Unified Multiple Zeta Functions and Their Integral Representations

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1 Introduction

Recently there has been a growing interest in finite multiple zeta values and symmetric multiple zeta values. Kaneko and Zagier conjectured that there is a certain correspondence between them and many supporting facts were found. In the previous papers [11, 12], we proposed unified multiple zeta functions, which are entire and interpolate Euler–Zagier multiple zeta values, $\mathcal{A}$-, $\widehat{\mathcal{A}}$-finite multiple zeta values, and $\mathcal{S}$-, $\widehat{\mathcal{S}}$-symmetric multiple zeta values [6–8, 17, 18]. Furthermore we gave the explicit form of finite multiple zeta values and symmetric multiple zeta values on all nonpositive indices.

In this article, we introduce further generalizations of the unified multiple zeta functions, which include, in addition to the previously included multiple zeta values, their star analogues, and multidimensional generalizations of p-adic Bernoulli measures constructed by Mazur as special cases. As applications, we establish multiple Kummer-type congruences of special values of the unified multiple zeta functions, which are regarded as multiple generalizations of the congruences given in [19].

Throughout this article, the empty sum should be understood as 0 and the empty product as 1 respectively. We also note that in this article, we provide intuitive descriptions instead of the precise ones. So we suggests that readers consult the forthcoming papers [13–15] for precise definitions.

2 Integral Representations of Unified Multiple Zeta Functions (I)

In [4], a generalization of symmetric multiple zeta values called refined symmetric multiple zeta values were introduced in terms of iterated contour integrals as follows.

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Definition 1 ([4])**.** For $k_1, \dots, k_r \in \mathbb{Z}_{\geq 1}$,

$$\zeta^{\mathrm{RS}}(k_1, \dots, k_r) := \frac{(-1)^r}{2\pi i} I(1, \underbrace{0, \dots, 0}_{k_1 - 1}, 1, \dots, 1, \underbrace{0, \dots, 0}_{k_r - 1}, 1),$$

where

$$I(a_1, \dots, a_n) := \int_{0<u_1<\cdots<u_n<1} \frac{d\gamma(u_1)}{\gamma(u_1) - a_1} \cdots \frac{d\gamma(u_n)}{\gamma(u_n) - a_n}$$

is a certain iterated contour integral.

Theorem 1 ([4])**.**

$$\zeta^{RS}(k_1, \dots, k_r) \equiv \zeta_{\mathcal{S}}(k_1, \dots, k_r) \mod \pi i \mathcal{Z}[\pi i].$$

The refined symmetric multiple zeta values are indeed a refinement of symmetric multiple zeta values in the sense that they satisfy double shuffle relations, duality relations, and reversal formula etc. (See [4] for details.)

As analytic generalizations of the refined symmetric multiple zeta values, we propose the following multiple zeta functions, which include extra variables $t_1, \dots, t_{r+1}$ besides the variables $s_1, \dots, s_r$ corresponding to indices. We call these multiple zeta functions the unified multiple zeta functions with parameters, since they are also generalizations of the previously introduced unified multiple zeta functions [12].

In the following, we simply refer to these multiple zeta functions as unified multiple zeta functions.

Definition 2. For $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{C}^r$ and $\mathbf{t} = (t_1, \dots, t_{r+1}) \in \mathbb{C}^{r+1}$ with sufficiently large $\Re s_1, \dots, \Re s_r$ and $\Re t_1, \dots, \Re t_{r+1} < 1$,

$$\zeta(s_1, \dots, s_q; t_1, \dots, t_q) := \sum_{0<n_1<\cdots<n_q} \frac{1}{(n_1 - t_1)^{s_1} \cdots (n_q - (t_1 + \cdots + t_q))^{s_q}},$$

$$\zeta_{\widehat{\mathcal{U}}}(\mathbf{s}; \mathbf{t}) := \sum_{i=0}^{r} (-1)^{s_{i+1} + \cdots + s_r} \zeta(s_1, \dots, s_i; t_1, \dots, t_i) \zeta(s_r, \dots, s_{i+1}; t_{r+1}, \dots, t_{i+2}).$$

It can be shown that $\zeta_{\widehat{\mathcal{U}}}(\mathbf{s}; \mathbf{t})$ is analytically continued to a holomorphic function except log branch points. Then we have the following integral representations.

Theorem 2. *For* $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{C}^r$, $\mathbf{t} = (t_1, \dots, t_{r+1}) \in \mathbb{C}^{r+1}$ *with* $\Re t_1, \dots, \Re t_{r+1} < 1$, *we have*

$$\begin{aligned} \zeta_{\widehat{\mathcal{U}}}(\mathbf{s}; \mathbf{t}) = & \frac{1}{2\pi\sqrt{-1}\Gamma(s_1) \cdots \Gamma(s_r)(e^{2\pi\sqrt{-1}s_1} - 1) \cdots (e^{2\pi\sqrt{-1}s_r} - 1)} \\ & \times \int_{0 \overset{\circ}{\succ} z_{r+1} \overset{\circ}{\succ} \cdots \overset{\circ}{\succ} z_1} dz_{r+1} \cdots dz_1 \frac{e^{t_1 z_1 + t_2 z_2 + \cdots + t_{r+1} z_{r+1}}}{(e^{z_1} - 1) \cdots (e^{z_{r+1}} - 1)} \\ & \times (z_1 - z_2)^{s_1 - 1} \cdots (z_r - z_{r+1})^{s_r - 1}, \end{aligned}$$

where the path $0 \overset{\circ}{\succ} z_{r+1} \overset{\circ}{\succ} \cdots \overset{\circ}{\succ} z_1$ *denotes a Hankel type analogue of iterated contour integrals given below.*

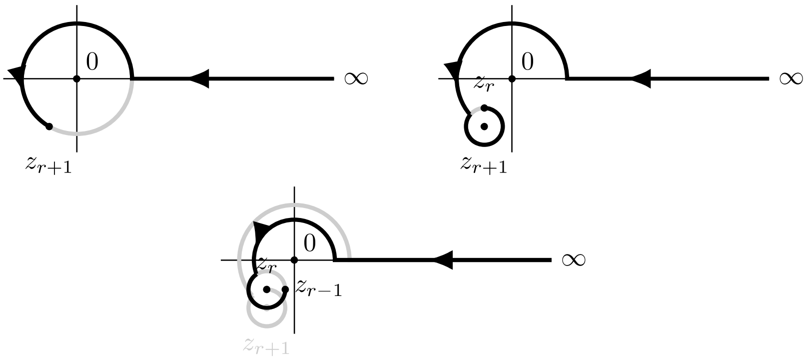

Figure 1: Hankel type analogue of iterated contours

In the theorem above, $a \overset{\circ}{\succ} b$ means that "for a fixed a, b starts at ∞, follows a, goes around a and goes back to ∞." (See Figure 1.) More precisely,

- $0 \overset{\circ}{\succ} z_{r+1}$ means that z_{r+1} goes on the ordinary Hankel contour.
- $z_{r+1} \overset{\circ}{\succ} z_r$ means that for a fixed z_{r+1}, z_r goes around z_{r+1} respecting the origin.
- Similarly, $z_j \overset{\circ}{\succ} z_{j-1}$ means that for a fixed z_j, z_{j-1} goes around z_j respecting the origin while ignoring $z_{r+1}, \ldots, z_{j+1}$.

It should be noted that the unified multiple zeta functions are also generalizations of (s,t)-adic symmetric multiple zeta values [5], which corresponds to the case $(t_1, \ldots, t_{r+1}) = (-s, 0, \ldots, 0, t)$.

Next we investigate the meanings of the variables $(t_1, \ldots, t_{r+1})$.

3 An interpretation of $(t_1, \ldots, t_{r+1})$ (I)

3.1 Previous Results

To study the meanings of $(t_1, \ldots, t_{r+1})$, we recall two previous results. The first previous result is related to the positions and sizes of running indices. To see it, we go back to the case of (s,t)-adic symmetric multiple zeta values or equivalently the original unified zeta functions with $(t', t) = (-s, t)$. Recall the original unified multiple zeta functions [12] defined by

$$\zeta_{\widehat{\mathcal{U}}}(s_1, \ldots, s_r; t', t) = \sum_{i=0}^{r} (-1)^{s_{i+1}+\cdots+s_r} \zeta(s_1, \ldots, s_i; t') \zeta(s_r, \ldots, s_{i+1}; t).$$

The refined Kaneko–Zagier conjecture ($\widehat{\mathrm{KZ}}$-conjecture) asserts that there is a one-to-one correspondence between $\zeta_{\widehat{\mathcal{S}}}(\mathbf{k})$ and $\zeta_{\widehat{\mathcal{A}}}(\mathbf{k})$ as follows.

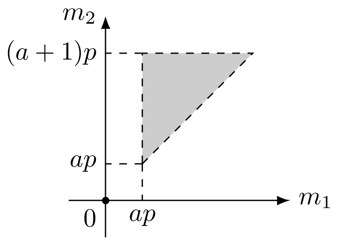

Figure 2: The region of the sum

Conjecture 3 ($\widehat{\text{KZ}}$-conjecture)**.** *There is a topological $\mathbb{Q}$-algebra isomorphism $\varphi : \mathcal{Z}_{\widehat{\mathcal{S}}} \to \mathcal{Z}_{\widehat{\mathcal{A}}}$ sending $\zeta_{\widehat{\mathcal{S}}}(\mathbf{k})$ to $\zeta_{\widehat{\mathcal{A}}}(\mathbf{k})$ for all $\mathbf{k} \in \mathbb{Z}^r_{\geq 1}$, and t to $\mathbb{p}$.*

Then the following was shown, which suggests that the variables $-t', t$ can be interpreted as the start point and the end point of the region of the sum respectively. (See Figure 2.)

Theorem 4 ([5])**.** *Let $k_1, \dots, k_r \in \mathbb{Z}_{\geq 1}$. Then the $\widehat{KZ}$-conjecture implies*

$$\varphi(\zeta_{\widehat{\mathcal{S}}}(k_1, \dots, k_r; a)) = \zeta_{\widehat{\mathcal{A}}}(k_1, \dots, k_r; a)$$

for any $a \in \mathbb{Z}$, where

$$\zeta_{\widehat{\mathcal{S}}}(k_1, \dots, k_r; a) = \zeta_{\widehat{\mathcal{U}}}(k_1, \dots, k_r; -at, (a+1)t) \bmod \pi i \mathcal{Z}[\pi i][[t]],$$
$$\zeta_{\widehat{\mathcal{A}}}(k_1, \dots, k_r; a) = \left(\left(\sum_{ap < n_1 < \cdots < n_r < (a+1)p} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \bmod p^n \right)_p \right)_n .$$

Remark. The $\widehat{\text{KZ}}$-conjecture corresponds to the case $a = 0$. Thus the above theorem shows that the $a = 0$ case automatically implies the general cases $a \in \mathbb{Z}$. The parameter a determines the diagonal position.

The second previous result is related to the region of indices. Let $\mathbb{p} = ((p \bmod p^n)_p)_n$ be the infinitely large prime.

Theorem 5 ([16])**.** *For any $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$, there exist rational polynomials $c_{\mathbf{k},\mathbf{m}}(x) \in \mathbb{Q}[x]$ indexed by a finite number of $\mathbf{m} \in \mathbb{Z}^{r'}_{\geq 1}$ ($r' \leq r$) such that*

$$\zeta_{\widehat{\mathcal{S}}}(\mathbf{k}) = \sum_{\mathbf{m}} c_{\mathbf{k},\mathbf{m}}(t) \zeta_{\widehat{\mathcal{S}}}(\mathbf{m}), \qquad \zeta_{\widehat{\mathcal{A}}}(\mathbf{k}) = \sum_{\mathbf{m}} c_{\mathbf{k},\mathbf{m}}(\mathbb{p}) \zeta_{\widehat{\mathcal{A}}}(\mathbf{m}) \qquad \textit{(finite sum).}$$

In particular, the $\widehat{KZ}$-conjecture implies

$$\varphi(\zeta_{\widehat{\mathcal{S}}}(k_1, \dots, k_r)) = \zeta_{\widehat{\mathcal{A}}}(k_1, \dots, k_r).$$

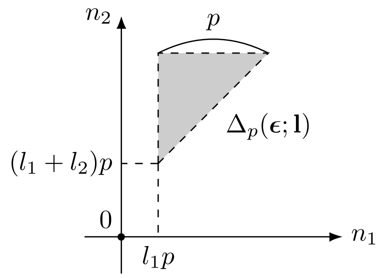

Figure 3: $\Delta_p(\boldsymbol{\epsilon};\mathbf{l})$

3.2 $\widehat{\mathrm{KZ}}$-conjecture for Any Positions, Any Indices and Any Inequalities

As shown above, the original $\widehat{\mathrm{KZ}}$-conjecture automatically implies that for any diagonal positions with a fixed size and that for any indices. We show that these results can be unified and generalized to the $\widehat{\mathrm{KZ}}$-conjecture for any positions, any indices and any inequalities.

To give the assertion, we first define the corresponding finite multiple zeta values. For $\boldsymbol{\epsilon}=(\epsilon_2,\dots,\epsilon_r)\in\{0,1\}^{r-1}$ and $\mathbf{l}=(l_1,\dots,l_r)\in\mathbb{Z}_p^r$ for a prime p, let

$$\begin{aligned}\Delta_p(\boldsymbol{\epsilon})&=\{(n_1,\dots,n_r)\in\{1,\dots,p-1\}^r \mid n_j<n_{j+1}+\epsilon_{j+1}\quad(j=1,\dots,r-1)\}\\&\qquad\subset(\mathbb{Z}_p^\times)^r,\\ \Delta_p(\boldsymbol{\epsilon};\mathbf{l})&=\Delta_p(\boldsymbol{\epsilon})-(l_1,l_1+l_2,\dots,l_1+\cdots+l_r)p\subset(\mathbb{Z}_p^\times)^r.\end{aligned}$$

(See Figure 3.)

Definition 3. For $\boldsymbol{\epsilon}\in\{0,1\}^{r-1}$, $\mathbf{k}\in\mathbb{Z}^r$ and $\mathbf{l}\in\mathbb{Q}^r$, we define

$$\begin{aligned}\zeta_{\widehat{\mathcal{A}}}(\mathbf{k};\boldsymbol{\epsilon};\mathbf{l})&:=\left(\left(\sum_{(n_1,\dots,n_r)\in\Delta_p(\boldsymbol{\epsilon};\mathbf{l})}\frac{1}{n_1^{k_1}\cdots n_r^{k_r}}\bmod p^n\right)_p\right)_n\\&=\sum_{(n_1,\dots,n_r)\in\Delta_{\mathbb{p}}(\boldsymbol{\epsilon};\mathbf{l})}\frac{1}{n_1^{k_1}\cdots n_r^{k_r}}\in\widehat{\mathcal{A}}.\end{aligned}$$

Note that for all sufficiently large prime p, we have $\mathbf{l}\in\mathbb{Z}_p^r$, which specifies the position and size of the running indices, while $a<b+\epsilon$ means $a<b$ if $\epsilon=0$ and $a\le b$ if $\epsilon=1$, i.e., $\boldsymbol{\epsilon}$ determines the inequalities. Note also that $\zeta_{\widehat{\mathcal{A}}}(\mathbf{k};\boldsymbol{\epsilon};\mathbf{l})$ can be described in terms of $\zeta_{\widehat{\mathcal{U}}}(\mathbf{s};\mathbf{t})$.

Next we define the corresponding symmetric counterpart. For $\boldsymbol{\epsilon}=(\epsilon_2,\dots,\epsilon_r)\in\{0,1\}^{r-1}$, and $\mathbf{l}=(l_1,\dots,l_r)\in\mathbb{C}^r$, let

$$\begin{aligned}\check{\boldsymbol{\epsilon}}&=(0,\epsilon_2,\dots,\epsilon_r,0)\in\mathbb{C}^{r+1},\\ \check{\mathbf{l}}&=(l_1,\dots,l_r,l_{r+1})\in\mathbb{C}^{r+1}\qquad(l_{r+1}=1-(l_1+\cdots+l_r)).\end{aligned}$$

Definition 4. For $\boldsymbol{\epsilon} \in \{0,1\}^{r-1}$, $\mathbf{k} \in \mathbb{Z}^r$ and $\mathbf{l} \in \mathbb{Q}^r$, we define

$$\zeta_{\widehat{\mathcal{S}}}(\mathbf{k};\boldsymbol{\epsilon};\mathbf{l}) := \zeta_{\widehat{\mathcal{U}}}(\mathbf{k};\check{\boldsymbol{\epsilon}} + \check{\mathbf{l}}t) \bmod \pi i \mathcal{Z}[\pi i][[t]] \in (\mathcal{Z}[\pi i]/\pi i \mathcal{Z}[\pi i])[[t]].$$

These definitions give rise to a generalization of the previous results.

Theorem 6. *For* $\boldsymbol{\epsilon} = (\epsilon_2, \dots, \epsilon_r) \in \{0,1\}^{r-1}$, $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$ *and* $\mathbf{l} = (l_1, \dots, l_r) \in \mathbb{Q}^r$, *the* $\widehat{KZ}$*-conjecture implies*

$$\varphi(\zeta_{\widehat{\mathcal{S}}}(\mathbf{k};\boldsymbol{\epsilon};\mathbf{l})) = \zeta_{\widehat{\mathcal{A}}}(\mathbf{k};\boldsymbol{\epsilon};\mathbf{l}).$$

Remark. Generalizations of the recursion relations in [16] can be explicitly given. See [14] for details.

4 Integral Representations of Unified Multiple Zeta Functions (II)

Compare the special values of $\zeta_{\widehat{\mathcal{U}}}$ in the $r = 1$ case given by

$$\zeta_{\widehat{\mathcal{U}}}(1-k;t_1,t_2) = \frac{1}{k}(B_k(t_2) - (-1)^{-k}B_k(t_1)), \tag{4.1}$$

where $B_k(t)$ denotes the k-th Bernoulli polynomial, and the p-adic Bernoulli measure $\mu_{k,\alpha}$, i.e., the regularization of Bernoulli distribution given by

$$\mu_{k,\alpha}(a + (p^N\mathbb{Z}_p)) = B_k\left(\frac{a}{p^N}\right) - \alpha^{-k}B_k\left(\left\{\frac{\alpha a}{p^N}\right\}\right), \tag{4.2}$$

where $\{x\}$ denotes the fractional part of $x \in \mathbb{Q}$. Then we see that (4.1) can be regarded as a degeneration of (4.2). Thus this comparison suggests that more parameters are still required.

In the following, we will show that it is indeed possible to construct multi-dimensional p-adic measures by defining further generalizations, which yields

$$\zeta_{\widehat{\mathcal{U}}}(1-k;t_1,t_2;\alpha_1;\beta_1) = \frac{1}{k}(\beta_1^{-k}B_k(t_2) - (-\alpha_1)^{-k}B_k(t_1))$$

in the $r = 1$ case.

4.1 Reformulation

For $\mathbf{s} = (s_1, \dots, s_q) \in \mathbb{C}^q$, let

$$\operatorname{wt}\mathbf{s} := s_1 + \cdots + s_q, \qquad \overleftarrow{\mathbf{s}} := (s_q, \dots, s_1),$$
$$\mathbf{s}_i := (s_1, \dots, s_i), \qquad \mathbf{s}^i := (s_{i+1}, \dots, s_q)$$

with $\mathbf{s}_0 := \mathbf{s}^q := \emptyset$. (Sometimes we use $\mathbf{s}_q$ instead of $\mathbf{s}$ to emphasize its dimension.)

For $\boldsymbol{\omega} = (\omega_1, \dots, \omega_{q-1}) \in (\mathbb{C}^\times)^{q-1}$, let $\omega_{i,j} := \omega_i \cdots \omega_{j-1}$ for $1 \le i \le j \le q$.

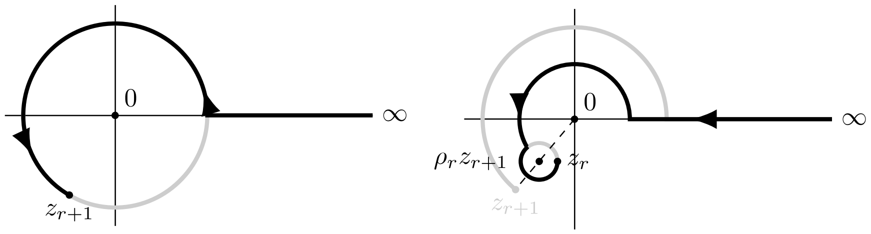

Figure 4: Hankel type analogue of iterated contours with ρ

Definition 5. For $\mathbf{s} = (s_1, \ldots, s_q), \mathbf{t} = (t_1, \ldots, t_q) \in \mathbb{C}^q$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r), \boldsymbol{\beta} = (\beta_1, \ldots, \beta_r) \in (\mathbb{C}^\times)^r$,

$$\zeta(\mathbf{s}_q; \mathbf{t}_q; \boldsymbol{\omega}_{q-1}) := \sum_{m_1, \ldots, m_q > 0} \frac{1}{((m_1 - t_1)\omega_{1,1})^{s_1} \cdots ((m_1 - t_1)\omega_{1,q} + \cdots + (m_q - t_q)\omega_{q,q})^{s_q}},$$

$$\zeta_{\widehat{\mathcal{U}}}(\mathbf{s}_r; \mathbf{t}_{r+1}; \boldsymbol{\alpha}_r; \boldsymbol{\beta}_r) := \sum_{j=0}^{r} (-1)^{\mathrm{wt}\, \mathbf{s}^j} \alpha_1^{s_1 - 1} \cdots \alpha_j^{s_j - 1} \beta_{j+1}^{s_{j+1}-1} \cdots \beta_r^{s_r - 1} \times \zeta(\mathbf{s}_j; \mathbf{t}_j; \boldsymbol{\beta}_{j-1}/\boldsymbol{\alpha}_{j-1}) \zeta(\underleftarrow{\mathbf{s}^j}; \underleftarrow{\mathbf{t}^{j+1}}; \underleftarrow{\boldsymbol{\alpha}^{j+1}/\boldsymbol{\beta}^{j+1}}).$$

It can be shown that $\zeta_{\widehat{\mathcal{U}}}(\mathbf{s}_r; \mathbf{t}_{r+1}; \boldsymbol{\alpha}_r; \boldsymbol{\beta}_r)$ is analytically continued to a holomorphic function except log branch points.

Theorem 7 (Reversal relation)**.** $\zeta_{\widehat{\mathcal{U}}}(\mathbf{s}; \mathbf{t}; \boldsymbol{\alpha}; \boldsymbol{\beta}) = (-1)^{\mathrm{wt}\, \mathbf{s}} \overline{\zeta_{\widehat{\mathcal{U}}}(\underleftarrow{\overline{\mathbf{s}}}; \underleftarrow{\overline{\mathbf{t}}}; \underleftarrow{\overline{\boldsymbol{\beta}}}; \underleftarrow{\overline{\boldsymbol{\alpha}}})}.$

Theorem 8. *For* $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$*,* $\mathbf{t} = (t_1, \ldots, t_{r+1}) \in \mathbb{C}^{r+1}$*,* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r), \boldsymbol{\beta} = (\beta_1, \ldots, \beta_r) \in \mathbb{R}_{>0}$ *with* $\Re t_1, \ldots, \Re t_{r+1} < 1$*, we have*

$$\zeta_{\widehat{\mathcal{U}}}(\mathbf{s}; \mathbf{t}; \boldsymbol{\alpha}; \boldsymbol{\beta}) = \frac{1}{2\pi\sqrt{-1}\Gamma(s_1) \cdots \Gamma(s_r)(e^{2\pi\sqrt{-1}s_1} - 1) \cdots (e^{2\pi\sqrt{-1}s_r} - 1)} \times \int_{0 \overset{\circ}{\succ} z_{r+1} \underset{\rho_r}{\overset{\circ}{\succ}} \cdots \underset{\rho_1}{\overset{\circ}{\succ}} z_1} dz_{r+1} \cdots dz_1 \frac{e^{t_1 z_1 + \cdots + t_{r+1} z_{r+1}}}{(e^{z_1} - 1) \cdots (e^{z_{r+1}} - 1)} \times (\alpha_1 z_1 - \beta_1 z_2)^{s_1 - 1} \cdots (\alpha_r z_r - \beta_r z_{r+1})^{s_r - 1},$$

where $\rho_j = \frac{\beta_j}{\alpha_j}$*, and the path* $0 \overset{\circ}{\succ} z_{r+1} \underset{\rho_r}{\overset{\circ}{\succ}} \cdots \underset{\rho_1}{\overset{\circ}{\succ}} z_1$ *denotes a Hankel type analogue of iterated contour integrals with* $\boldsymbol{\rho}$ *given below.*

In the theorem above, $a \underset{\rho}{\overset{\circ}{\succ}} b$ denotes $\rho a \overset{\circ}{\succ} b$, i.e., a slight modification of $a \overset{\circ}{\succ} b$. (See Figure 4.)

4.2 Special Values at Negative Integers

In order to give multidimensional p-adic measures, we need the special values on negative regions, which corresponds to Bernoulli polynomials.

Let $P(\mathbf{k}_r;\mathbf{t}_{r+1};\boldsymbol{\alpha}_r;\boldsymbol{\beta}_r) := \zeta_{\widehat{\mathcal{U}}}(-\mathbf{k}_r;\mathbf{t}_{r+1};\boldsymbol{\alpha}_r;\boldsymbol{\beta}_r)$ and

$$\mathcal{F}(\mathbf{w}_r;\mathbf{t}_{r+1};\boldsymbol{\alpha}_r;\boldsymbol{\beta}_r) := \sum_{j=0}^{r}\prod_{l=1}^{j}\alpha_l^{-1}\frac{e^{-t_l(\rho_{l,l}w_l\alpha_l^{-1}+\cdots+\rho_{l,j}w_j\alpha_j^{-1})}}{e^{-(\rho_{l,l}w_l\alpha_l^{-1}+\cdots+\rho_{l,j}w_j\alpha_j^{-1})}-1}$$
$$\times\prod_{l=j+1}^{r}\beta_l^{-1}\frac{e^{t_{l+1}(\rho_{j+2,l+1}^{-1}w_{j+1}\beta_{j+1}^{-1}+\cdots+\rho_{l+1,l+1}^{-1}w_l\beta_l^{-1})}}{e^{\rho_{j+2,l+1}^{-1}w_{j+1}\beta_{j+1}^{-1}+\cdots+\rho_{l+1,l+1}^{-1}w_l\beta_l^{-1}}-1},$$

where $\rho_j = \frac{\beta_j}{\alpha_j}, \rho_{i,j} = \rho_i\cdots\rho_{j-1}$. Then we have two types of expressions for $P(\mathbf{k};\mathbf{t};\boldsymbol{\alpha};\boldsymbol{\beta})$: one due to the generating function and the other due to the iterated residues.

Theorem 9.

$$\mathcal{F}(\mathbf{w};\mathbf{t};\boldsymbol{\alpha};\boldsymbol{\beta}) = \sum_{\mathbf{k}=(k_1,\ldots,k_r)\in\mathbb{Z}_{\geq 0}^r} P(\mathbf{k};\mathbf{t};\boldsymbol{\alpha};\boldsymbol{\beta})\frac{w_1^{k_1}\cdots w_r^{k_r}}{k_1!\cdots k_r!}, \tag{4.3}$$

$$P(\mathbf{k};\mathbf{t};\boldsymbol{\alpha};\boldsymbol{\beta}) = (-1)^{k_1+\cdots+k_r}k_1!\cdots k_r! \tag{4.4}$$
$$\times \operatorname*{Res}_{z_{r+1}=0}\operatorname*{Res}_{z_r=\rho_r z_{r+1}}\cdots\operatorname*{Res}_{z_1=\rho_1 z_2}\frac{e^{t_1z_1+\cdots+t_{r+1}z_{r+1}}}{(e^{z_1}-1)\cdots(e^{z_{r+1}}-1)}$$
$$\times(\alpha_1z_1-\beta_1z_2)^{-k_1-1}\cdots(\alpha_rz_r-\beta_rz_{r+1})^{-k_r-1}.$$

From these expressions, we have the following properties.

Theorem 10. *We have* $P(\mathbf{k};\mathbf{t};\boldsymbol{\alpha};\boldsymbol{\beta}) \in \mathbb{Q}[\alpha_1^{\pm1},\ldots,\alpha_r^{\pm1},\beta_1^{\pm1},\ldots,\beta_r^{\pm1}][t_1,\ldots,t_{r+1}]$ *of total degree exactly* $\operatorname{wt}\mathbf{k}+r$ *in* $t_1,\ldots,t_{r+1}$*. The degrees of* $P(\mathbf{k};\mathbf{t};\boldsymbol{\alpha};\boldsymbol{\beta})$ *in* t_1 *and* t_{r+1} *are* $\operatorname{wt}\mathbf{k}+r$ *respectively.*

Example 1. The weight 0 and degree 1 case:

$$P(0;t_1,t_2;\alpha_1;\beta_1) = \frac{t_1}{\alpha_1}-\frac{1}{2\alpha_1}+\frac{t_2}{\beta_1}-\frac{1}{2\beta_1} = \alpha_1^{-1}B_1(t_1)+\beta_1^{-1}B_1(t_2).$$

Example 2. The weight 0 and degree 2 case:

$$\begin{aligned}
&P(0,0;t_1,t_2,t_3;\alpha_1,\alpha_2;\beta_1,\beta_2)\\
&= \frac{\beta_1t_1^2}{2\alpha_1^2\alpha_2}-\frac{\beta_1t_1}{2\alpha_1^2\alpha_2}-\frac{t_1}{2\alpha_1\alpha_2}-\frac{t_1}{2\alpha_1\beta_2}+\frac{t_1t_2}{\alpha_1\alpha_2}+\frac{t_1t_3}{\alpha_1\beta_2}\\
&\quad+\frac{t_2^2}{2\alpha_2\beta_1}-\frac{t_2}{2\alpha_1\alpha_2}-\frac{t_2}{2\beta_1\beta_2}-\frac{t_2}{2\alpha_2\beta_1}+\frac{t_2t_3}{\beta_1\beta_2}\\
&\quad+\frac{\alpha_2t_3^2}{2\beta_1\beta_2^2}-\frac{\alpha_2t_3}{2\beta_1\beta_2^2}-\frac{t_3}{2\alpha_1\beta_2}-\frac{t_3}{2\beta_1\beta_2}\\
&\quad+\frac{\beta_1}{12\alpha_1^2\alpha_2}+\frac{1}{4\alpha_1\alpha_2}+\frac{1}{4\alpha_1\beta_2}+\frac{\alpha_2}{12\beta_1\beta_2^2}+\frac{1}{12\alpha_2\beta_1}+\frac{1}{4\beta_1\beta_2}.
\end{aligned}$$

5 An Interpretation of $(t_1, \ldots, t_{r+1})$ (II)

5.1 Multidimensional p-Adic Measures

Let $\{x\}_p \in [0,1) \cap \mathbb{Q}$ be the p-adic fractional part of $x \in \mathbb{Q}_p$, i.e., the negative power part in the p-adic expansion and $[x]_p = x - \{x\}_p$ be its p-adic integer part. Let $\{\mathbf{x}\}_p = (\{x_1\}_p, \ldots, \{x_{r+1}\}_p) \in \mathbb{Q}^{r+1}$ for $\mathbf{x} \in \mathbb{Q}_p^{r+1}$.

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r), \boldsymbol{\beta} = (\beta_1, \ldots, \beta_r) \in (\mathbb{Z}_p^\times)^r$, $\boldsymbol{\lambda} \in \mathbb{Z}_p^{r+1}$ and

$$\Phi = \Phi(\boldsymbol{\alpha}_r; \boldsymbol{\beta}_r) = \begin{pmatrix} -\alpha_1 & 0 & \cdots & \cdots & 0 \\ \beta_1 & -\alpha_2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \beta_{r-1} & -\alpha_r \\ 0 & \cdots & \cdots & 0 & \beta_r \end{pmatrix} \in (\mathbb{Z}_p^\times)^{(r+1)\times r}.$$

Theorem 11. *The following map extends to a measure on $\mathbb{Z}_p^r$:*

$$\mu_{\mathbf{k};\boldsymbol{\lambda};\boldsymbol{\alpha};\boldsymbol{\beta}}(\mathbf{a} + (p^n\mathbb{Z}_p)^r) = p^{n \operatorname{wt} \mathbf{k}} P\left(\mathbf{k}; \left\{\frac{\Phi\mathbf{a} - \boldsymbol{\lambda}}{p^n}\right\}_p + \frac{\boldsymbol{\lambda}}{p^n}; \boldsymbol{\alpha}; \boldsymbol{\beta}\right) \in \mathbb{Q}_p,$$

where $\mathbf{a} \in \{0, \ldots, p^n - 1\}^r$.

Remark. The distribution property follows from the expression of $P(\mathbf{k}; \mathbf{t}; \boldsymbol{\alpha}; \boldsymbol{\beta})$ via the generating function (4.3) while the boundedness follows from that via iterated residues (4.4). Note that product measures are used in the construction of the multiple p-adic L-functions [1–3].

Example 3. For $n \in \mathbb{Z}_{\geq 0}$ and $a_1 \in \{0, \ldots, p^n - 1\}$,

$$\mu_{0;0;\alpha_1;\beta_1}(a_1 + (p^n\mathbb{Z}_p)) = -\frac{1}{\alpha_1}\left[\frac{-\alpha_1 a_1}{p^n}\right]_p - \frac{1}{2\alpha_1} - \frac{1}{\beta_1}\left[\frac{\beta_1 a_1}{p^n}\right]_p - \frac{1}{2\beta_1}.$$

The case $\alpha_1 = -\alpha, \beta_1 = 1$ coincides with the Mazur measure [10].

Example 4. For $n \in \mathbb{Z}_{\geq 0}$ and $a_1, a_2 \in \{0, \ldots, p^n - 1\}$,

$$\begin{aligned}
&\mu_{0,0;0,0;\alpha_1,\alpha_2;\beta_1,\beta_2}(a_1 + (p^n\mathbb{Z}_p), a_2 + (p^n\mathbb{Z}_p)) \\
&= \frac{\beta_1}{2\alpha_1^2\alpha_2}\left[\frac{-\alpha_1 a_1}{p^n}\right]_p^2 - \left(\frac{\beta_1}{2\alpha_1^2\alpha_2} + \frac{1}{2\alpha_2} + \frac{1}{2\beta_2}\right)\left[\frac{-\alpha_1 a_1}{p^n}\right]_p \\
&\quad + \frac{1}{\alpha_1\alpha_2}\left[\frac{-\alpha_1 a_1}{p^n}\right]_p\left[\frac{\beta_1 a_1 - \alpha_2 a_2}{p^n}\right]_p \\
&\quad + \frac{1}{\alpha_1\beta_2}\left[\frac{-\alpha_1 a_1}{p^n}\right]_p\left[\frac{\beta_2 a_2}{p^n}\right]_p + \frac{1}{2\alpha_2\beta_1}\left[\frac{\beta_1 a_1 - \alpha_2 a_2}{p^n}\right]_p^2 \\
&\quad - \left(\frac{1}{2\alpha_1\alpha_2} + \frac{1}{2\beta_1\beta_2} + \frac{1}{2\alpha_2\beta_1}\right)\left[\frac{\beta_1 a_1 - \alpha_2 a_2}{p^n}\right]_p \\
&\quad + \frac{1}{\beta_1\beta_2}\left[\frac{\beta_1 a_1 - \alpha_2 a_2}{p^n}\right]_p\left[\frac{\beta_2 a_2}{p^n}\right]_p \\
&\quad + \frac{\alpha_2}{2\beta_1\beta_2^2}\left[\frac{\beta_2 a_2}{p^n}\right]_p^2 - \left(\frac{\alpha_2}{2\beta_1\beta_2^2} + \frac{1}{2\alpha_1\beta_2} + \frac{1}{2\beta_1\beta_2}\right)\left[\frac{\beta_2 a_2}{p^n}\right]_p \\
&\quad + \frac{\beta_1}{12\alpha_1^2\alpha_2} + \frac{1}{4\alpha_1\alpha_2} + \frac{1}{4\alpha_1\beta_2} + \frac{\alpha_2}{12\beta_1\beta_2^2} + \frac{1}{12\alpha_2\beta_1} + \frac{1}{4\beta_1\beta_2}.
\end{aligned}$$

From these examples, we see that the denominators may include some explicit primes. However they are p-adic integers for any prime p.

Theorem 12. *For $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{Z}_{\geq 0}^r$, $\boldsymbol{\lambda} \in \mathbb{Z}_p^{r+1}$, $\boldsymbol{\alpha}, \boldsymbol{\beta} \in (\mathbb{Z}_p^\times)^r$, we have*

$$P(\mathbf{k}; \boldsymbol{\lambda}; \boldsymbol{\alpha}; \boldsymbol{\beta}) = \int_{\mathbb{Z}_p^r} x_1^{k_1} \cdots x_r^{k_r} \mu_{\mathbf{0};\boldsymbol{\lambda};\boldsymbol{\alpha};\boldsymbol{\beta}}.$$

Example 5.

$$\frac{1}{k+1}(\beta_1^{-k-1} B_{k+1}(\lambda_2) - (-\alpha_1)^{-k-1} B_{k+1}(\lambda_1)) = \int_{\mathbb{Z}_p} x_1^k \mu_{0,0;\lambda_1,\lambda_2;\alpha_1;\beta_1}.$$

Theorem 13. *For $\mathbf{k} \in \mathbb{Z}_{\geq 0}^r$, $\boldsymbol{\lambda} \in \mathbb{Z}_p^{r+1}$, $\boldsymbol{\alpha}, \boldsymbol{\beta} \in (\mathbb{Z}_p^\times)^r$, $n \in \mathbb{Z}_{\geq 0}$ and $\mathbf{a} \in \{0, \ldots, p^n - 1\}^r$,*

$$\mu_{\mathbf{k};\boldsymbol{\lambda};\boldsymbol{\alpha};\boldsymbol{\beta}}(\mathbf{a} + (p^n\mathbb{Z}_p)^r) \in \mathbb{Z}_p.$$

In particular, we have

$$P(\mathbf{k}; \boldsymbol{\lambda}; \boldsymbol{\alpha}; \boldsymbol{\beta}) \in \mathbb{Z}_p.$$

Example 6. In the $p = 2$ and Mazur measure case, this theorem implies that the measure is indeed p-adic integer though 2 appears explicitly in the denominator.

$$\mu_{0;0;-\alpha;1}(a_1 + (2^n\mathbb{Z}_2)) = \frac{1}{\alpha}\left[\frac{\alpha a_1}{2^n}\right]_2 + \frac{\alpha^{-1} - 1}{2} \in \mathbb{Z}_2.$$

Remark. More general p-adic measures can be constructed. Other constructions such as Koblitz's construction [9], construction via the associated power series $\in \mathbb{Z}_p[[\mathbf{T}]]$ may also work. However it may not be so easy due to the singularities of the generating function $\mathcal{F}(\mathbf{w}; \mathbf{t}; \boldsymbol{\alpha}; \boldsymbol{\beta})$.

5.2 Applications

Once we obtain p-adic measures, the routine such as Kummer-type congruences etc. works well. Here we give one result as an example. Let ω be the Teichmüller character and we write $x = \omega(x)\langle x\rangle$ for $x \in \mathbb{Z}_p^\times$.

Definition 6. For $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{Z}_p^r$, $\boldsymbol{\lambda} \in \mathbb{Z}_p^r$, $\boldsymbol{\alpha}, \boldsymbol{\beta} \in (\mathbb{Z}_p^\times)^r$ and $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r$, we define

$$L_{\widehat{\mathcal{U}},p}(\mathbf{s}; \boldsymbol{\lambda}; \omega^{\mathbf{k}}; \boldsymbol{\alpha}; \boldsymbol{\beta}) := \int_{(\mathbb{Z}_p^\times)^r} \langle x_1\rangle^{-s_1} \cdots \langle x_r\rangle^{-s_r} \omega^{k_1}(x_1) \cdots \omega^{k_r}(x_r) \mu_{\mathbf{0};\boldsymbol{\lambda};\boldsymbol{\alpha};\boldsymbol{\beta}} \in \mathbb{Z}_p$$

and

$$Q(\mathbf{k}; \boldsymbol{\lambda}; \boldsymbol{\alpha}; \boldsymbol{\beta}) := L_{\widehat{\mathcal{U}},p}(-\mathbf{k}; \boldsymbol{\lambda}; \omega^{\mathbf{k}}; \boldsymbol{\alpha}; \boldsymbol{\beta}).$$

Theorem 14. *For $\boldsymbol{\lambda} \in \mathbb{Z}_p^r$, $\boldsymbol{\alpha}, \boldsymbol{\beta} \in (\mathbb{Z}_p^\times)^r$, $n \in \mathbb{Z}_{\geq 0}$ and $\mathbf{k} \in \mathbb{Z}_{\geq 0}^r$, we have*

$$\sum_{j_1=0}^{n_1} \cdots \sum_{j_r=0}^{n_r} (-1)^{j_1+\cdots+j_r} \binom{n_1}{j_1} \cdots \binom{n_r}{j_r} Q(\mathbf{k}+\mathbf{j}(p-1); \boldsymbol{\lambda}; \boldsymbol{\alpha}; \boldsymbol{\beta}) \equiv 0 \pmod{p^{n_1+\cdots+n_r}}.$$

A special case reduces to the congruence shown in [19]:

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \Bigg(\frac{B_{j(p-1)+k}(x) - B_{j(p-1)+k}}{j(p-1)+k} - p^{j(p-1)+k-1} \frac{B_{j(p-1)+k}\left(\left\{ \frac{-\lambda}{p} \right\}_p + \frac{\lambda}{p} \right) - B_{j(p-1)+k}}{j(p-1)+k} \Bigg) \equiv 0 \pmod{p^n}.$$

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