SYMMETRY BREAKING OPERATORS FOR $(\mathrm{GL}(n+1,\mathbb{R}),\ \mathrm{GL}(n,\mathbb{R}))$ AND RIESZ DISTRIBUTIONS

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1. Introduction

In this short exposition, we highlight several results on symmetry breaking operators between principal series representations of $\mathrm{GL}(n+1,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{R})$, originating from joint work with Jan Frahm [DF24] and Quentin Labriet [DL25].

Following the ideas developed in by Kobayashi and Speh [KS15], these operators can be described in terms of distributional kernels, and the results in this paper are established in that setting. Although the proofs are carried out in a broader setting, the essential ideas can already be illustrated in the simpler case of the Riesz distributions $|x|^{\mu}_{\varepsilon} = \operatorname{sgn}(x)|x|^{\mu}$. The purpose of this note is therefore to present these results and, at the same time, to illustrate the underlying methods by discussing the analogous situation for the simpler Riesz distributions. We consider a family of distributions $K^{\eta,\nu}_{\xi,\lambda}$ on $\mathrm{GL}(n+1,\mathbb{R})$ depending on parameters $(\xi,\lambda,\eta,\nu) \in (\mathbb{Z}/2\mathbb{Z})^{n+1} \times \mathbb{C}^{n+1} \times (\mathbb{Z}/2\mathbb{Z})^{n} \times \mathbb{C}^{n}$. For instance, in the case $n=2$, it takes the form

$$K^{\eta,\nu}_{\xi,\lambda}\begin{pmatrix} 1 & & \\ x & 1 & \\ z & y & 1 \end{pmatrix} = |z|^{\lambda_1-\nu_2-\frac{1}{2}}_{\xi_1+\eta_2}|x|^{\nu_2-\lambda_2-\frac{1}{2}}_{\eta_2+\xi_2}|z-xy|^{\lambda_2-\nu_1-\frac{1}{2}}_{\xi_2+\eta_1}, \qquad (x,y,z\in\mathbb{R}).$$

1.1. **Notation.** $\mathbb{N} = \{1,2\ldots\}$, $\mathbb{N}_0 = \{0,1,2,\ldots,\}$, $[\varepsilon] \in \{0,1\}$ denotes the unique representative of $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$. Moreover, let e_i denote the standard basis of $\mathbb{C}^n$ and $E_{i,j}$ the standard basis of $n \times n$ matrices.

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2. Setup

2.1. **Principal series representations.** Let $G_k = \mathrm{GL}(k,\mathbb{R})$, and consider the minimal parabolic subgroup P_k of G_k, consisting of upper triangular matrices. Then P_k has Langlands decomposition $P_k = M_kA_kN_k$, where M_k is the group of diagonal matrices with entries in $\{-1,1\}$, A_k is the group of diagonal matrices with entries in $\mathbb{R}_{>0}$ and N_k consists of unipotent upper triangular matrices. We let $\mathfrak{a}_k$ denote the Lie algebra of A and $\mathfrak{n}_k$ the Lie algebra of N_k.

The irreducible representations of M_k are one-dimensional and are given by the characters

$$M_k \to \{-1,1\}, \qquad \operatorname{diag}(\varepsilon_1,\ldots,\varepsilon_k) \mapsto \varepsilon_1^{\delta_1}\cdots\varepsilon_k^{\delta_k}$$

where $\delta = (\delta_1,\ldots,\delta_k) \in (\mathbb{Z}/2\mathbb{Z})^k$. We identify δ with the corresponding character of M_k. Furthermore, we identify $(\mathfrak{a}_k)^*_{\mathbb{C}} \simeq \mathbb{C}^k$ via $\mu \mapsto (\mu(E_{1,1}),\cdots,\mu(E_{k,k}))$. Then the half sum of positive roots $\rho_k = \frac{1}{2}\operatorname{tr}\operatorname{ad}|_{\mathfrak{n}_k}$ corresponds to $\frac{1}{2}(k-1,k-3,\ldots,3-k,1-k)$. For $\mu \in \mathbb{C}^k$ we write e^{μ} for the character of A_k given by $e^{\mu}(e^H) = e^{\mu(H)}$ $(H \in \mathfrak{a}_k)$.

For $\delta \in (\mathbb{Z}/2\mathbb{Z})^k$ and $\mu \in \mathbb{C}^k$ we extend the character $\delta\otimes e^{\mu}$ of M_kA_k trivially to $P_k = M_kA_kN_k$ and write $\delta \otimes e^{\mu} \otimes 1$ for it. Using smooth (normalized) parabolic induction from P_k to G_k, we obtain the principal series representation $\operatorname{Ind}_{P_k}^{G_k}(\delta \otimes e^{\mu} \otimes 1)$, as the left-regular representation of G_k on

$$\{f \in \mathbb{C}^{\infty}(G_k) \mid f(gman) = \delta(m)^{-1}a^{-\mu-\rho_k}f(g)\,\forall man \in P_k\,\forall g \in G_k\}.$$

For $k = n+1$, we denote the principal series representation by

$$\pi_{\xi,\lambda} = \operatorname{Ind}_{P_{n+1}}^{G_{n+1}}(\xi \otimes e^{\lambda} \otimes 1), \qquad (\xi \in (\mathbb{Z}/2\mathbb{Z})^{n+1},\ \lambda \in \mathbb{C}^{n+1}).$$

Similarly, for $k = n$, we denote the principal series representation by

$$\tau_{\eta,\nu} = \operatorname{Ind}_{P_n}^{G_n}(\eta \otimes e^{\nu} \otimes 1), \qquad (\eta \in (\mathbb{Z}/2\mathbb{Z})^{n}, \nu \in \mathbb{C}^{n}).$$

2.2. Symmetry breaking operators in terms of distributions.

Let $G = G_{n+1}$ and $H = G_n$, where we consider H as a subgroup of G by embedding it in the upper left block. We denote the minimal parabolic subgroups P_{n+1} and P_n by P_G and P_H, respectively.

Symmetry breaking operators are intertwining operators between the principal series representations $\pi_{\xi,\lambda}|_H$ and $\tau_{\eta,\nu}$, that is elements in

$$\operatorname{Hom}_H(\pi_{\xi,\lambda}|_H, \tau_{\eta,\nu}).$$

We define a G-invariant pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ on $\pi_{\xi,-\lambda} \times \pi_{\xi,\lambda}$ given by

$$\langle \varphi, \psi \rangle = \int_{G/P_G} \varphi(g)\psi(g)\, d(gP_G), \qquad (\varphi \in \pi_{\xi,-\lambda},\ \psi \in \pi_{\xi,\lambda}).$$

Here $d(gP_G)$ denotes the unique (up to scalar multiples) G-invariant integral on $C(G \times_{P_G} \mathbb{C}_{2\rho_G})$. Since $\overline{N}_G \simeq \mathbb{R}^{n(n+1)/2}$ is open and dense in G/P_G, the pairing $\langle\cdot,\cdot\rangle$ may equivalently be expressed as an integral over $\overline{N}_G$. Using this pairing, we can identify the space of symmetry breaking operators with distributions on G satisfying certain equivariance properties in the sense that $A \in \operatorname{Hom}_H(\pi_{\xi,\lambda}|_H, \tau_{\eta,\nu})$ is given by

$$Af = \langle K_A, f\rangle, \qquad (f \in \pi_{\xi,\lambda})$$

where K_A is a distribution on G (for a detailed discussion on this see [KS15] and [Fra23]).

2.3. An explicit family of symmetry breaking operators.

Consider the following polynomials on G, defined by certain minor determinants:

$$\Phi_p(g) = \det(g_{ij})_{n+2-p \le i \le n+1,\, 1 \le j \le p} \qquad \Psi_q(g) = \det(g_{ij})_{n+1-q \le i \le n,\, 1 \le j \le q} \qquad (g \in G),$$

where $1 \le p \le n+1$ and $1 \le q \le n$. We then define the function

$$K_{\xi,\lambda}^{\eta,\nu}(g) = |\Phi_1(g)|_{\delta_1}^{s_1} \cdots |\Phi_{n+1}(g)|_{\delta_{n+1}}^{s_{n+1}} |\Psi_1(g)|_{\varepsilon_1}^{t_1} \cdots |\Psi_n(g)|_{\varepsilon_n}^{t_n},$$

where

$$s_i = \lambda_i - \nu_{n+1-i} - \frac{1}{2}, \qquad t_i = \nu_{n+1-i} - \lambda_{i+1} - \frac{1}{2}, \qquad s_{n+1} = \lambda_{n+1} + \frac{n}{2},$$

and

$$\delta_i = \xi_i + \eta_{n+1-i}, \qquad \varepsilon_i = \eta_{n+1-i} + \xi_{i+1}, \qquad \delta_{n+1} = \xi_{n+1},$$

for $i = 1, \ldots, n$. We refer to the parameters $(\xi, \lambda, \eta, \nu)$ as *principal series parameters* and to $(\delta, s, \varepsilon, t)$ as *spectral parameters.* We will use the principal series parameters and spectral parameters interchangeably in the sense that $K_{\xi,\lambda}^{\eta,\nu} = K_{\delta,s}^{\varepsilon,t}$.

The function $K_{\xi,\lambda}^{\eta,\nu}$ satisfies the appropriate equivariance properties ensuring it defines a symmetry breaking operator by

$$A_{\xi,\lambda}^{\eta,\nu} f = \langle K_{\xi,\lambda}^{\eta,\nu}, f\rangle, \qquad (f \in \pi_{\xi,\lambda})$$

for the parameters where $K_{\xi,\lambda}^{\eta,\nu}$ is locally integrable on G.

Proposition 2.1 ([DF24, Proposition 2.1])**.** *The function $K_{\xi,\lambda}^{\eta,\nu}$ is locally integrable on G if and only if* $\operatorname{Re}(\lambda_i - \nu_j + \frac{1}{2}) > 0$ *for all $i + j \le n+1$ and* $\operatorname{Re}(\nu_j - \lambda_i + \frac{1}{2}) > 0$ *for all $i + j > n+1$.*

3. Analytic continuation

The family $(K^{\eta,\nu}_{\xi,\lambda})$ defines a holomorphic family of distributions for the parameters (λ,ν) in the domain of $\mathbb{C}^{n+1}\times\mathbb{C}^n$ described in Proposition 2.1. In this section we aim to analytically extend this family beyond this initial domain. To this end, we find a meromorphic function $\gamma(\xi,\lambda,\eta,\nu)$ such that

$$\gamma(\xi,\lambda,\eta,\nu)^{-1}K^{\eta,\nu}_{\xi,\lambda}$$

is holomorphic in $(\lambda,\nu)\in\mathbb{C}^{n+1}\times\mathbb{C}^n$. Such a function γ is called a *normalization* of $K^{\eta,\nu}_{\xi,\lambda}$. Consequently, the normalized holomorphic family of distributions give rise to a holomorphic family of symmetry breaking operators.

Consider the meromorphic function

$$n(\xi,\lambda,\eta,\nu)=\prod_{i+j\le n+1}\Gamma\big(\tfrac{\lambda_i-\nu_j+\frac{1}{2}+[\xi_i+\eta_j]}{2}\big)\prod_{i+j>n+1}\Gamma\big(\tfrac{\nu_j-\lambda_i+\frac{1}{2}+[\eta_j+\xi_i]}{2}\big),$$

and the normalized family of distributions

$$\mathbf{K}^{\eta,\nu}_{\xi,\lambda}:=n(\xi,\lambda,\eta,\nu)^{-1}K^{\eta,\nu}_{\xi,\lambda}.$$

Theorem 3.1 ([DF24, Theorem A])**.** *The family of distributions* $(\mathbf{K}^{\eta,\nu}_{\xi,\lambda})$ *depends holomorphically on* $(\lambda,\nu)\in\mathbb{C}^{n+1}\times\mathbb{C}^n$ *for all* $(\xi,\eta)\in(\mathbb{Z}/2\mathbb{Z})^{n+1}\times(\mathbb{Z}/2\mathbb{Z})^n$.

The process of finding this normalization and performing the analytic continuation proceeds in two steps:

(1) Find Bernstein–Sato identities for $K^{\eta,\nu}_{\xi,\lambda}$.

- Use the corresponding Bernstein–Sato polynomials to construct an initial normalization $\tilde{n}(\xi,\lambda,\eta,\nu)$ of $K^{\eta,\nu}_{\xi,\lambda}$.
- Apply these identities to analytically extend the family $(\tilde{n}(\xi,\lambda,\eta,\nu)^{-1}\times K^{\eta,\nu}_{\xi,\lambda})$ to all parameters $(\lambda,\nu)\in\mathbb{C}^{n+1}\times\mathbb{C}^n$.

(2) Identify singularities of $\tilde{n}(\xi,\lambda,\eta,\nu)$ for which $\tilde{n}(\xi,\lambda,\eta,\nu)^{-1}K^{\eta,\nu}_{\xi,\lambda}$ vanishes.

- Refine the normalization of $K^{\eta,\nu}_{\xi,\lambda}$ by removing these singularities.

Example 3.2. We will now illustrate these steps in the simple case of the Riesz distribution $u^{\mu}_{\varepsilon}(x):=|x|^{\mu}_{\varepsilon}$, which we consider as a distribution on Schwartz functions. The Riesz distribution is locally integrable whenever $\mathrm{Re}(\mu)>-1$.

In this case, the Bernstein–Sato identity takes the form

$$\frac{d}{dx}u^{\mu}_{\varepsilon}=\mu\cdot u^{\mu-1}_{\varepsilon-1}.$$

From this we introduce the normalization $\tilde{n}(\varepsilon,\mu)=\Gamma(\mu+1)$ and define $\tilde{u}^{\mu}_{\varepsilon}:=\tilde{n}(\varepsilon,\mu)^{-1}u^{\mu}_{\varepsilon}$. Then

$$\frac{d}{dx}\tilde{u}^{\mu}_{\varepsilon}=\mu\cdot\frac{\Gamma(\mu)}{\Gamma(\mu+1)}\cdot\tilde{u}^{\mu-1}_{\varepsilon-1}=\tilde{u}^{\mu-1}_{\varepsilon-1}.$$

This identity allows us to nalytically extend $\tilde{u}^{\mu}_{\varepsilon}$ from $\mathrm{Re}(\mu)>-1$ to $\mu\in\mathbb{C}$. Specifically, if $\mathrm{Re}(\mu)\in(-n-1,-n]$ then

$$\tilde{u}^{\mu}_{\varepsilon}=\frac{d^n}{dx^n}\tilde{u}^{\mu+n}_{\varepsilon+n}.$$

To refine the normalization, note that $\tilde{u}_0^0 = 1$ and hence $\tilde{u}_{-1}^{-1} = \frac{d}{dx}\tilde{u}_0^0 = 0$. Similarly, $\tilde{u}_{-n}^{-n} = 0$ for $n \in \mathbb{N}$. These zeroes occur precisely at the poles of $\Gamma(\frac{\mu+2-[\varepsilon]}{2})$, and since

$$\tilde{n}(\varepsilon,\mu) = \Gamma(\mu+1) = \frac{2^\mu}{\sqrt{\pi}}\Gamma(\tfrac{\mu+1+[\varepsilon]}{2})\Gamma(\tfrac{\mu+2-[\varepsilon]}{2}),$$

it suffices to use the simplified normalization $n(\varepsilon,\mu) = \Gamma(\frac{\mu+1+[\varepsilon]}{2})$ so that

$$\mathbf{u}_\varepsilon^\mu = n(\varepsilon,\mu)^{-1}u_\varepsilon^\mu,$$

forms a holomorphic family of distributions.

△

As this example shows, the two steps provide a concrete procedure for finding a normalization and to analytically extending the distribution. For the Riesz distribution, step (1), that is finding a Bernstein–Sato identity, is straight forward as there is only one variable and step (2) we get from the observation $u_0^0 = 1$. However, how to perform step (1) and (2) is less clear in context of $K_{\xi,\lambda}^{\eta,\nu}$. Guessing Bernstein–Sato identities directly is infeasible, and simply fixing the parameters such that $K_{\xi,\lambda}^{\eta,\nu} = 1$ does not guarantee we get any zeros, since the explicit form of the associated differential operators is unknown.

We now revisit Example 3.2, deriving steps (1) and (2) in an alternative way that is better suited for generalization to $K_{\xi,\lambda}^{\eta,\nu}$.

Example 3.3. Consider the Fourier transform of the Riesz distribution

$$\mathcal{F}u_\varepsilon^\mu = i^{[\varepsilon]}\sqrt{\pi}2^{\mu+1}\frac{\Gamma(\frac{\mu+1+[\varepsilon]}{2})}{\Gamma(\frac{-\mu+[\varepsilon]}{2})}u_\varepsilon^{-\mu-1},$$

see, e.g., [GS64, Chapter 2, Section 3, Equations (12) and (13)]. (Note that this formula alone suffices to analytically continue u_ε^μ but this is not so in the corresponding situation for $K_{\xi,\lambda}^{\eta,\nu}$.) Now consider the multiplication operator M defined by $f \mapsto xf$. It is immediate that $Mu_\varepsilon^\mu = u_{\varepsilon+1}^{\mu+1}$. Moreover, conjugatinging M by $\mathcal{F}$ and applying it to u_ε^μ yields

$$(\mathcal{F}\circ M\circ\mathcal{F}^{-1})u_\varepsilon^\mu = -i\mu \times u_{\varepsilon-1}^{\mu-1}.$$

Since $\mathcal{F}\circ M\circ\mathcal{F}^{-1} = -i\frac{d}{dx}$ we thereby obtain a Bernstein–Sato identity for u_ε^μ using only information about $\mathcal{F}u_\varepsilon^\mu$ and Mu_ε^μ.

For step (2), recall that $\tilde{u}_\varepsilon^\mu = \Gamma(\mu+1)^{-1}u_\varepsilon^\mu$ and note that

$$\mathcal{F}\tilde{u}_\varepsilon^\mu = i^{[\varepsilon]}\sqrt{\pi}2^{-\mu}\frac{\Gamma(\frac{-\mu+1-[\varepsilon]}{2})}{\Gamma(\frac{\mu+2-[\varepsilon]}{2})}\tilde{u}_\varepsilon^{-\mu-1}.$$

Observe that the left-hand side is holomorphic in μ. For this to hold for the right-hand side, we must have $\tilde{u}_\varepsilon^{-\mu-1} = 0$ at the poles of $\Gamma(\frac{1-[\varepsilon]-\mu}{2})$. Hence $\tilde{u}_\varepsilon^\mu = 0$ at the poles of $\Gamma(\frac{\mu+2-[\varepsilon]}{2})$ and we recover the correct normalization for $\mathbf{u}_\varepsilon^\mu$. We emphasize that, to obtain step (1) and (2), it suffices to know $\mathcal{F}u_\varepsilon^\mu$ and Mu_ε^μ. Intuitively, the multiplication operator M makes u_ε^μ *more regular*, whereas conjugating M with the Fourier transform produces an operator that makes u_ε^μ *less regular.*

△

Returning to the case of $K_{\xi,\lambda}^{\eta,\nu}$, we are interested in finding operators that plays an analogous role to M and $\mathcal{F}$. The analogue of M is clear: is is given by the multiplication operators M_{Φ_i} and M_{Ψ_j} which maps $M_{\Phi_i}K_{\delta,t}^{\varepsilon,s} = \Phi_i K_{\delta,s}^{\varepsilon,t} = K_{\delta+e_i,s+e_i}^{\varepsilon,t}$ and similarly for M_{Ψ_j}.

In this setting, the role of the Fourier transform $\mathcal{F}$ is played by the Knapp–Stein intertwining operators $\mathbf{T}_{\xi,\lambda}^i : \pi_{\xi,\lambda} \to \pi_{w_i(\xi,\lambda)}$ defined by the integral

$$\mathbf{T}_{\xi,\lambda}^i f(g) = \frac{1}{\Gamma(\frac{\lambda_i-\lambda_{i+1}+[\xi_i+\xi_{i+1}]}{2})}\int_{\mathbb{R}} |x|_{\xi_i+\xi_{i+1}}^{\lambda_i-\lambda_{i+1}-1} f(g\overline{n}_i(x))\,dx, \qquad (i=1,\dots,n).$$

Here $\overline{n}_i(x)=I_{n+1}+xE_{i+1,i}=e^{tE_{i+1,i}}$, and w_i acts on ξ and λ by swapping the i-th and $(i+1)$-th entries. Similarly, we introduce the Knapp–Stein intertwining operators for H, defined by

$$\mathbf{S}^i_{\eta,\nu}f(g)=\frac{1}{\Gamma(\frac{\nu_i-\nu_{i+1}+[\eta_i+\eta_{i+1}]}{2})}\int_{\mathbb{R}}|x|^{\nu_i-\nu_{i+1}-1}_{\eta_i+\eta_{i+1}}f(\overline{n}_i(x)g)\,dx,\qquad (i=1,\dots,n-1).$$

We then obtain a result analogous to the formula for $\mathcal{F}u^\mu_\varepsilon$, namely:

Theorem 3.4 ([DF24, Theorem 4.2]). *There exists explicit scalars $c_i(\xi,\lambda,\eta,\nu)$ and $d_i(\xi,\lambda,\eta,\nu)$ such that*

$$\mathbf{T}^i_{\xi,\lambda}K^{\eta,\nu}_{\xi,\lambda}=c_i(\xi,\lambda,\eta,\nu)K^{\eta,\nu}_{w_i(\xi,\lambda)},\qquad (i=1,\dots,n),$$

and

$$\mathbf{S}^i_{\eta,\nu}K^{\eta,\nu}_{\xi,\lambda}=d_i(\xi,\lambda,\eta,\nu)K^{w_i(\eta,\nu)}_{\xi,\lambda},\qquad (i=1,\dots,n-1).$$

Just as we found that $\mathcal{F}\circ M\circ\mathcal{F}^{-1}$ is a differential operator, we now conjugate the multiplication operators M_{Φ_i} and M_{Ψ_i} by the Knapp–Stein intertwining operators and find that these, too, yield differential operators. However, in contrast to the one-variable situation, where there is only a single way to conjugate the multiplication operator by the integral transform, we now have $2n+1$ different multiplication operators and $2n-1$ different integral transforms, which can be combined in a number of different ways. In [DF24] we did not find a canonical way of making these combinations and so we made a choice of some combination that makes the analytic extension of the normalized $K^{\eta,\nu}_{\xi,\lambda}$ work.

Theorem 3.5 ([DF24, Proposition 5.2 and Theorem 5.3] and [DL25, Theorem 3.4]). *Let $\mathcal{D}_1=M_{\Phi_1}$ and $\mathcal{F}_{n+1}=M_{\Psi_{n+1}}$. The following operators are differential operators with non-constant coefficients:*

$$\mathcal{D}_{i+1}(\lambda)=\mathbf{T}^i_{w_i(\xi,\lambda)+(e_i,e_i)}\circ\mathcal{D}_i(w_i\lambda)\circ\left(\mathbf{T}^i_{w_i(\xi,\lambda)}\right)^{-1},\qquad (i=1,\dots,n),$$
$$\mathcal{F}_i(\lambda)=\mathbf{T}^i_{w_i(\xi,\lambda)+(e_{i+1},e_{i+1})}\circ\mathcal{F}_{i+1}(w_i\lambda)\circ\left(\mathbf{T}^i_{w_i(\xi,\lambda)}\right)^{-1},\qquad (i=1,\dots,n).$$

Furthermore they yield Bernstein–Sato identities of the form

$$\mathcal{F}_i(\lambda)\mathbf{K}^{\eta,\nu}_{\xi,\lambda}=p_i(\xi,\lambda,\eta,\nu)\mathbf{K}^{\eta+\mathbb{1},\nu+\mathbb{1}}_{\xi+\hat{e}_i,\lambda+\hat{e}_i},\tag{1}$$

and

$$\mathcal{D}_i(\lambda)\mathbf{K}^{\eta,\nu}_{\xi,\lambda}=q_i(\xi,\lambda,\eta,\nu)\mathbf{K}^{\eta,\nu}_{\xi+e_i,\lambda+e_i},\tag{2}$$

where p_i and q_i are scalar functions, $\mathbb{1}=(1,\dots,1)$ and $\hat{e}_i=\mathbb{1}-e_i$.

Note that in [DL25] the forms of $\mathcal{F}_i$, $\mathcal{D}_j$, p_i and q_j are given explicitly, along with other differential operators and Bernstein–Sato identities for $K^{\eta,\nu}_{\xi,\lambda}$. Using these Bernstein–Sato identities and their corresponding Bernstein–Sato polynomials we can find a normalization $\tilde{n}(\xi,\lambda,\eta,\nu)$ and carry out the analytic continuation for $\tilde{n}(\xi,\lambda,\eta,\nu)^{-1}K^{\eta,\nu}_{\xi,\lambda}$ see [DF24, Proposition 6.1]. In order to carry out step (2) we can then use the identities from Theorem 3.4 to find zeroes for $\tilde{n}(\xi,\lambda,\eta,\nu)^{-1}K^{\eta,\nu}_{\xi,\lambda}$ by inspecting $c_i(\xi,\lambda,\eta,\nu)$ and $d_i(\xi,\lambda,\eta,\nu)$. Both of these steps are more technical versions of the arguments in Example 3.3.

4. Residues, zeros and evaluation

4.1. **Residues.** We now turn to the study of residues of the holomorphic family of distributions $(\mathbf{K}^{\eta,\nu}_{\xi,\lambda})$. To gain intuition for this problem, we return to our example.

Example 4.1. Recall that $\mathbf{u}^\mu_\varepsilon=|x|^\mu_\varepsilon/\Gamma(\frac{\mu+1+[\varepsilon]}{2})$, and now accounting for the normalization we get the identity:

$$x\mathbf{u}^\mu_\varepsilon=\frac{\Gamma(\frac{\mu+2+[\varepsilon+1]}{2})}{\Gamma(\frac{\mu+1+[\varepsilon]}{2})}\cdot\mathbf{u}^{\mu+1}_{\varepsilon+1}=\left(\tfrac{\mu+1}{2}\right)^{1-[\varepsilon]}\cdot\mathbf{u}^{\mu+1}_{\varepsilon+1}.$$

Hence, by induction we get

$$x^k\mathbf{u}^\mu_\varepsilon = \big(\tfrac{\mu+1+[\varepsilon]}{2}\big)_{\frac{k+(-1)^\varepsilon[k]}{2}} \cdot \mathbf{u}^{\mu+k}_{\varepsilon+k},$$

where $(x)_n = \Gamma(x+n)/\Gamma(x)$ is the Pochhammer symbol. From this we observe that, since $\mathbf{u}^{\mu+k}_\varepsilon$ is holomorphic, $x^k\mathbf{u}^\mu_\varepsilon = 0$ whenever $\mu + [\varepsilon] = -1, -3, \ldots, 1-(k+(-1)^\varepsilon[k])$, and thus the support of $\mathbf{u}^\mu_\varepsilon$ is contained in $\{x = 0\}$ whenever $\mu + [\varepsilon]$ is a negative odd integer. Combining this information with the degree of homogeneity of $\mathbf{u}^\mu_\varepsilon$, we obtain

$$\mathbf{u}^\mu_\varepsilon\big|_{\mu+[\varepsilon]=-2\ell-1} = c_{\ell,\varepsilon}\cdot\delta^{(2\ell+[\varepsilon])}(x),$$

for some non-zero constants $c_{\ell,\varepsilon}$.

△

Applying the same argument to $\mathbf{K}^{\eta,\nu}_{\xi,\lambda}$, we can also identify subsets that contains its support. However, since more variables are involved, the residues cannot be identified as easily from the support of $\mathbf{K}^{\eta,\nu}_{\xi,\lambda}$, and we therefore obtain the following weaker result.

Proposition 4.2 ([DF24, Proposition 7.2]). *If $i + j \le n$ and $\lambda_i - \nu_j + \frac{1}{2} + [\xi_i + \eta_j] \in -2\mathbb{N}_0$, then the support of $\mathbf{K}^{\eta,\nu}_{\xi,\lambda}$ is contained in*

$$\{\Phi_i = 0\}\cap\{\Psi_i = 0\}\cap\{\Phi_{i+1} = 0\}\cap\cdots\cap\{\Psi_{n-j} = 0\}\cap\{\Phi_{n+1-j} = 0\}.$$

If $i + j > n + 1$ and $\nu_j - \lambda_i + \frac{1}{2} + [\eta_j + \xi_i] \in -2\mathbb{N}_0$, then the support of $\mathbf{K}^{\eta,\nu}_{\xi,\lambda}$ is contained in

$$\{\Psi_{n+1-j} = 0\}\cap\{\Phi_{n+2-j} = 0\}\cap\cdots\cap\{\Phi_{i-1} = 0\}\cap\{\Psi_{i-1} = 0\}.$$

To obtain this result, we used the multiplication maps M_{Φ_i} and M_{Ψ_j}, but instead we could try to use the Fourier transform and the Bernstein–Sato identities again.

Example 4.3. Recall the Bernstein–Sato identity

$$\frac{d}{dx}\mathbf{u}^\mu_\varepsilon = 2\Big(\frac{\mu}{2}\Big)^{1-[\varepsilon]}\cdot\mathbf{u}^{\mu-1}_{\varepsilon-1},$$

and by induction we obtain its iterated form

$$\frac{d^k}{dx^k}\mathbf{u}^\mu_\varepsilon = 2^k\Big(\frac{\mu+1-k+[\varepsilon+1+k]}{2}\Big)_{\frac{k+(-1)^\varepsilon[k]}{2}}\cdot\mathbf{u}^{\mu-k}_{\varepsilon-k}.$$

Furthermore, recall the formula for the Fourier Transform of $\mathbf{u}^\mu_\varepsilon$:

$$\mathcal{F}\mathbf{u}^\mu_\varepsilon = i^{[\varepsilon]}\sqrt{\pi}2^{\mu+1}\mathbf{u}_\varepsilon^{-\mu-1}.$$

Using that $\mathcal{F}[1] = 2\pi\delta$, we can then notice that

$$2\sqrt{\pi}\mathbf{u}_0^{-1} = \mathcal{F}\mathbf{u}_0^0 = \mathcal{F}[\frac{1}{\sqrt{\pi}}] = 2\sqrt{\pi}\delta$$

giving us $\mathbf{u}_0^{-1} = \delta$. Then, using the iterate form of the Bernstein–Sato identity, we obtain the remaining residues

$$\delta^{(k)} = \frac{d^k}{dx^k}\mathbf{u}_0^{-1} = 2^k\big(\tfrac{[1+k]-k}{2}\big)_{\frac{k+[k]}{2}}\cdot\mathbf{u}^{-k-1}_{[k]}.$$

△

From this example we can notice that if we know the "first" residue, where the Dirac delta distribution appears, then we can use the Bernstein–Sato identities to obtain the rest of the residues. So, to apply this idea in the setting of $\mathbf{K}^{\eta,\nu}_{\xi,\lambda}$, we first find parameters such that $\mathbf{K}^{\eta,\nu}_{\xi,\lambda} = \text{const}\times\delta$, where here δ is the Dirac delta distribution at the identity of G/P_G.

Theorem 4.4 ([DL25, Theorem 4.5]). *For $\eta_k = \xi_k$ and $\nu_k = \lambda_k + \frac{1}{2}$ for $k = 1, \ldots, n$ we have that*

$$\mathbf{K}^{\eta,\nu}_{\xi,\lambda} = \text{const}\times e_G(\xi,\lambda)e_H(\eta,-\nu)\delta,$$

where e_G and e_H are Harish-Chandra's e-function for G and H respectively.

In [DL25] we also find that, for other parameters $\mathbf{K}^{\eta,\nu}_{\xi,\lambda}$ has Dirac delta distributions supported at other points in G/P_G as residues. Using the Bernstein–Sato identities we get the following.

Theorem 4.5 ([DL25, Theorem 4.8])**.** *Let* $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}_0^n$. *Then for* $\eta_k=\xi_k+\alpha_k$ *and* $\nu_k=\lambda_k+\alpha_k+\frac{1}{2}$ *for* $k=1,\dots,n$ *we have that*

$$\mathbf{K}^{\eta,\nu}_{\xi,\lambda}=b_\alpha(\xi,\lambda)\times\mathcal{F}_1(\lambda)^{\alpha_1}\mathcal{F}_2(\lambda)^{\alpha_2}\cdots\mathcal{F}_n(\lambda)^{\alpha_n}\delta,$$

for an explicit scalar function $b_\alpha(\xi,\lambda)$ *where* $\mathcal{F}_i(\lambda)^{\alpha_i}$ *is the composition of the differential operator* $\mathcal{F}_i(\lambda)$ *with itself* α_i *times. Furthermore, the right-hand side is non-vanishing for generic* $\lambda\in\mathbb{C}^{n+1}$.

4.2. **Zeros.** We now aim to find a set of zeros for the family $(\mathbf{K}^{\eta,\nu}_{\xi,\lambda})$. To gain some intuition for this problem, we consider the following idea:

Example 4.6. If $\mathbf{u}^{\mu_0}_{\varepsilon_0}=0$ for some parameters (ε_0,μ_0), then by the Bernstein–Sato identity

$$\frac{d}{dx}\mathbf{u}^{\mu}_{\varepsilon}=2\Big(\frac{\mu}{2}\Big)^{1-[\varepsilon]}\cdot\mathbf{u}^{\mu-1}_{\varepsilon-1}.$$

We then get that if $(\varepsilon_0,\mu_0)\neq(1,0)$, then $\mathbf{u}^{\mu_0-1}_{\varepsilon_0-1}=0$. In this sense, we can propagate any zero of $\mathbf{u}^{\mu}_{\varepsilon}$ to a whole family of zeros, as long as the Bernstein–Sato polynomial does not vanish. △

Using this idea in the case of $\mathbf{K}^{\eta,\nu}_{\xi,\lambda}$ and so we first obtain a zero of $\mathbf{K}^{\eta,\nu}_{\xi,\lambda}$. We do this by noting that when $\mathbf{K}^{\eta,\nu}_{\xi,\lambda}$ is integrable, it contains the factors $\mathbf{u}^{s_1}_{\delta_1}(g_{n+1,1})$ and $\mathbf{u}^{t_1}_{\varepsilon_1}(g_{n,1})$. Taking a residue of these factors, one can pick the parameters in such a way that the remaining factors in $\mathbf{K}^{\eta,\nu}_{\xi,\lambda}$ are locally integrable, but one of the gamma-factors in $n(\xi,\lambda,\eta,\nu)^{-1}$ vanishes, making the whole thing vanish. Using the Bernstein–Sato identities, we can then propagate these zeros to a whole family of zeros. Furthermore, using the transformation identities with the Knapp–Stein intertwining operators (Theorem 3.4), we can then obtain an even larger set of zeros, which gives the following:

Theorem 4.7 ([DF24, Theorem 7.3])**.** *The distribution* $\mathbf{K}^{\eta,\nu}_{\xi,\lambda}$ *vanishes for all parameters* (ξ,λ,η,ν) *in the sets*

$$\mathcal{N}_{i,j,k}=\{\lambda_i-\nu_k+\tfrac{1}{2}+[\xi_i+\eta_k]\in-2\mathbb{N}_0\}\cap\{\nu_k-\lambda_j+\tfrac{1}{2}+[\eta_k+\xi_j]\in-2\mathbb{N}_0\},$$

where $i<j$ *and* $k\in\{1,\dots,n\}$ *and*

$$\mathcal{M}_{i,j,k}=\{\nu_j-\lambda_k+\tfrac{1}{2}+[\eta_j+\xi_k]\in-2\mathbb{N}_0\}\cap\{\lambda_k-\nu_i+\tfrac{1}{2}+[\xi_k+\eta_i]\in-2\mathbb{N}_0\},$$

where $i<j$ *and* $k\in\{1,\dots,n+1\}$.

4.3. **Evaluation on the spherical vector.** For $\xi=(0,\dots,0)$, the representation $\pi_{0,\lambda}$ contains a unique $O(n+1)$-fixed vector $\mathbf{1}_\lambda$ such that $\mathbf{1}_\lambda(e)=1$. By abuse of notation, we denote by $\mathbf{1}_\nu$ the analogous defined $O(n)$-fixed vector in $\tau_{0,\nu}$. We then have the following result:

Theorem 4.8 ([DF24, Theorem 8.5])**.** *For all* $(\lambda,\nu)\in\mathbb{C}^{n+1}\times\mathbb{C}^n$ *we have*

$$\mathbf{A}^{0,\nu}_{0,\lambda}\mathbf{1}_\lambda=\langle\mathbf{K}^{0,\nu}_{0,\lambda},\mathbf{1}_\lambda\rangle=\mathrm{const}\times e_G(\lambda)e_H(-\nu)\mathbf{1}_\nu,$$

where the non-zero constant is independent of (λ,ν) *and* e_G *and* e_H *are the Harish-Chandra e-functions for* G *and* H.

This result is shown by relating the symmetry breaking operators $\mathbf{A}^{\eta,\nu}_{\xi,\lambda}$ to the local archimedean Rankin–Selberg integrals involving the Whittaker models of $\pi_{\xi,\lambda}$ and $\tau_{\eta,\nu}$. Since the evaluation of Rankin–Selberg integrals on spherical Whittaker vectors was carried out by Ishii and Stade [IS13] we obtain the result.

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