A new proof of q-Schur duality without the assumption of BZ-duality

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Abstract

In this note, we aim to provide an overview of (q-)MSW formula, which were formulated by Maesaka–Seki–Watanabe ([7], understood as $q = 1$ case), Yamamoto ([17], for Schur MZVs case) and the author ([14], for Schur q-MZVs case), and to present the new proof techniques of several types of the "DUALITY". A large part of this note is based on a forthcoming paper [13], a joint work with Yoshihiro Takeyama (University of Tsukuba).

1 Preliminaries

For $\boldsymbol{k} = (k_1, \ldots, k_r) \in (\mathbb{Z}_{>0})^r$ with $k_r > 1$, the *multiple zeta values* is defined by the limit of the *multiple harmonic sums*

$$\zeta(\boldsymbol{k}) = \lim_{N\to\infty} \zeta_{<N}(\boldsymbol{k}) = \lim_{N\to\infty} \sum_{0<m_1<\cdots<m_r<N} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \in \mathbb{R}_{>0}.$$

The condition for convergence of this series is $k_r > 1$. Multiple zeta values (MZVs for short) were apparently first investigated by Euler for the cases $\zeta(k, l)$. In recent years, they have attracted renewed attention because of their emergence in various areas of mathematics and physics. One of the main interests behind the study of MZVs are to analyze a large amount of linear (resp. algebraic) relations and its transcendence.

Besides the topics mentioned above, there exist many other interesting studies about MZVs. This abundance of research lies the following property of MZVs:

Theorem 1.1 (Drinfeld, Kontsevich, Le–Murakami etc.)**.** *For any* $\boldsymbol{k} = (k_1, \ldots, k_r)$ *with* $k_r > 1$*,* $\zeta(\boldsymbol{k})$ *has the integral representation*

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$$\zeta(\boldsymbol{k}) = \int\cdots\int_{0<t_{1,1}<\cdots<t_{1,k_1}<\cdots<t_{r,1}<\cdots<t_{r,k_r}<1} \frac{\mathrm{d}t_{1,1}}{1-t_{1,1}}\frac{\mathrm{d}t_{1,2}}{t_{1,2}}\cdots\frac{\mathrm{d}t_{1,k_1}}{t_{1,k_1}}\cdots\frac{\mathrm{d}t_{r,1}}{1-t_{r,1}}\frac{\mathrm{d}t_{r,2}}{t_{r,2}}\cdots\frac{\mathrm{d}t_{r,k_r}}{t_{r,k_r}}.$$

Maesaka, Seki and Watanabe ([7]) made a remarkable discovery by studying its certain Riemann sums.

Theorem 1.2 (MSW formula, [7, Theorem 1.3]). *For any $\boldsymbol{k} = (k_1, \ldots, k_r) \in (\mathbb{Z}_{>0})^r$ and $N \in \mathbb{Z}_{>0}$, define $\zeta^\flat_{<N}(\boldsymbol{k})$ by*

$$\zeta^\flat_{<N}(\boldsymbol{k}) = \sum_{\substack{0<n_{j,1}\le\cdots\le n_{j,k_j}<N\ (1\le j\le r)\\ n_{j,k_j}<n_{(j+1),1}\ (1\le j<r)}} \prod_{j=1}^{r} \frac{1}{(N-n_{j,1})n_{j,2}\cdots n_{j,k_j}} \in \mathbb{Q}_{>0}.$$

Then it holds

$$\zeta_{<N}(\boldsymbol{k}) = \zeta^\flat_{<N}(\boldsymbol{k}).$$

Theorem 1.2 states that suitable finite sums arising from two types of representation always have the same value. We call $\zeta^\flat_{<N}(\boldsymbol{k})$ *$\flat$-value of* $\zeta(\boldsymbol{k})$ and call this phenomenon the *discretization of MZVs*.

Example 1.3 ($\boldsymbol{k} = (2,3)$). $\flat$-value is written as

$$\zeta^\flat_{<N}(\boldsymbol{k}) = \sum_{0<n_{1,1}\le n_{1,2}<n_{2,1}\le n_{2,2}\le n_{2,3}<N} \frac{1}{(N-n_{1,1})n_{1,2}(N-n_{2,1})n_{2,2}n_{2,3}}.$$

It is easy to see that

$$\lim_{N\to\infty}\zeta^\flat_{<N}(\boldsymbol{k}) = \int\cdots\int_{0<t_{1,1}<t_{1,2}<t_{2,1}<t_{2,2}<t_{2,3}<1} \frac{\mathrm{d}t_{1,1}}{1-t_{1,1}}\frac{\mathrm{d}t_{1,2}}{t_{1,2}}\frac{\mathrm{d}t_{2,1}}{1-t_{2,1}}\frac{\mathrm{d}t_{2,2}}{t_{2,2}}\frac{\mathrm{d}t_{2,3}}{t_{2,3}}.$$

That is because we have

$$\lim_{N\to\infty}\zeta_{<N}(\boldsymbol{k}) = \zeta(\boldsymbol{k}) = \lim_{N\to\infty}\zeta^\flat_{<N}(\boldsymbol{k}). \tag{1}$$

In general (1) does not occur so that the definition of the $\flat$-value is surprising.

As an application of [7], Maesaka, Seki and Watanabe provided a new proof of the following theorem.

Theorem 1.4 (Ordinary duality). *For any $a_j,\ b_j,\ s \ge 1,\ (1 \le j \le s)$, every tuple of positive integer is written as $\boldsymbol{k} = (\underbrace{1,\ldots,1}_{a_1-1}, b_1+1, \ldots, \underbrace{1,\ldots,1}_{a_s-1}, b_s+1)$. If we define the dual $\boldsymbol{k}^\dagger$ as*

$$\boldsymbol{k}^\dagger = (\underbrace{1,\ldots,1}_{b_s-1}, a_s+1, \ldots, \underbrace{1,\ldots,1}_{b_1-1}, a_1+1),$$

then, it holds

$$\zeta(\boldsymbol{k}) = \zeta(\boldsymbol{k}^\dagger).$$

Theorem 1.5 (Hoffman duality, [6, Theorem 4.6]). *For any $\boldsymbol{k} = (k_1, \ldots, k_r)$, it holds*

$$\zeta_{\mathcal{A}}(\boldsymbol{k}) = (-1)^r \sum_{\boldsymbol{k} \preceq \boldsymbol{l}} \zeta_{\mathcal{A}}(\boldsymbol{l}).$$

Here, $\boldsymbol{k} \preceq \boldsymbol{l}$ means the partial order by replacing "+" to "," (e.g. $(4) \preceq (1,3)$).

Remark 1.6. In Theorem 1.5, $\zeta_{\mathcal{A}}(\boldsymbol{k})$ denotes the *Finite MZVs*

$$\zeta_{\mathcal{A}}(\boldsymbol{k}) = \left(\sum_{0<m_1<\cdots<m_r<p} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \mod p \right)_{p:\text{prime}} \in \mathcal{A},$$

$$\mathcal{A} = \prod_{p:\text{prime}} \mathbb{F}_p \Big/ \bigoplus_{p:\text{prime}} \mathbb{F}_p \, .$$

In [7], the proofs of Theorem 1.5 and Theorem 1.4 are slightly different from each other. By calculating the difference $\zeta^\flat_{<N}(\boldsymbol{k}) - \zeta^\flat_{<N}(\boldsymbol{k}^\dagger)$, we have Theorem 1.4, and applying $1/(p-n) \equiv -1/n \pmod p$, we have Theorem 1.5. Both process is clear.

Theorem 1.2 is generalized to Schur type by Yamamoto ([17]). In this note, we only state the definitions of *Schur multiple zeta values*, and omit other definitions. For further information, the readers are referred to [14, 15].

Definition 1.7 (Schur MZVs ([9]), content parametrized index ([14, 17])). For any Young tableau $\boldsymbol{k} = (k_{i,j})$ satisfying certain condition (cf. [9]), define the Schur MZVs by

$$\zeta(\boldsymbol{k}) = \lim_{N\to\infty} \zeta_{<N}(\boldsymbol{k}) = \lim_{N\to\infty} \sum_{(m_{i,j}) \in \mathrm{SSYT}_{<N}(D)} \prod_{(i,j)\in D} \frac{1}{m_{i,j}^{k_{i,j}}} \in \mathbb{R}_{>0},$$

where D is a Young diagram as shape of $\boldsymbol{k}$ and $\mathrm{SSYT}_{<N}(D)$ is the set of semi-standard Young tableaux of shape D whose entries are positive and less than N. If $\boldsymbol{k} = (k_{i,j})_{(i,j)\in D}$ satisfies $k_{i,j} = k_{i-1,j-1}$ for any $(i,j) \in D$, then $\boldsymbol{k}$ is referred to as the *content parametrized index.*

Example 1.8. We give one example of content parametrized Schur MZVs. Let $\boldsymbol{k} = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 2 & 2 \\ \hline\end{array}$, then $\zeta(\boldsymbol{k})$ is

$$\zeta\left(\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 2 & 2 \\ \hline\end{array}\right) = \sum_{\substack{0<m_{11}\le m_{12} \\ \wedge \quad \wedge \\ m_{21}\le m_{22}}} \frac{1}{m_{11}^2 m_{12}^3 m_{21}^2 m_{22}^2}.$$

Theorem 1.9 (Schur MSW formula, [17, Theorem 3.6]). *For any content parametrized index $\boldsymbol{k}$ and $N \in \mathbb{Z}_{>0}$, define the $\flat$-value $\zeta^\flat_{<N}(\boldsymbol{k})$ by*

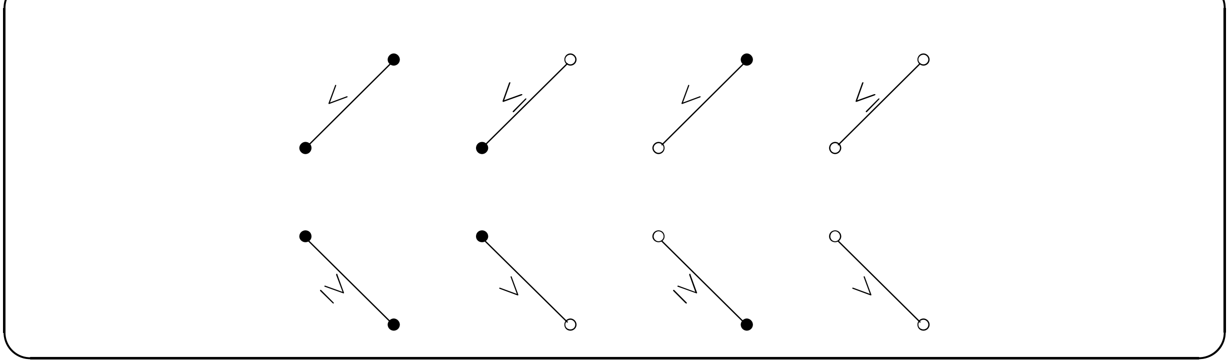
$$\zeta^\flat_{<N}(\boldsymbol{k}) = \sum_{\substack{\boldsymbol{n}_p^{(l)} \in \{1,\ldots,N-1\}^{J_p} \\ \boldsymbol{n}_p^{(l)} \trianglelefteq \boldsymbol{n}_p^{(l+1)} \ (1\le l<k_p) \\ \boldsymbol{n}_p^{(k_p)} \triangleleft \boldsymbol{n}_{p+1}^{(1)} \ (p_0\le p<p_1)}} \frac{1}{\prod(\boldsymbol{N} - \boldsymbol{n}_p^{(1)}) \prod(\boldsymbol{n}_p^{(2)}) \cdots \prod(\boldsymbol{n}_p^{(k_p)})} \in \mathbb{Q}_{>0}.$$

Then it holds

$$\zeta_{<N}(\boldsymbol{k}) = \zeta^{\flat}_{<N}(\boldsymbol{k}).$$

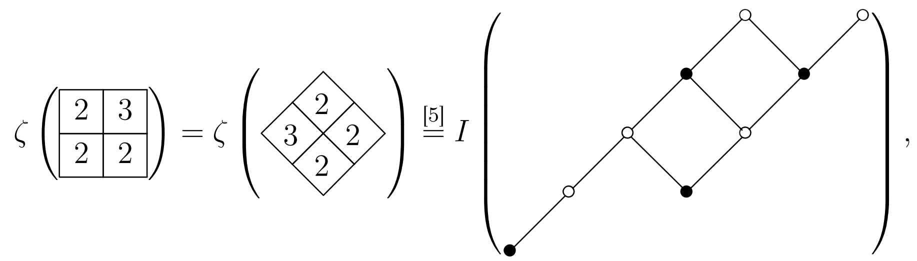
Remark 1.10 (cf. [5, 14])**.** The notation in Theorem 1.9 essentially comes from the 2-*poset representation for MZVs*, originally defined by Yamamoto ([16]). Based on [16], Hirose–Murahara–Onozuka ([5]) formulated the 2-poset representation for content parametrized Schur MZVs. So, the order of tuples $\lhd$, $\unlhd$ and the symbols in $\zeta^{\flat}_{<N}(\boldsymbol{k})$ denote the order of integral variables. Since the definition of $\lhd$, $\unlhd$ is somewhat complicated, we define them in analogy with 2-poset and call it *summation order* as bellow:

Summation order



Although 2-posets are left-right commutative in general, the summation order reflects information on sets of Lebesgue measure zero, and hence summation order is **not left–right commutative**.

Example 1.11 (cf. [16])**.** Let $\boldsymbol{k} = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 2 & 2 \\ \hline \end{array}$. we have



where $I : \{\text{admissible 2-posets}\} \to \mathbb{R}_{>0}$ is defined by

$$I(X) = \int_{\Delta(X)} \prod_{x \in X} \omega_{\delta_X(x)}(t_x).$$

Here, $\Delta(X) = \big\{(t_x)_x \in (0,1)^X \mid t_x < t_y \text{ if } x \prec y\big\}$, $\delta_X : X \to \{0,1\}$ and

$$\frac{\mathrm{d}t}{t} \leftrightarrow \circ, \quad = \frac{\mathrm{d}t}{1-t} \leftrightarrow \bullet.$$

By the above summation order, we have

$$\zeta^{\flat}_{<N}\left(\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 2 & 2 \\ \hline \end{array}\right) = \sum_{\boldsymbol{n}\in X} \frac{1}{(N-n_1)n_2n_3(N-n_4)n_5(N-n_6)n_7(N-n_8)n_9},$$
$$X = \left\{ \boldsymbol{n} \in \{1,\dots,N-1\}^9 \;\middle|\; \begin{smallmatrix} n_1 \le n_2 \le n_3 < n_4 \le n_5 \\ \vee\!| \quad \vee \quad \vee\!| \\ n_6 \le n_7 < n_8 \le n_9 \end{smallmatrix} \right\}.$$

Now, we introduce the q-analogue of Theorem 1.2. For $m \in \mathbb{Z}_{\geq 0}$, *$q$-integer* is given by $[m] = (1-q^m)/(1-q)$. We define the *q-analogue of MZVs* as follows:

Definition 1.12 (Bradley–Zhao type q-MZVs (BZ-MZVs for short))**.** For $\boldsymbol{k} = (k_1,\dots,k_r) \in (\mathbb{Z}_{>0})^r$ with $k_r > 1$, $\zeta^{BZ}(\boldsymbol{k})$ is defined by

$$\zeta^{BZ}(\boldsymbol{k}) = \lim_{N\to\infty} \zeta^{BZ}_{<N}(\boldsymbol{k}) = \lim_{N\to\infty} \sum_{0<m_1<\cdots<m_r<N} \frac{q^{(k_1-1)m_1+\cdots+(k_r-1)m_r}}{[m_1]^{k_1}\cdots[m_r]^{k_r}} \in \mathbb{Q}[\![q]\!].$$

Remark 1.13. By definition, we have

$$[m] = \frac{1-q^m}{1-q} = 1 + q + q^2 + \cdots q^{m-1} \to m \text{ as } q \uparrow 1,$$

so that BZ-MZVs satisfy

$$\zeta^{BZ}(\boldsymbol{k}) \to \zeta(\boldsymbol{k}) \text{ as } q \uparrow 1,$$

and so does $\zeta^{q\flat}_{<N}(\boldsymbol{k})$.

Next theorem is the q-analgue of Theorem 1.2, called the *q-MSW formula*.

Theorem 1.14 (q-MSW formula, [14, Theorem 1.2])**.** *For any $N > 0$ and $\boldsymbol{k} = (k_1,\dots,k_r) \in (\mathbb{Z}_{>0})^r$, we have*

$$\zeta^{BZ}_{<N}(\boldsymbol{k}) = \zeta^{q\flat}_{<N}(\boldsymbol{k}),$$

where

$$\zeta^{q\flat}_{<N}(\boldsymbol{k}) = \sum_{\substack{0<n_{j,1}\le\cdots\le n_{j,k_j}<N (1\le j\le r) \\ n_{j,k_j}<n_{(j+1),1} (1\le j<r)}} \prod_{j=1}^{r} \frac{q^{n_{j,2}+\cdots+n_{j,k_j}}}{[N-n_{j,1}][n_{j,2}]\cdots[n_{j,k_j}]} \in \mathbb{Q}[\![q]\!].$$

2 A new proof of the q-version of Hoffman duality

Based on the study in Bachmann–Takeyama–Tasaka [1, 2], K. Hessami Pilehrood et. al. ([4]) formulated the "q-analogue" of Theorem 1.5.

Theorem 2.1 ([4, Theorem 3.1])**.** *Assume $q = \eta_N$ (primitive N-th root of unity), then it holds*

$$\zeta^{BZ}_{<N}(\boldsymbol{k}) = (-1)^r \sum_{\boldsymbol{k} \preceq \boldsymbol{l}} \zeta^{SZ}_{<N}(\boldsymbol{l})$$

for all $\boldsymbol{k} = (k_1, \ldots, k_r) \in (\mathbb{Z}^r_{>0})$. Here, $\zeta^{SZ}_{<N}(\boldsymbol{k})$ denotes the Schlesinger–Zudilin type q-MZVs

$$\zeta^{SZ}_{<N}(\boldsymbol{k}) = \sum_{0<m_1<\cdots<m_r<N} \frac{q^{k_1 m_1 + \cdots + k_r m_r}}{[m_1]^{k_1} \cdots [m_r]^{k_r}} \in \mathbb{Q}[\![q]\!].$$

Proof. (cf. [14]) The equation

$$\frac{1}{[N-n]} = -\frac{q^n}{[n]},\ n < N$$

almost obviously holds if and only if $q = \eta_N$. In this situation, $\zeta^{q\flat}_{<N}(\boldsymbol{k})$ is equivalent to the right hand side of Theorem 2.1. □

Remark 2.2. By the shape of these equations

$$\zeta_{\mathcal{A}}(\boldsymbol{k}) = (-1)^r \sum_{\boldsymbol{k} \preceq \boldsymbol{l}} \zeta_{\mathcal{A}}(\boldsymbol{l}),$$
$$\zeta^{BZ}_{<N}(\boldsymbol{k}) = (-1)^r \sum_{\boldsymbol{k} \preceq \boldsymbol{l}} \zeta^{SZ}_{<N}(\boldsymbol{l}),$$

Theorem 2.1 is regarded as a q-analogue of Theorem 1.5.

3 Bradley–Zhao type duality via q-MSW formula

BZ-MZVs satisfy the same form of the ordinary duality (Theorem 1.4).

Theorem 3.1 (BZ-duality, [3, Corollary 3])**.** *For any $a_j,\ b_j,\ s \in \mathbb{Z}_{>0}$ $(1 \le j \le s)$, it holds*

$$\zeta^{BZ}(\underbrace{1, \ldots, 1}_{a_1 - 1}, b_1 + 1, \ldots, \underbrace{1, \ldots, 1}_{a_s - 1}, b_s + 1) = \zeta^{BZ}(\underbrace{1, \ldots, 1}_{b_s - 1}, a_s + 1, \ldots, \underbrace{1, \ldots, 1}_{b_1 - 1}, a_1 + 1).$$

Example of Theorem 3.1 (cf. [13]). We will introduce the process of Theorem 3.1 in the case $\boldsymbol{k} = (1, 2)$. In this case, $\boldsymbol{k}^\dagger = (3)$. By definition, we have

$$\zeta^{q\flat}_{<N}(\boldsymbol{k}) = \sum_{0<n_1<n_2 \le n_3<N} \frac{q^{n_3}}{[N-n_1][N-n_2][n_3]}. \tag{2}$$

Applying

$$\frac{1}{[n]} = \frac{q^n}{[n]} + 1 - q \tag{3}$$

to (2) yields

$$\zeta^{q\flat}_{<N}(\boldsymbol{k}) = (1-q) \sum_{0<n_1<n_2\le n_3<N} \frac{q^{n_3}}{[N-n_1][n_3]} + \sum_{0<n_1<n_2\le n_3<N} \frac{q^N \cdot q^{n_3-n_2}}{[N-n_1][N-n_2][n_3]}. \tag{4}$$

We also apply (3) to (4), then we have

$$\begin{aligned}\zeta^{q\flat}_{<N}(\boldsymbol{k}) = (1-q)^2 \sum_{0<n_1<n_2\le n_3<N} \frac{q^{n_3}}{[n_3]} &+ \sum_{0<n_1<n_2\le n_3<N} \frac{q^N \cdot q^{n_3-n_2}}{[N-n_1][N-n_2][n_3]} \\ &+ \sum_{0<n_1<n_2\le n_3<N} \frac{q^N \cdot q^{n_3-n_1}}{[N-n_1][n_3]}.\end{aligned} \tag{5}$$

By (5), we obtain

$$\zeta^{q\flat}_{<N}(\boldsymbol{k}) = (1-q)^2 \sum_{0<n_1<n_2\le n_3<N} \frac{q^{n_3}}{[n_3]} + q^N O(N^3) \text{ as } N\to\infty. \tag{6}$$

Therefore, we can take a limit of $\zeta^{q\flat}_{<N}(\boldsymbol{k})$ and obtain

$$\zeta^{q\flat}(\boldsymbol{k}) = \lim_{N\to\infty} \zeta^{q\flat}_{<N}(\boldsymbol{k}) = (1-q)^2 \sum_{0<n_1<n_2\le n_3} \frac{q^{n_3}}{[n_3]}.$$

Now, we replace ${}^{\forall}n_j$ with

$$n_1 = \ell_1, \quad n_2 = \ell_1+\ell_2, \quad n_3 = \ell_1+\ell_2+\ell_3, \ (\ell_1,\ell_2>0,\ \ell_3\ge 0). \tag{7}$$

Then, we have

$$\begin{aligned}\zeta^{q\flat}(\boldsymbol{k}) &= (1-q)^3 \sum_{m=1}^{\infty} \sum_{\substack{\ell_1,\ell_2>0 \\ \ell_3\ge 0}} q^{m(\ell_1+\ell_2+\ell_3)} \\ &= \sum_{m>0} \frac{q^m(1-q)}{1-q^m}\frac{q^m(1-q)}{1-q^m}\frac{1-q}{1-q^m} = \zeta^{BZ}(3) = \zeta^{BZ}(\boldsymbol{k}^\dagger).\end{aligned}$$

The general case can be proved inductively. □

Remark 3.2 (Takeyama–T. [13]). In general, (6) is written by

$$\begin{aligned}\frac{\zeta^{q\flat}_{<N}(\boldsymbol{k})}{(1-q)^{k_1+\cdots+k_r}} = \sum_{\substack{0<n_{j,1}\le\cdots\le n_{j,k_j}<N(1\le j\le r) \\ n_{j,k_j}<n_{(j+1),1}(1\le j<r)}} \sum_{\substack{m_{j,2},\ldots,m_{j,k_j}>0 \\ (1\le j\le r)}} \prod_{j=1}^{r} q^{n_{j,2}m_{j,2}+\cdots+n_{j,k_j}m_{j,k_j}} \\ + q^N O(N^J) \text{ as } N\to\infty\end{aligned} \tag{8}$$

for any $\boldsymbol{k} = (k_1,\ldots,k_r) \in (\mathbb{Z}_{>0})^r$.

Remark 3.3 (cf. [11, 12])**.** In the proof of Theorem 3.1, we call (7) the *resummation*. It is surprising that the resummation essentially coincides with the technique of the *resummation duality*, originally formulated by Takeyama (cf, [11, 12]):

$$(e_1 - g_1)^{\alpha_1} g_{\beta_1+1} \cdots (e_1 - g_1)^{\alpha_r} g_{\beta_r+1} \\ - (e_1 - g_1)^{\beta_r} g_{\alpha_r+1} \cdots (e_1 - g_1)^{\beta_1} g_{\alpha_1+1} \in \ker Z_q.$$

Here, for $^\forall \alpha_j,\ \beta_j\ (1 \leq j \leq r),\ e_j,\ g_j$ and $\ker Z_q$ denotes the element of the $\mathbb{Q}$-algebra and the kernel of evaluation map

$$Z_p :\ \{\text{words}\} \to \{q\text{-MZVs}\}.$$

Although we are currently unable to determine whether BZ-duality is contained in resummation duality, it is very interesting that the proof methods are remarkably similar.

4 Nakasuji–Ohno duality via q-Schur MSW formula

Based on Definition 1.7, we define the *q-Schur MZVs* and formulate the q-Schur version of Theorem 1.2 (i.e. q-analogue of Theorem 1.9) as follows.

Definition 4.1 (BZ type of q-Schur MZVs)**.** For any Young tableau $\boldsymbol{k} = (k_{i,j})$ satisfying the convergent condition (i.e. corner condition), define the q-Schur MZVs by

$$\zeta(\boldsymbol{k}) = \lim_{N\to\infty} \zeta_{<N}(\boldsymbol{k}) = \lim_{N\to\infty} \sum_{(m_{i,j})\in \mathrm{SSYT}_{<N}(D)} \prod_{(i,j)\in D} \frac{q^{(k_{i,j}-1)m_{i,j}}}{[m_{i,j}]^{k_{i,j}}} \in \mathbb{Q}[\![q]\!].$$

Theorem 4.2 (q-Schur MSW formula, [14, Theorem 1.5])**.** *For any content parametrized index $\boldsymbol{k}$ and $N \in \mathbb{Z}_{>0}$, define the $q\flat$-value $\zeta^{q\flat}_{<N}(\boldsymbol{k})$ by*

$$\zeta^{q\flat}_{<N}(\boldsymbol{k}) = \sum_{\substack{\boldsymbol{n}_p^{(l)} \in \{1,\dots,N-1\}^{J_p} \\ \boldsymbol{n}_p^{(l)} \trianglelefteq \boldsymbol{n}_p^{(l+1)} (1\leq l<k_p) \\ \boldsymbol{n}_p^{(k_p)} \triangleleft \boldsymbol{n}_{p+1}^{(1)} (p_0 \leq p < p_1)}} \prod_{p=p_0}^{p_1} \frac{\prod'(q^{\boldsymbol{n}_p^{(1)}}) \cdots \prod'(q^{\boldsymbol{n}_p^{(k_p)}})}{\prod([\boldsymbol{N} - \boldsymbol{n}_p^{(1)}]_q) \prod([\boldsymbol{n}_p^{(2)}]_q) \cdots \prod([\boldsymbol{n}_p^{(k_p)}]_q)} \in \mathbb{Q}[\![q]\!].$$

Here, $\prod'(q^{\boldsymbol{n}}) = q^{n_2+\cdots+n_k}$ for any tuple $\boldsymbol{n} = (n_1, \dots, n_k)$. Then it holds

$$\zeta^{BZ}_{<N}(\boldsymbol{k}) = \zeta^{q\flat}_{<N}(\boldsymbol{k}).$$

In the study by Nakasuji and Ohno ([8]), we have the Schur type of duality as follows.

Theorem 4.3 (Schur duality, [8, Theorem 3.5])**.** *For any content parametrized index $\boldsymbol{k} = (k_{i,j})_{(i,j)\in D}$ satisfying certain condition, we can define the dual index of $\boldsymbol{k}$ and write it as $\boldsymbol{k}^\dagger$. Then it holds*

$$\zeta(\boldsymbol{k}) = \zeta(\boldsymbol{k}^\dagger).$$

Remark 4.4 (cf. [5, 8, 9])**.** The algorithm to compute the dual index (see [8]) is very complicated, so that we give an concrete example.

Let $\boldsymbol{k} = \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 2 & 3 \\ \hline 2 & 4 & 2 \\ \hline\end{array}$. We divide $\boldsymbol{k}$ column by column as follows.

$$\begin{array}{|c|}\hline 2 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline\end{array}.$$

The dual of each column can be written as

$$\begin{array}{|c|}\hline 2 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline 2 \\ \hline 2 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline 2 \\ \hline\end{array}.$$

by Theorem 1.4. $\boldsymbol{k}^\dagger$ is obtained by gluing the three Young tableaux together in a suitable way:

$$\boldsymbol{k}^\dagger = \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 1 & 2 \\ \cline{2-3} \multicolumn{1}{c|}{} & 1 & \multicolumn{1}{c}{} \\ \hline 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} 1 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}.$$

Now, we give a new proof of Theorem 4.3 by using Theorem 4.2. We generalize (8) to Schur type.

Lemma 4.5 (Takeyama–Tsuruta, [13])**.** *For any content parametrized index $\boldsymbol{k}$, there exists a positive constant $J > 0$ such that*

$$\frac{\zeta^{q\flat}_{<N}(\boldsymbol{k})}{(1-q)^{\sum_{i,j} k_{i,j}}} = \sum_{\substack{\boldsymbol{n}_p^{(l)} \in \{1,\dots,N-1\}^{J_p} \\ \boldsymbol{n}_p^{(l)} \trianglelefteq \boldsymbol{n}_p^{(l+1)} (1 \le l < k_p) \\ \boldsymbol{n}_p^{(k_p)} \triangleleft \boldsymbol{n}_{p+1}^{(1)} (p_0 \le p < p_1)}} \sum_{\substack{\widetilde{\boldsymbol{m}}_p^{(l)} \in (\mathbb{Z}_{>0})^{J_p} \\ 1 \le l < k_p \\ p_0 \le p < p_1}} \prod_{p=p_0}^{p_1} q^{\widetilde{\boldsymbol{m}}_p^{(1)} \oplus \widetilde{\boldsymbol{n}}_p^{(1)} + \cdots + \widetilde{\boldsymbol{m}}_p^{(k_p)} \oplus \widetilde{\boldsymbol{n}}_p^{(k_p)}}$$
$$+ q^N O(N^J) \text{ as } N \to \infty.$$

Here, for any tuples $\boldsymbol{n} = (n_1, \dots, n_a)$ and $\boldsymbol{m} = (m_1, \dots, m_a)$, we define

$$\widetilde{\boldsymbol{n}} = \begin{cases} \varnothing & (a = 1), \\ (n_2, \dots, n_a) & (a \ge 2). \end{cases}$$

and

$$\widetilde{\boldsymbol{m}} \oplus \widetilde{\boldsymbol{n}} = \begin{cases} \varnothing & (a = 1), \\ m_2 n_2 + \cdots + m_a n_a & (a \ge 2). \end{cases}$$

In what follows, we present a new proof of Theorem 4.3 using resummation (i.e. Theorem 4.2). However, we restrict ourselves to the partial results currently available, rather than treating the general case.

Example of Theorem 4.3. Let $\boldsymbol{k} = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 2 & 2 \\ \hline \end{array}$. The dual index is $\boldsymbol{k}^\dagger = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$. By Lemma 4.5, we have

$$
\begin{aligned}
\frac{\zeta^{q\flat}(\boldsymbol{k})}{(1-q)^9} &= \sum_{\substack{m_2,m_3,m_5,\\ m_7,m_9>0}} \sum_{\substack{n_1 \le n_2 \le n_3 < n_4 \le n_5 \\ \hphantom{n_1 \le n_2 \le{}} \vee\!| \quad \vee \quad \vee\!| \\ n_6 \le n_7 < n_8 \le n_9}} q^{m_2n_2+m_3n_3+m_5n_5+m_7n_7+m_9n_9} \\
&= \sum_{\substack{{}^\forall m_j>0 \\ 0<n_1\le n_2\le n_3<n_4 \\ 1\le n_6\le n_3}} \frac{q^{m_9}}{(1-q^{m_9})^2} \cdot q^{m_2n_2+m_3n_3} \\
&\quad \times \left\{ \frac{\boldsymbol{q^{m_5n_4}}}{1-q^{m_5}} \frac{q^{(m_7+m_9)n_6} - \boldsymbol{q^{(m_7+m_9)n_4}}}{1-q^{m_7+m_9}} - \frac{\boldsymbol{q^{(m_5+m_9)n_4}}}{1-q^{m_5+m_9}} \frac{q^{m_7n_6} - \boldsymbol{q^{m_7n_4}}}{1-q^{m_7}} \right\}.
\end{aligned}
$$

By the symmetry of m_5 and m_7, **boldface terms** vanish taking a limit ${}^\forall m_j \to \infty$. Then we have

$$
\begin{aligned}
\frac{\zeta^{q\flat}(\boldsymbol{k})}{(1-q)^9} &= \cdots \qquad (9) \\
&= \left(\sum_{\substack{m_5 \; m_9 \\ \wedge \;\; \wedge \\ m_3 \; m_7 \\ \wedge \\ m_2}} - \sum_{\substack{m_5 > m_9 \\ \wedge \\ m_3 \quad m_7 \\ \wedge \\ m_2}} \right) \frac{q^{m_2}(1-q)^2}{(1-q^{m_2})^2} \frac{1-q}{1-q^{m_3}} \frac{q^{m_5}(1-q)^2}{(1-q^{m_5})^2} \frac{q^{m_7}(1-q)^2}{(1-q^{m_7})^2} \frac{q^{m_9}(1-q)^2}{(1-q^{m_9})^2}.
\end{aligned}
$$

In (9), it coincides with the *Jacobi–Trudi formula for (q-)MZVs*

$$
\sum_{\substack{m_5 \; m_9 \\ \wedge \;\; \wedge \\ m_3 \; m_7 \\ \wedge \\ m_2}} - \sum_{\substack{m_5 > m_9 \\ \wedge \\ m_3 \quad m_7 \\ \wedge \\ m_2}} = \begin{vmatrix} \displaystyle\sum_{m_5<m_3<m_2} & \displaystyle\sum_{m_9<m_5<m_3<m_2} \\ \displaystyle\sum_{m_7} & \displaystyle\sum_{m_9<m_7} \end{vmatrix} = \zeta^{BZ}\left(\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right).
$$

Thus, we obtain the desired conclusion. □

Remark 4.6 (Conjectual observation)**.** It is known that the same type of calculation as for (9) can be carried out even when $\boldsymbol{k}$ is *ribbon type* or 3×3 *type*. It should be possible to formulate (9) for general (skew-)Young tableaux.

5 Another open problems

In this note, we stated a example proof of Theorem 4.3 by using Theorem 4.2 and resummation only for a special case.

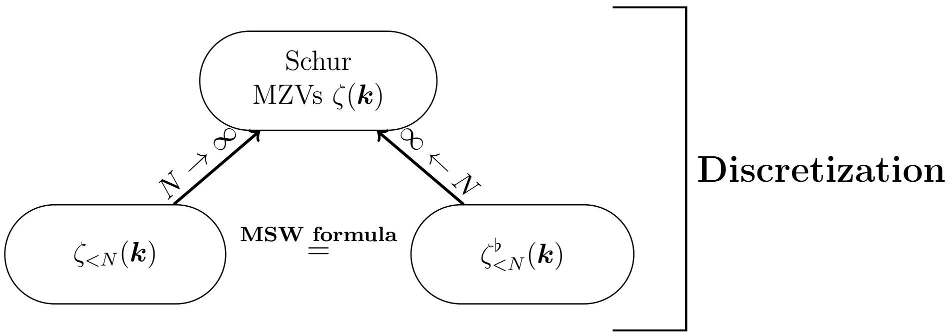
In the study by Nakasuji and Ohno ([8]), They proved Theorem 4.3 in the assumption of Theorem 1.4. On the other hand, our new proof **does not** need the assumption of Theorem 1.4.

Problem 5.1. Give a general proof of Theorem 4.3 by using Theorem 4.2 and resummation.

In Theorem 4.3, almost proofs we currently known rely heavily on combinatorial arguments, whereas in the study of MZVs, algebraic aspects are very often taken into account. Thus, we pose the following problem:

Problem 5.2. Give an algebraic proof of Theorem 4.3.

Discretization for MZVs can be understood via the following diagram:



On the other hand, integral representation for q-MZVs is based on the *Jackson q-integrals*, which completely differ from the Riemann integrals. By explicit computation, Zhao's iterated Jackson q-integrals (cf. [18], Cor.12.2.14) recover $q\flat$-values; however, it is hard to interpret them as arising from objects corresponding to Riemann sums. Thus we pose the following problem:

Problem 5.3. Consider the suitable integral representations for q-MZVs corresponding to Riemann sums (i.e. Theorem 1.1) and satisfying the above diagram.

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