all the structure and properties of the categories involved in it.*

Theorem 1.2 (Theorem 4.3)**.** *There is a characterization of when twisted and cotwisted tensor products inherit a Frobenius algebra structure from their components. This characterization states that one single morphism controls both the deformation of the multiplication as well as the deformation of the comultiplication.*

Theorem 1.3 (Theorem 5.3)**.** *There is a characterization of when warped tensor products inherit a Frobenius algebra structure from their components. This characterization essentially states that an invertible central morphism controls both the deformation of the pairing as well as the deformation of the copairing.*

As an application, in Theorem 5.4 we provide a recipe to construct what ought to be noncommutative 2d TQFTs.

Notation. We denote by k a field (of arbitrary characteristic) and by 1_k its unit. All the categories we consider are symmetric monoidal, and we will often abbreviate their structure $(\mathsf{C}, \otimes, a, \mathbf{1}, l, r, c)$ as $(\mathsf{C}, \otimes, \mathbf{1}, c)$ or simply C when there is no ambiguity.

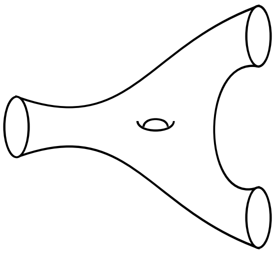


Figure 1: Example of a 2 cobordism.

2 Topological quantum field theories

In this section we recall the very basics of topological quantum field theories. We refer the reader to [Koc04] for more details.

Definition 2.1. Let $n \in \mathbb{N}$ and let M_1 and M_2 be n manifolds. An $n+1$ *cobordism* between M_1 and M_2 is an $n+1$ manifold N such that $\partial N = M_1 \amalg M_2$.

For example, the three punctured torus $N = \mathbb{T}^2 \setminus \{*_1, *_2, *_3\}$ is a 2 cobordism between one single circle $M_1 = \mathbb{S}^1$ and two disjoint circles $M_2 = \mathbb{S}^1 \amalg \mathbb{S}^1$, as illustrated in Figure 1.

Given n manifolds M_1, M_2, and M_3, together with $n+1$ cobordisms N_1 and N_2 with $\partial N_1 = M_1 \amalg M_2$ and $\partial N_2 = M_2 \amalg M_3$, the $n+1$ cobordism $(N_1 \amalg N_2)/M_2$ is known as *horizontal composition*. Given n manifolds M_1, M_2, M_3, and M_4, together with $n+1$ cobordisms N_1 and N_2 with $\partial N_1 = M_1 \amalg M_2$ and $\partial N_2 = M_3 \amalg M_4$, the $n+1$ cobordism $N_1 \amalg N_2$ is known as *vertical composition.*

Definition 2.2. Let $n \in \mathbb{N}$. The category whose objects are $n-1$ manifolds and whose morphisms are n cobordisms is called the *n cobordism category*, and denoted nCob.

We will be interested in $\mathsf{2Cob}$, whose objects are a (possibly empty) finite disjoint unions of $\mathbb{S}^1$, and whose morphisms are generated by the cylinder, cap, cup, both pairs of pants, and the swap.

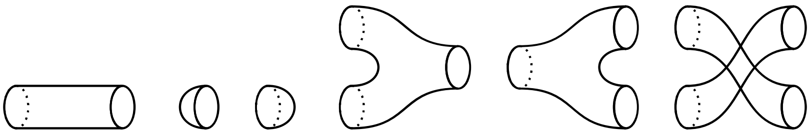


Figure 2: Generators of morphisms in $\mathsf{2Cob}$.

Indeed, any other morphism in $\mathsf{2Cob}$ can be obtained as a horizontal composition of the above by the classification of surfaces. We illustrate this in Figure 3.

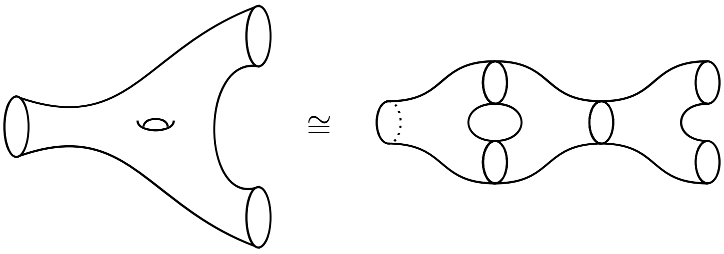
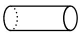


Figure 3: The 2 cobordism in Figure 1 as a composition of 2 cobordisms in Figure 2.

Definition 2.3. Let $n \in \mathbb{N}$. An *n dimensional topological quantum field theory*, or *nd TQFT*, is a symmetric monoidal functor $F : \mathsf{nCob} \to \mathrm{Vect}_k$. The category $\mathsf{SymMonFun}(\mathsf{nCob}, \mathrm{Vect}_k)$ whose objects are nd TQFTs and whose morphisms are the symmetric monoidal natural transformations between them is called the *category of nd TQFTs*.

To every Frobenius algebra A we can assign a 2d TQFT as follows.

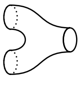
$$F\colon \mathsf{2Cob} \longrightarrow \mathrm{Vect}_k$$
$$\emptyset \longmapsto k$$
$$\mathbb{S}^1 \longmapsto A$$



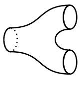
$\longmapsto \mathrm{id}_A : A \to A$

$\longmapsto u : k \to A$

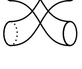
$\longmapsto \epsilon : A \to k$



$\longmapsto m : A \otimes A \to A$



$\longmapsto \Delta : A \to A \otimes A$



$\longmapsto \sigma : A \otimes A \to A \otimes A$

3 Frobenius algebras

In this section we define Frobenius algebras and relate them with 2d TQFTs.

Definition 3.1. Let $(\mathsf{C}, \otimes, \mathbf{1}, c)$ be a symmetric monoidal category. A *Frobenius algebra* in C is a tuple $(A, m, u, \Delta, \epsilon)$ where A is an object in C, $m : A \otimes A \to A$, $u : \mathbf{1} \to A$, $\Delta : A \to A \otimes A$, $\epsilon : A \to \mathbf{1}$ are morphisms in C, and the following diagrams commute.

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}_A \otimes u"] \arrow[dr, "\mathrm{id}_A"'] & A \otimes A \arrow[d, "m"] & A \arrow[l, "u \otimes \mathrm{id}_A"'] \arrow[dl, "\mathrm{id}_A"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mathrm{id}_A \otimes m"] \arrow[d, "m \otimes \mathrm{id}_A"'] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"] & A
\end{tikzcd}
\tag{1}$$

$$\begin{tikzcd}
A & A \otimes A \arrow[l, "\mathrm{id}_A \otimes \epsilon"'] \arrow[r, "\epsilon \otimes \mathrm{id}_A"] & A \\
& A \arrow[ul, "\mathrm{id}_A"] \arrow[u, "\Delta"'] \arrow[ur, "\mathrm{id}_A"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \otimes A & A \otimes A \arrow[l, "\mathrm{id}_A \otimes \Delta"'] \\
A \otimes A \arrow[u, "\Delta \otimes \mathrm{id}_A"] & A \arrow[l, "\Delta"'] \arrow[u, "\Delta"']
\end{tikzcd}
\tag{2}$$

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\Delta \otimes \mathrm{id}_A"] \arrow[dr, "m"] \arrow[dd, "\mathrm{id}_A \otimes \Delta"'] & & A \otimes A \otimes A \arrow[dd, "\mathrm{id}_A \otimes m"] \\
& A \arrow[dr, "\Delta"] & \\
A \otimes A \otimes A \arrow[rr, "m \otimes \mathrm{id}_A"] & & A
\end{tikzcd}
\tag{3}$$

The commutativity of diagram (3) is known as the *Frobenius condition*. It is a topological restriction that captures the two diffeomorphisms of Figure 4.

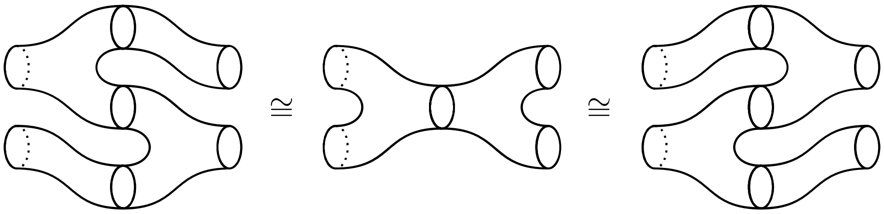


Figure 4: The topological interpretation of the Frobenius condition.

A Frobenius algebra A in C is said to be *commutative* when the following diagram commutes.

$$\begin{tikzcd}
A \otimes A \arrow[rr, "c_{A,A}"] \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}
\tag{4}$$

We denote by $\mathsf{cFrob}(\mathsf{C})$ the category of commutative Frobenius algebras in C and by $\mathsf{Frob}(\mathsf{C})$ the category of all Frobenius algebras in C. We have the following equivalence.

Theorem 3.2 ([Abr96], [Koc04], [Oca24])**.** *Let* C *be a symmetric monoidal category. There is a symmetric monoidal equivalence of categories* $\mathsf{SymMonFun}(\mathsf{nCob}, \mathsf{C}) \simeq \mathsf{cFrob}(\mathsf{C})$ *given by the following assignment.*

$$\begin{aligned}
\mathsf{SymMonFun}(\mathsf{nCob}, \mathsf{C}) &\longrightarrow \mathsf{cFrob}(\mathsf{C}) \\
F &\longmapsto F(\mathbb{S}^1) \\
\eta \colon F \to G &\longmapsto \eta_{\mathbb{S}^1} : F(\mathbb{S}^1) \to G(\mathbb{S}^1)
\end{aligned}
\tag{5}$$

4 Twisted and cotwisted tensor products

In this section we define of twisted and cotwisted tensor products and characterize when they are Frobenius algebras. These objects were introduced in [ČSV95].

Definition 4.1. Let (A, m_A, u_A) and (B, m_B, u_B) be algebras in C a symmetric monoidal category, let $\tau : B \otimes A \to A \otimes B$ be a morphism in C, and set $A \otimes_\tau B := A \otimes B$, $m_{A\otimes_\tau B} := (m_A \otimes m_B)(\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_B)$, and $u_{A\otimes_\tau B} := u_A \otimes u_B$. The *twisted tensor product* of A and B is the tuple $(A \otimes_\tau B, m_{A\otimes_\tau B}, u_{A\otimes_\tau B})$.

Theorem 4.2 ([ČSV95]). *Let (A, m_A, u_A) and (B, m_B, u_B) be algebras in C a symmetric monoidal category, let $\tau : B\otimes A \to A\otimes B$ be an isomorphism in C, and let the following diagrams commute.*

$$\begin{tikzcd}
B \otimes A \arrow[rd, "\tau"] & A \arrow[l, "u_B\otimes \mathrm{id}_A"'] \arrow[d, "\mathrm{id}_A\otimes u_B"] \\
B \arrow[u, "\mathrm{id}_B\otimes u_A"] \arrow[r, "u_A\otimes \mathrm{id}_B"] & A \otimes B
\end{tikzcd} \tag{6}$$

$$\begin{tikzcd}
B\otimes B\otimes A\otimes A \arrow[r, "\mathrm{id}_B\otimes\tau\otimes\mathrm{id}_A"] \arrow[d, "m_B\otimes m_A"'] & B\otimes A\otimes B\otimes A \arrow[r, "\tau\otimes\tau"] & A\otimes B\otimes A\otimes B \arrow[d, "\mathrm{id}_A\otimes\tau\otimes\mathrm{id}_B"] \\
B\otimes A \arrow[r, "\tau"] & A\otimes B & A\otimes A\otimes B\otimes B \arrow[l, "m_A\otimes m_B"']
\end{tikzcd} \tag{7}$$

Then the twisted tensor product $(A \otimes_\tau B, m_{A\otimes_\tau B}, u_{A\otimes_\tau B})$ is an algebra in C.

This definition and theorem can be replicated by inverting all the arrows (or by working in the opposite category C^{op}, which is symmetric monoidal in a canonical way) to obtain the notion of cotwisted tensor product, which we denote by $(A \otimes^\theta B, \Delta_{A\otimes^\theta B}, \epsilon_{A\otimes^\theta B})$.

The author, together with Oswald, Das, and Plavnik, has studied when the deformations $A \otimes_\tau B$ and $A \otimes^\theta B$ inherit the structure of a Frobenius algebra from A and B. This becomes increasingly interesting when both deformations are applied simultaneously, obtaining a *twisted and cotwisted* tensor product $(A \otimes^\theta_\tau B, m_{A\otimes_\tau B}, u_{A\otimes_\tau B}, \Delta_{A\otimes^\theta B}, \epsilon_{A\otimes^\theta B})$.

Theorem 4.3. *Let $(A, m_A, u_A, \Delta_A, \epsilon_A)$ and $(B, m_B, u_B, \Delta_B, \epsilon_B)$ be algebras and coalgebras in C a symmetric monoidal category, let $\tau : B \otimes A \to A \otimes B$ be an isomorphism in C satisfying the commutativity of diagrams* (6) *and* (7)*, and let $\theta : A \otimes B \to B \otimes A$ be an isomorphism in C satisfying the commutativity of the analogous diagrams.*

1. *The twisted and cotwisted tensor product $A \otimes^{\tau^{-1}}_\tau B$ is a Frobenius algebra if τ^{-1} makes certain diagrams commute, see [OO24].*
2. *The twisted and cotwisted tensor product $A \otimes^\theta_{c_{B,A}} B$ is a Frobenius algebra if and only if $\theta = c_{A,B}$, see [DOP25].*
3. *The twisted and cotwisted tensor product $A \otimes^\theta_\tau B$ is a Frobenius algebra if and only if $\theta = \tau^{-1}$, see [OO25].*

Unfortunately, Theorem 4.3 highlights a fundamental problem with the naive attempt at defining a noncommutative TQFT by replacing the tensor product of two Frobenius algebras with a twisted and cotwisted tensor product: such a twisted and cotwisted tensor product will almost never be commutative, so it will not yield interesting examples of the usual 2d TQFTs. Requiring that a twisted and cotwisted tensor product is a Frobenius algebra imposes too strong of a relation between the morphisms τ and θ. To remedy this, in the following section we propose to deform a different structure map.

5 Warped tensor products

In this section we introduce warped tensor products of Frobenius algebras and characterize when they are Frobenius algebras. This is based on the upcoming preprint [DOP25].

The definition of Frobenius algebra that best suits our purposes follows.

Definition 5.1. Let C be a symmetric monoidal category. A *Frobenius algebra* in C is a tuple (A, m, u, β, α) where (A, m, u) is an algebra in C, $\beta : A\otimes A \to \mathbf{1}$ and $\alpha : \mathbf{1} \to A\otimes A$ are morphisms in C, and the following diagrams commute.

$$\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "m\otimes\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A\otimes m"'] & A\otimes A \arrow[d, "\beta"] \\
A\otimes A \arrow[r, "\beta"] & \mathbf{1}
\end{tikzcd} \tag{8}$$

$$\begin{tikzcd}
A \arrow[r, "\alpha\otimes\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A\otimes\alpha"'] \arrow[dr, "\mathrm{id}_A"] & A\otimes A\otimes A \arrow[d, "\mathrm{id}_A\otimes\beta"] \\
A\otimes A\otimes A \arrow[r, "\beta\otimes\mathrm{id}_A"] & A
\end{tikzcd} \tag{9}$$

Given an object A in C, a morphism $\beta : A \otimes A \to \mathbf{1}$ is said to be a *pairing*, and a morphism $\alpha : \mathbf{1} \to A \otimes A$ is said to be a *copairing*. This reinterpretation allows us to consider deformations of the pairing.

Definition 5.2. Let $(A, m_A, u_A, \beta_A, \alpha_A)$ and $(B, m_B, u_B, \beta_B, \alpha_B)$ be Frobenius algebras in C a symmetric monoidal category, let $\omega : B \otimes A \to A \otimes B$ be an isomorphism in C, and set $A \mathbin{\overline{\times}}_\omega B := A \otimes B$, $m_{A\overline{\times}_\omega B} := (m_A \otimes m_B)(\mathrm{id}_A \otimes c_{B,A} \otimes \mathrm{id}_B)$, $u_{A\overline{\times}_\omega B} := u_A \otimes u_B$, $\beta_{A\overline{\times}_\omega B} := (\beta_A \otimes \beta_B)(\mathrm{id}_A \otimes \omega \otimes \mathrm{id}_B)$, and $\alpha_{A\overline{\times}_\omega B} := (\mathrm{id}_A \otimes \omega^{-1} \otimes \mathrm{id}_B)(\alpha_A \otimes \alpha_B)$. The *warped tensor product* of A and B is the tuple $(A \mathbin{\overline{\times}}_\omega B, m_{A\overline{\times}_\omega B}, u_{A\overline{\times}_\omega B}, \beta_{A\overline{\times}_\omega B}, \alpha_{A\overline{\times}_\omega B})$.

Once again, we can ask whether $A \mathbin{\overline{\times}}_\omega B$ inherits a Frobenius algebra structure.

Theorem 5.3 ([DOP25])**.** *Let $(A, m_A, u_A, \beta_A, \alpha_A)$ and $(B, m_B, u_B, \beta_B, \alpha_B)$ be Frobenius algebras in C a symmetric monoidal category, C, and let $\omega : B \otimes A \to A \otimes B$ be an isomorphism in C. The warped tensor product $A \mathbin{\overline{\times}}_\omega B$ is a Frobenius algebra if and only if there exist $\psi : \mathbf{1} \to A \otimes B$ and $\varphi : \mathbf{1} \to A \otimes B$ morphisms in C such that the following diagrams commute.*

$$\begin{tikzcd}
B\otimes A \arrow[r, "\omega"] \arrow[d, "c_{B,A}"'] & A\otimes B \\
A\otimes B \arrow[r, "\mathrm{id}_A\otimes\mathrm{id}_B\otimes\psi"] & A\otimes B\otimes A\otimes B \arrow[u, "m_{A\overline{\times}_\omega B}"']
\end{tikzcd} \tag{10}$$

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "\psi\otimes\varphi"] \arrow[d, "\varphi\otimes\psi"'] \arrow[dr, "u_{A\overline{\times}_\omega B}"] & A\otimes B\otimes A\otimes B \arrow[d, "m_{A\overline{\times}_\omega B}"] \\
A\otimes B\otimes A\otimes B \arrow[r, "m_{A\overline{\times}_\omega B}"] & A\otimes B
\end{tikzcd} \tag{11}$$

$$\begin{tikzcd}
A\otimes B \arrow[r, "\mathrm{id}_A\otimes\mathrm{id}_B\otimes\psi"] \arrow[d, "\psi\otimes\mathrm{id}_A\otimes\mathrm{id}_B"] & A\otimes B\otimes A\otimes B \arrow[d, "m_{A\overline{\times}_\omega B}"'] & A\otimes B \arrow[r, "\mathrm{id}_A\otimes\mathrm{id}_B\otimes\varphi"] \arrow[d, "\varphi\otimes\mathrm{id}_A\otimes\mathrm{id}_B"] & A\otimes B\otimes A\otimes B \arrow[d, "m_{A\overline{\times}_\omega B}"'] \\
A\otimes B\otimes A\otimes B \arrow[r, "m_{A\overline{\times}_\omega B}"] & A\otimes B & A\otimes B\otimes A\otimes B \arrow[r, "m_{A\overline{\times}_\omega B}"] & A\otimes B
\end{tikzcd} \tag{12}$$

As an application of this characterization, we can construct (braided or symmetric) monoidal structures on $\mathsf{Frob}(\mathsf{C})$ that descend to $\mathsf{cFrob}(\mathsf{C})$. In other words, warped tensor products induce symmetric monoidal structures on $\mathsf{Frob}(\mathsf{C})$.

Theorem 5.4 ([DOP25])**.** *Let C be a symmetric monoidal category, for each $(A, m_A, u_A, \beta_A, \alpha_A)$ in $\mathsf{Frob}(\mathsf{C})$ choose $\psi_A : \mathbf{1} \to A$ a morphism in C such that for all $(B, m_B, u_B, \beta_B, \alpha_B)$ in $\mathsf{Frob}(\mathsf{C})$ we have the following.*

1. *$\omega_{B,A} := (m_A \otimes m_B)(\mathrm{id}_A \otimes c_{B,A} \otimes \mathrm{id}_B)(\mathrm{id}_A \otimes \mathrm{id}_B \otimes \psi_A \otimes \psi_B)(c_{B,A})$ makes the warped tensor product $A \mathbin{\overline{\times}}_{\omega_{B,A}} B$ into a Frobenius algebra.*
2. *$\psi_{A\otimes B} = \psi_A \otimes \psi_B$.*

Then the following assignment makes $(\mathsf{Frob}(\mathsf{C}), \overline{\times}, \mathbf{1}, c)$ into a symmetric monoidal category.

$$\begin{aligned} \overline{\times} \colon \mathsf{Frob}(\mathsf{C}) \times \mathsf{Frob}(\mathsf{C}) &\longrightarrow \mathsf{Frob}(\mathsf{C}) \\ (A, B) &\longmapsto A \mathbin{\overline{\times}}_{\omega_{B,A}} B \end{aligned} \tag{13}$$

Remark 5.5. Let $\mathsf{C} = \mathrm{Vect}_k$ and for each A in $\mathsf{Frob}(\mathsf{C})$ let $z_A \in A$ be an invertible central element. Such a choice can always be made because $u(1_k) \in A$ is invertible and central. Then the linear maps defined by $\psi_A(1_k) = z_A$ satisfy the hypothesis of Theorem 5.4, yielding a symmetric monoidal structure on Vect_k which (depending on the choices of z_A) may be different from the standard one.

The braided monoidal categories obtained in this way are the first approach to the requirements of noncommutative 2d TQFTs proposed by the author in his work.

References

[Abr96] L. Abrams. Two-dimensional topological quantum field theories and Frobenius algebras. *J. Knot Theory Ramifications*, 5(5):569–587, 1996.

[ČSV95] A. Čap, H. Schichl, and J. Vanžura. On twisted tensor products of algebras. *Comm. Algebra*, 23(12):4701–4735, 1995.

[DOP25] R. Das, P. S. Ocal, and J. Plavnik. On warped tensor products of frobenius algebras, 2025. To appear.

[Koc04] J. Kock. *Frobenius algebras and 2D topological quantum field theories*, volume 59 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2004.

[Oca24] P. S. Ocal. Symmetric monoidal equivalences of topological quantum field theories in dimension two and Frobenius algebras. *Proc. Amer. Math. Soc.*, 152(5):2261–2265, 2024.

[OO24] P. S. Ocal and A. Oswald. A dichotomy between twisted tensor products of bialgebras and Frobenius algebras. *J. Algebra*, 644:351–380, 2024.

[OO25] P. S. Ocal and A. Oswald. On twisted and cotwisted tensor products of frobenius algebras, 2025. To appear.

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