

DERIVATIVES OF MULTIPLE EISENSTEIN SERIES

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(Based on a joint work with HAYATO KANNO [BK])

ABSTRACT. In this survey article, we summarize the results of [BK], in which the authors investigated the structure of multiple Eisenstein series. Specifically, it presents a large family of linear relations among these series alongside a conjectural formula for their derivatives. In this note, we provide an explicit derivative formula for the associated q -series and a self-contained proof of the derivation formula in depth one.

1. INTRODUCTION

Multiple Eisenstein series are natural generalizations of classical Eisenstein series and provide a bridge between the theory of modular forms and multiple zeta values. They are holomorphic functions on the complex upper half-plane \mathbb{H} , defined for integers $k_1, \dots, k_r \geq 2$ by

$$G_{k_1, \dots, k_r}(\tau) := \lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \sum_{\substack{\lambda_1, \dots, \lambda_r \in \mathbb{Z}_M \tau + \mathbb{Z}_N \\ \lambda_1 \succ \dots \succ \lambda_r \succ 0}} \frac{1}{\lambda_1^{k_1} \dots \lambda_r^{k_r}},$$

where the order \succ is the standard lexicographical order defined by

$$\lambda_1 \succ \lambda_2 :\iff \lambda_1 - \lambda_2 \in \mathbb{H} \cup \mathbb{R}_{>0}$$

and where $\mathbb{Z}_M = \{m \in \mathbb{Z} \mid |m| < M\}$. They are a hybrid of classical Eisenstein series and multiple zeta values, defined for a larger set of indices, namely $k_1 \geq 2, k_2, \dots, k_r \geq 1$, by

$$\zeta(k_1, \dots, k_r) := \sum_{n_1 \succ \dots \succ n_r \succ 0} \frac{1}{n_1^{k_1} \dots n_r^{k_r}}.$$

While the algebraic structure of multiple zeta values is conjecturally well-understood via the extended double shuffle relations ([IKZ]), the structure of the space of multiple Eisenstein series was less clear until the work [BK]. Denote the space of all multiple Eisenstein series by

$$\mathcal{E} := \mathbb{Q} + \text{Span}_{\mathbb{Q}}\{G_{k_1, \dots, k_r}(\tau) \mid r \geq 1, k_1, \dots, k_r \geq 2\}.$$

In a previous work [BI], formal multiple Eisenstein series were introduced to model these objects, suggesting an underlying \mathfrak{sl}_2 -algebra structure. As a consequence of this work, we have the following conjecture.

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Conjecture 1.1 ([BI]). (i) The maps W, D, δ defined on the generators of \mathcal{E} by

$$\begin{aligned} W &: G_{k_1, \dots, k_r}(\tau) \mapsto (k_1 + \dots + k_r)G_{k_1, \dots, k_r}(\tau), \\ D &: G_{k_1, \dots, k_r}(\tau) \mapsto 2\pi i \frac{d}{d\tau} G_{k_1, \dots, k_r}(\tau), \\ \delta &: G_{k_1, \dots, k_r}(\tau) \mapsto \begin{cases} -\frac{1}{2}G_{k_2, \dots, k_r}(\tau) & k_1 = 2, \\ 0 & k_1 > 2, \end{cases} \end{aligned}$$

give well-defined \mathbb{Q} -linear maps $\mathcal{E} \rightarrow \mathcal{E}$.

(ii) The maps W, D, δ are derivations on \mathcal{E} .

(iii) (W, D, δ) forms an \mathfrak{sl}_2 -triple, i.e. we have the commutator relations

$$[W, D] = 2D, \quad [W, \delta] = -2\delta, \quad [\delta, D] = W$$

and thus \mathcal{E} is an \mathfrak{sl}_2 -algebra.

In particular, this conjecture implies that the space \mathcal{E} is closed under $D = 2\pi i \frac{d}{d\tau}$. These series possess a Fourier expansion with coefficients in the ring of multiple zeta values \mathcal{Z} extended by $2\pi i$. The building blocks of this Fourier expansion are given by the following q -series, which can be defined for $k_1, \dots, k_r \geq 1$ by

$$g(k_1, \dots, k_r) = \sum_{\substack{m_1 > \dots > m_r > 0 \\ n_1, \dots, n_r > 0}} \frac{n_1^{k_1-1}}{(k_1-1)!} \cdots \frac{n_r^{k_r-1}}{(k_r-1)!} q^{m_1 n_1 + \dots + m_r n_r} \in \mathbb{Q}[[q]].$$

In the Fourier expansion of multiple Eisenstein series, the q -series g always appear together with a power of $-2\pi i$, and therefore we set for $k_1, \dots, k_r \geq 1$

$$\hat{g}(k_1, \dots, k_r) := (-2\pi i)^{k_1 + \dots + k_r} g(k_1, \dots, k_r) \in \mathbb{Q}[\pi i][[q]].$$

Then the Fourier expansion of multiple Eisenstein series has the following form.

Theorem 1.2 ($r = 1, 2$ [GKZ], $r \geq 1$ [Ba]). For $k_1 \geq 3, k_2, \dots, k_r \geq 2$ there exist explicit $\alpha_{l_1, \dots, l_r, j}^{k_1, \dots, k_r} \in \mathbb{Z}$, such that for $q = e^{2\pi i \tau}$ we have

$$\mathbb{G}_{k_1, \dots, k_r}(\tau) = \zeta(k_1, \dots, k_r) + \sum_{\substack{0 < j < r \\ l_1 + \dots + l_r = k_1 + \dots + k_r \\ l_1 \geq 2, l_2, \dots, l_r \geq 1}} \alpha_{l_1, \dots, l_r, j}^{k_1, \dots, k_r} \zeta(l_1, \dots, l_j) \hat{g}(l_{j+1}, \dots, l_r) + \hat{g}(k_1, \dots, k_r).$$

In particular, $\mathbb{G}_{k_1, \dots, k_r}(\tau) = \zeta(k_1, \dots, k_r) + \sum_{n > 0} a_{k_1, \dots, k_r}(n) q^n$ for some $a_{k_1, \dots, k_r}(n) \in \mathcal{Z}[\pi i]$.

The purpose of this note is to outline recent progress on the algebraic structure of multiple Eisenstein series and to provide explicit, self-contained proofs for specific derivative formulas. Previous work [BI] introduced formal multiple Eisenstein series to model these objects, which suggested an underlying \mathfrak{sl}_2 -algebra structure. In the joint work [BK], we aimed to make this structure explicit. We will first review the main results of [BK], namely:

- (1) A large family of relations among multiple Eisenstein series (Theorem 2.1).
- (2) A conjectured formula for the derivative $2\pi i \frac{d}{d\tau} G_{k_1, \dots, k_r}(\tau)$ (Conjecture 2.3).

In addition to summarizing these findings, this note details concrete derivative evaluations. For the depth one case $2\pi i \frac{d}{d\tau} G_k(\tau)$, Turan already gave an explicit formula in his bachelor's thesis ([Tu]). Here, we provide a self-contained proof of this result (Theorem 4.1). Furthermore, we present an explicit formula for the derivative of the q -series $g(k_1, \dots, k_r)$ (Theorem 3.1), a result implicitly contained in [BKu] but not previously written down.

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2. RELATIONS AMONG MES & CONJECTURAL DERIVATIONS

We start by recalling the basic algebraic setup. Let $\mathfrak{H}^1 = \mathbb{Q}\langle z_1, z_2, \dots \rangle$ be the non-commutative polynomial algebra in the variables z_k . We identify indices (k_1, \dots, k_r) with words $z_{k_1} \dots z_{k_r}$. The subspace of indices with $k_1 \geq 2$ is denoted by $\mathfrak{H}^{\geq 2} = \mathbb{Q}\langle z_2, z_3, \dots \rangle$, and the subspace of admissible indices $(k_1 \geq 2, k_2, \dots, k_r \geq 1)$ is denoted by $\mathfrak{H}^0 = \mathbb{Q} + \sum_{k \geq 2} z_k \mathfrak{H}^1$. In \mathfrak{H}^1 , define the harmonic product for words $w, v \in \mathfrak{H}^1$ and $r, s \geq 1$ by

$$z_r w * z_s v = z_r (w * z_s v) + z_s (z_r w * v) + z_{r+s} (w * v)$$

and $\mathbf{1} * w = w = w * \mathbf{1}$, where $\mathbf{1}$ denotes the empty word. Notice that the subspaces \mathfrak{H}^0 and $\mathfrak{H}^{\geq 2}$ are closed under $*$, and we obtain commutative \mathbb{Q} -algebras. With this, multiple Eisenstein series can be viewed as algebra homomorphisms

$$\begin{aligned} G : (\mathfrak{H}^{\geq 2}, *) &\longrightarrow \mathcal{E} \\ w = z_{k_1} \cdots z_{k_r} &\longmapsto G(w) := G_{k_1, \dots, k_r}. \end{aligned}$$

2.1. The Relation Family R . Define the operator $\varphi : \mathfrak{H}^1 \rightarrow \mathfrak{H}^1$ by the failure of the double shuffle relation with z_2 :

$$\varphi(w) := w * z_2 - w \sqcup z_2.$$

Using φ , we define a bilinear map $R : \mathfrak{H}^1 \times \mathfrak{H}^1 \rightarrow \mathfrak{H}^1$ which measures the failure of φ to be a derivation with respect to $*$:

$$R(u, v) := \varphi(u * v) - \varphi(u) * v - u * \varphi(v).$$

It turns out that for $u, v \in \mathfrak{H}^{\geq 2}$, we have $R(u, v) \in \mathfrak{H}^{\geq 2}$ ([BK, Lemma A.5]). One of the main results in [BK] is that R gives relations among multiple Eisenstein series.

Theorem 2.1 ([BK]). *For any $u, v \in \mathfrak{H}^{\geq 2}$, we have*

$$G(R(u, v)) = 0.$$

This provides a large family of explicit linear relations among multiple Eisenstein series. For example, in weight 6, $R(z_2, z_2)$ yields the relation $6G_{3,3} - 3G_{4,2} - G_6 = 0$.

2.2. Derivatives and the Drop1 Operator. In [BI], it was conjectured that \mathcal{E} is an \mathfrak{sl}_2 -algebra, implying the existence of a derivation corresponding to $2\pi i \frac{d}{d\tau}$. Theorem 2.1 suggests that φ “should” give a derivation for multiple Eisenstein series, as the error R of being a derivation is in the kernel of G . The problem is that $\varphi(\mathfrak{H}^{\geq 2}) \not\subset \mathfrak{H}^{\geq 2}$, since the shuffle product $w \sqcup z_2$ usually introduces a letter z_1 . In [BK], we propose that we can remove this z_1 by giving an explicit formula for this derivative using the “Drop1” operator introduced by Hirose, Maesaka, Seki, and Watanabe [HMSW]. Their statement is as follows.

Theorem 2.2 ([HMSW]). *There exists an explicit \mathbb{Q} -linear map $\mathcal{D} : \mathfrak{H}^0 \rightarrow \mathfrak{H}^2$ such that*

- (i) *For $w \in \mathfrak{H}^0$ we have $\zeta(w) = \zeta(\mathcal{D}(w))$.*
- (ii) *For $w \in \mathfrak{H}^2$ we have $\mathcal{D}(w) = w$.*

Moreover, $\mathcal{D}(z_{k_1} \cdots z_{k_r}) \in \mathbb{Z}\langle z_2, z_3, \dots \rangle$.

For a precise definition, see [BK, Section 4.1]. Now define the map $\theta : \mathfrak{H}^0 \rightarrow \mathfrak{H}^{\geq 2}$ by combining Drop1 with the operator φ :

$$\theta(w) := -\mathcal{D}(\varphi(w)).$$

We conjecture that θ gives the derivative on the space of multiple Eisenstein series.

Conjecture 2.3 ([BK]). *For any $w \in \mathfrak{H}^{\geq 2}$, we have*

$$2\pi i \frac{d}{d\tau} G(w) = G(\theta(w)).$$

This formula has been verified for depth 1 and coincides with known results for $G_{\{2\}^r}$.

Example 2.4. As an example for the derivatives predicted by Conjecture 2.3, we have

$$\begin{aligned} 2\pi i \frac{d}{d\tau} G_2 &= 3G_4 - 4G_{2,2}, \\ 2\pi i \frac{d}{d\tau} G_3 &= 5G_5 - 4G_{3,2} - 6G_{2,3}, \\ 2\pi i \frac{d}{d\tau} G_4 &= 7G_6 - 5G_{4,2} - 6G_{3,3} - 8G_{2,4}, \\ 2\pi i \frac{d}{d\tau} G_{2,2} &= 3G_{4,2} + 3G_{2,4} - 12G_{2,2,2}. \end{aligned}$$

3. DERIVATIVES AND RELATIONS OF q -ANALOGUES OF MZV

The q -series $g(k_1, \dots, k_r) \in \mathbb{Q}[[q]]$ can also be defined via its generating series

$$\mathbf{g}(X_1, \dots, X_r) = \sum_{k_1, \dots, k_r \geq 1} g(k_1, \dots, k_r) X_1^{k_1-1} \cdots X_r^{k_r-1} := \sum_{m_1 > \dots > m_r > 0} \prod_{j=1}^r \frac{e^{X_j} q^{m_j}}{1 - e^{X_j} q^{m_j}},$$

from which one immediately obtains the explicit expression from the introduction:

$$g(k_1, \dots, k_r) = \sum_{\substack{m_1 > \dots > m_r > 0 \\ n_1, \dots, n_r > 0}} \prod_{j=1}^r \frac{n_j^{k_j-1}}{(k_j-1)!} q^{m_j n_j}.$$

These q -series first appeared in the author's master's thesis and were subsequently studied in detail in [BKu]. There, it was shown that the space

$$\mathcal{Z}_q = \mathbb{Q} + \text{span}_{\mathbb{Q}}\{g(k_1, \dots, k_r) \mid r \geq 1, k_1, \dots, k_r \geq 1\}$$

is closed under the operator $q \frac{d}{dq}$. We provide the following explicit expression for the derivative, which was not given in [BKu].

Theorem 3.1. *For $k_1, \dots, k_r \geq 1$ we have*

$$\begin{aligned} q \frac{d}{dq} g(k_1, \dots, k_r) &= g(2, k_1, \dots, k_r) + \sum_{j=2}^r g(k_1, \dots, k_{j-1}, 2, k_j, \dots, k_r) + \sum_{j=1}^r g(k_1, \dots, k_j + 2, \dots, k_r) \\ &\quad - \sum_{\substack{1 \leq j \leq r \\ a+b=k_j+2}} (a-1)g(k_1, \dots, k_{j-1}, a, b, k_{j+1}, \dots, k_r) \\ &\quad - \sum_{\substack{1 \leq i < j \leq r \\ a+b=k_j+1}} k_i g(k_1, \dots, k_i + 1, \dots, k_{j-1}, a, b, k_{j+1}, \dots, k_r) \\ &\quad - \sum_{j=1}^r k_j g(k_1, \dots, k_j + 1, \dots, k_r, 1) \\ &\quad + \sum_{j=1}^r (r-j+1)k_j g(k_1, \dots, k_j + 1, \dots, k_r) \\ &\quad + \sum_{j=1}^r \sum_{a=1}^{k_j+1} \beta_{a,k_j} g(k_1, \dots, k_{j-1}, a, k_{j+1}, \dots, k_r), \end{aligned}$$

where the coefficients $\beta_{a,k} \in \mathbb{Q}$ are given for $1 \leq a \leq k+1$ by

$$\beta_{a,k} = \left((-1)^{k-1} \binom{k+1-a}{2-a} - \binom{k+1-a}{k-a} \right) \frac{B_{k+2-a}}{(k+2-a)!}.$$

Proof. In [BKu, (3.2)] it is shown that the action of the operator $q \frac{d}{dq}$ on \mathfrak{g} can be written as¹

$$\begin{aligned} q \frac{d}{dq} \mathfrak{g}(X_1, \dots, X_r) &= g(2) \mathfrak{g}(X_1, \dots, X_r) - \sum_{j=1}^{r+1} \frac{d}{dY} \mathfrak{g}(X_1 + Y, \dots, X_j + Y, X_j, \dots, X_r) \Big|_{Y=0} \\ &\quad + \sum_{j=1}^r \frac{d}{dY} \mathfrak{g}(X_1 + Y, \dots, X_j + Y, X_{j+1}, \dots, X_r) \Big|_{Y=0}. \end{aligned}$$

¹Here we set $X_{r+1} = 0$, i.e., the case $j = r+1$ gives the term $\mathfrak{g}(X_1 + Y, \dots, X_r + Y, Y)$ for the first sum on the right.

Notice that this is equivalent to

$$q \frac{d}{dq} \mathfrak{g}(X_1, \dots, X_r) = g(2) \mathfrak{g}(X_1, \dots, X_r) - \sum_{j=1}^{r+1} \frac{d}{dY} \mathfrak{g}(X_1 + Y, \dots, X_j + Y, X_j, \dots, X_r) \Big|_{Y=0} \\ + \sum_{j=1}^r (r - j + 1) \frac{d}{dX_j} \mathfrak{g}(X_1, \dots, X_r).$$

Therefore, the action of $q \frac{d}{dq}$ on the q -series g is given by

$$q \frac{d}{dq} g(k_1, \dots, k_r) = g(2)g(k_1, \dots, k_r) - g(k_1, \dots, k_r, 2) \\ - \sum_{\substack{1 \leq j \leq r \\ a+b=k_j+2}} (a-1)g(k_1, \dots, k_{j-1}, a, b, k_{j+1}, \dots, k_r) \\ - \sum_{\substack{1 \leq i < j \leq r \\ a+b=k_j+1}} k_i g(k_1, \dots, k_i + 1, \dots, k_{j-1}, a, b, k_{j+1}, \dots, k_r) \\ - \sum_{j=1}^r k_j g(k_1, \dots, k_j + 1, \dots, k_r, 1) \\ + \sum_{j=1}^r (r - j + 1) k_j g(k_1, \dots, k_j + 1, \dots, k_r).$$

Moreover, the product can be evaluated by [BKu, Lemma 2.6] as

$$g(2)g(k_1, \dots, k_r) = \sum_{j=0}^r g(k_1, \dots, k_j, 2, k_{j+1}, \dots, k_r) + \sum_{j=1}^r g(k_1, \dots, k_j + 2, \dots, k_r) \\ + \sum_{j=1}^r \sum_{a=1}^{k_j+1} \left((-1)^{k_j-1} \binom{k_j+1-1}{2-a} - \binom{k_j+1-a}{k_j-a} \right) \frac{B_{k_j+2-a}}{(k_j+2-a)!} g(k_1, \dots, k_{j-1}, a, k_{j+1}, \dots, k_r).$$

Combining this with the above yields the statement in the theorem. \square

Example 3.2. We provide a few examples for the derivatives of $g(k_1, \dots, k_r)$ from Theorem 3.1.

$$q \frac{d}{dq} g(2) = g(4) - 4g(3, 1) + 2g(3) - \frac{1}{6}g(2), \\ q \frac{d}{dq} g(3) = g(5) - 2g(3, 2) - 6g(4, 1) + 3g(4) - \frac{1}{12}g(3), \\ q \frac{d}{dq} g(2, 2) = g(2, 4) + g(4, 2) - 4g(2, 3, 1) - 4g(3, 1, 2) - 4g(3, 2, 1) \\ + 2g(2, 3) + 4g(3, 2) - \frac{1}{3}g(2, 2).$$

In the Fourier expansion of multiple Eisenstein series, the q -series g only appear with all entries ≥ 2 . We therefore define the space

$$\mathcal{Z}_q^{\geq 2} = \mathbb{Q} + \text{span}_{\mathbb{Q}}\{g(k_1, \dots, k_r) \mid r \geq 1, k_1, \dots, k_r \geq 2\}$$

One can show that $\mathcal{Z}_q^{\geq 2}$ is also closed under multiplication (which is indeed not the case when replacing 2 by any larger integer, in contrast to the usual harmonic product). By Theorem 1.2, one directly obtains that $\mathbb{C} \otimes \mathcal{E} = \mathbb{C} \otimes \mathcal{Z}_q^{\geq 2}$. Therefore, the question of whether \mathcal{E} is closed under $(2\pi i) \frac{d}{d\tau}$ is closely related to the question of the closure of $\mathcal{Z}_q^{\geq 2}$ under $q \frac{d}{dq}$. The latter was also conjectured by Okounkov for a different model of q -multiple zeta values.

Conjecture 3.3. *The space $\mathcal{Z}_q^{\geq 2}$ is closed under $q \frac{d}{dq}$.*

Note that the formula in Theorem 3.1 does not give elements in $\mathcal{Z}_q^{\geq 2}$, even when one considers the case $k_1, \dots, k_r \geq 2$. However, similar to multiple zeta values, there exist various relations among the q -series g arising from an analogue of the double shuffle relations.

Proposition 3.4 ([BKu]). *(i) For $k_1, k_2 \geq 1$, we have*

$$g(k_1)g(k_2) = g(k_1, k_2) + g(k_2, k_1) + g(k_1 + k_2) + \sum_{j=1}^{k_1+k_2-1} (\lambda_{k_1, k_2}^j + \lambda_{k_2, k_1}^j) g(j),$$

where the rational numbers λ_{k_1, k_2}^j are given by

$$\lambda_{k_1, k_2}^j = (-1)^{k_2-1} \binom{k_1 + k_2 - 1 - j}{k_1 - j} \frac{B_{k_1+k_2-j}}{(k_1 + k_2 - j)!},$$

and where we use the convention $\binom{n}{k} = 0$ for $k < 0$.

(ii) For $k_1, k_2 \geq 1$ and $k = k_1 + k_2$ we have

$$\begin{aligned} g(k_1)g(k_2) &= \sum_{j=1}^{k-1} \left(\binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) g(j, k-j) \\ &\quad + \binom{k-2}{k_1-1} \left(q \frac{d}{dq} \frac{g(k-2)}{k-2} - g(k-1) \right) + \delta_{k_1,1} \delta_{k_2,1} g(2), \end{aligned}$$

where $\delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$ denotes the Kronecker delta.

As a corollary, we obtain the following formula, which shows that Conjecture 3.3 is true in depth one.

Corollary 3.5. *For $k \geq 1$ we have*

$$\begin{aligned} q \frac{d}{dq} g(k) &= (2k-1)g_{k+2}(q) - \sum_{j=2}^k (k+j-1)g(j, k+2-j) - g(k, 2) \\ &\quad + \sum_{j=2}^k \frac{B_{k+2-j}}{(k+2-j)!} (3k-j+1)g(j) + (-1)^k \frac{B_k}{k!} g(2). \end{aligned}$$

In particular $q \frac{d}{dq} g(k) \in \mathcal{Z}_q^{\geq 2}$ for $k \geq 2$.

Proof. This can be shown by combining Proposition 3.4 for the cases $k_1 = 1, k_2 = k - 1$ and $k_1 = 2, k_2 = k - 1$ and canceling out the appearing $g(k - 1, 1)$ term. \square

4. DERIVATIVES OF DEPTH ONE MULTIPLE EISENSTEIN SERIES

In this section, we present a self-contained proof of the derivative formula for multiple Eisenstein series in depth one. This result was established in the bachelor's thesis of Can Turan [Tu]. Since this thesis is written in German, we give a self-contained proof here using the results mentioned above.

Theorem 4.1. *For any integer $k \geq 2$, we have*

$$2\pi i \frac{d}{d\tau} G_k(\tau) = (2k - 1)G_{k+2}(\tau) - \sum_{j=2}^k (k + j - 1)G_{k+2-j,j}(\tau) - G_{2,k}(\tau). \quad (4.1)$$

Proof. The proof relies on the comparison of the Fourier expansions on both sides of (4.1). Recall that for $k \geq 2$, the Fourier expansion is given by

$$G_k(\tau) = \zeta(k) + \hat{g}_k(q),$$

where $\hat{g}_k(q) = (-2\pi i)^k \sum_{n=1}^{\infty} \sigma_{k-1}(n) \frac{q^n}{(k-1)!}$. Applying the derivative $2\pi i \frac{d}{d\tau} = q \frac{d}{dq}$ to $G_k(\tau)$, we use Corollary 3.5 to obtain

$$q \frac{d}{dq} \hat{g}_k = (2k - 1)\hat{g}_{k+2} - \sum_{j=2}^k (k + j - 1)\hat{g}_{k+2-j,j} - \hat{g}_{2,k} + \mathcal{E}_k(q), \quad (4.2)$$

where $\mathcal{E}_k(q)$ denotes the error term involving products of single \hat{g} 's given by

$$\mathcal{E}_k(q) = \sum_{j=2}^k \lambda_j \zeta(k + 2 - j) \hat{g}_j + \lambda_2 \zeta(k) \hat{g}_2,$$

and where the coefficients vanish for odd weights of ζ . Now consider the right-hand side of (4.1). We define the linear combination

$$L_k(\tau) := (2k - 1)G_{k+2}(\tau) - \sum_{j=2}^k (k + j - 1)G_{k+2-j,j}(\tau) - G_{2,k}(\tau).$$

Using the Fourier expansion of double Eisenstein series (Theorem 1.2), we can decompose G_{k_1,k_2} into a constant term, an “odd” part (involving $\zeta(l)$ for odd l), and an “even” part (involving $\zeta(l)$ for even l and \hat{g}_{k_1,k_2}). The constant term of $L_k(\tau)$ is given by

$$(2k - 1)\zeta(k + 2) - \sum_{j=2}^k (k + j - 1)\zeta(k + 2 - j, j) - \zeta(2, k)$$

which vanishes as a simple consequence of the double shuffle relations. This can also be seen by multiplying (4.2) by $(1 - q)^{k+2}$ and taking the limit $q \rightarrow 1$. We examine the coefficients of

$\hat{g}_{l_1}\zeta(l_2)$ in the expansion of $L_k(\tau)$, where $l_1 + l_2 = k + 2$. The coefficient of $\hat{g}_{l_1}\zeta(l_2)$ in G_{k_1, k_2} is given by $C_{k_1, k_2}^{l_2}$. Thus, the coefficient of $\hat{g}_{l_1}\zeta(l_2)$ in $L_k(\tau)$ is

$$\mathcal{C}(l_2) := (2k - 1)\delta_{k+2, l_2} - \sum_{j=2}^k (k + j - 1)C_{k+2-j, j}^{l_2} - C_{2, k}^{l_2}.$$

If l_2 is odd, we have, as a consequence of [GKZ, Corollary, p. 8], that

$$\sum_{j=2}^k (k + j - 1)C_{k+2-j, j}^l = -C_{2, k}^l.$$

Since G_{k+2} has no $\zeta(\text{odd})$ terms, the first term $(2k - 1)\delta_{k+2, l_2}$ vanishes. The remaining terms cancel exactly, so $\mathcal{C}(l_2) = 0$ for all odd l_2 . If l_2 is even, the terms correspond exactly to the error terms $\mathcal{E}_k(q)$ arising in the derivative of the generating functions \hat{g}_k . The remaining part of the Fourier expansion comes from the \hat{g}_{k_1, k_2} terms. We have

$$L_k(\tau) = \cdots + (2k - 1)\hat{g}_{k+2} - \sum_{j=2}^k (k + j - 1)\hat{g}_{k+2-j, j} - \hat{g}_{2, k}.$$

This matches exactly the formula (4.2) for $q \frac{d}{dq} \hat{g}_k$ (modulo the product terms which were handled by the even l_2 case). Thus, both the constant term and the q -expansion of the LHS and RHS of (4.1) coincide. \square

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