Algebraic relations for ninth variations of Schur functions and their applications

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1 Introduction

Schur functions s_λ and their skew generalizations $s_{\lambda/\mu}$ are fundamental objects in the theory of symmetric functions. They play important roles in various fields of mathematics, ranging from combinatorics to representation theory, integrable systems, and mathematical physics. In 1992, Macdonald [M92] introduced the so-called "ninth variation" of Schur functions defined via the Jacobi-Trudi formula, which includes various variants of Schur functions such as factorial and flagged Schur functions. Subsequently, Nakagawa, Noumi, Shirakawa, and Yamada [NNSY01] studied a class of the ninth variation, denoted by $S^{(r)}_{\lambda/\mu}(X)$, using the Gauss decomposition of the matrix of variables X. More recently, Foley and King [FoKi21] introduced another class, $S^{\mathrm{FK},M}_{\lambda/\mu}(\boldsymbol{w})$, via tableau expressions.

The aim of this note is to establish algebraic (mainly quadratic) relations for $S^{(r)}_{\lambda/\mu}(X)$, based on joint work [TY] with Wataru Takeda. These results yield, as specializations, corresponding relations for $S^{\mathrm{FK},M}_{\lambda/\mu}(\boldsymbol{w})$ and, furthermore, for the "diagonally constant" Schur multiple zeta functions $\zeta^{M}_{\lambda/\mu}(\boldsymbol{a})$ introduced in [NPY18]. In particular, we investigate a generalization of the quadratic relation for s_λ obtained by Kleber [K01] via Plücker relations to the case of $S^{(r)}_{\lambda}(X)$. For detailed proofs of the results presented in this note, see [TY].

2 Schur functions and related variations

2.1 Notations

A partition is a non-increasing sequence $\lambda = (\lambda_1, \dots, \lambda_k)$ of positive integers. The length and weight of λ are defined and denoted by $\ell(\lambda) := k$ and $|\lambda| := \lambda_1 + \cdots + \lambda_k$, respectively. We sometimes write $\lambda = (1^{m_1(\lambda)} 2^{m_2(\lambda)} \cdots)$, where $m_i(\lambda)$ is the multiplicity of i in λ. The Young diagram associated with λ is defined by $D(\lambda) := \{(i,j) \in \mathbb{Z}^2 \,|\, 1 \le i \le k, 1 \le j \le \lambda_i\}$, which is depicted as a collection of square boxes with the i-th row having λ_i boxes. The conjugate partition of λ is denoted by $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_{k'})$, where $\lambda'_i = \#\{j \,|\, \lambda_j \ge i\}$. A skew partition λ/μ is a pair of partitions $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (\mu_1, \dots, \mu_l)$ satisfying $\lambda \supset \mu$, that is, $k \ge l$ and $\lambda_i \ge \mu_i$ for all i. If μ is the empty partition $\varnothing$, we identify λ/μ with λ. We also associate λ/μ with the skew Young diagram $D(\lambda/\mu) := D(\lambda) \setminus D(\mu)$. We say that $(i,j) \in D(\lambda/\mu)$ is a corner of λ/μ if $(i+1, j) \notin D(\lambda/\mu)$ and $(i, j+1) \notin D(\lambda/\mu)$, and denote by $C(\lambda/\mu)$ the set of all corners of λ/μ. A Young tableau of shape λ/μ over a set S is a filling $T = (t_{i,j})_{(i,j)\in D(\lambda/\mu)}$ of the boxes of $D(\lambda/\mu)$ with $t_{i,j} \in S$. We denote by $\mathrm{T}(\lambda/\mu, S)$ the set of all Young tableaux of shape λ/μ over S. In particular, a Young tableau $(t_{i,j}) \in \mathrm{T}(\lambda/\mu, \mathbb{Z}_{>0})$ is called semi-standard if $t_{i,j} \le t_{i,j+1}$ and $t_{i,j} < t_{i+1,j}$ for all i, j. We denote by $\mathrm{SSYT}(\lambda/\mu)$ the set of all semi-standard Young tableaux of shape λ/μ. Moreover, for $M \in \mathbb{N}$, $\mathrm{SSYT}_M(\lambda/\mu)$ denotes the subset of $\mathrm{SSYT}(\lambda/\mu)$ consisting of all $(t_{i,j})$ satisfying $t_{i,j} \in [M] := \{1, 2, \dots, M\}$ for all i, j.

Throughout this note, λ/μ is a skew partition such that λ is contained in the $r \times s$ rectangle (s^r) for some non-negative integers r and s. We set $N = r + s$. For a set S, we denote by $\mathbb{C}[S]$ the ring of polynomials over $\mathbb{C}$ in the indeterminates indexed by the elements of S.

2.2 Schur functions

For variables $\boldsymbol{x} = \{x_k \mid k \in [M]\}$, the skew Schur function is defined by the tableau expression

$$s_{\lambda/\mu} = s_{\lambda/\mu}(\boldsymbol{x}) := \sum_{(t_{ij})\in \mathrm{SSYT}_M(\lambda/\mu)} \prod_{(i,j)\in D(\lambda/\mu)} x_{t_{i,j}} \in \mathbb{C}[\boldsymbol{x}].$$

It is known that $s_{\lambda/\mu}$ satisfies various algebraic relations expressed as determinants. Among them, the following Jacobi-Trudi, dual Jacobi-Trudi, and Giambelli formulas are fundamental.

- Jacobi-Trudi and dual Jacobi-Trudi formulas:

$$s_{\lambda/\mu} = \det \left[h_{\lambda_i-\mu_j-i+j}\right]_{1\le i,j\le \ell(\lambda)},$$
$$s_{\lambda/\mu} = \det \left[e_{\lambda'_i-\mu'_j-i+j}\right]_{1\le i,j\le \ell(\lambda')},$$

  where, $h_d := s_{(d)}$ and $e_d := s_{(1^d)}$ for $d \in \mathbb{Z}_{>0}$ are the complete symmetric functions and the elementary symmetric functions, respectively, with the convention that $h_0 = e_0 = 1$ and $h_d = e_d = 0$ for $d \in \mathbb{Z}_{<0}$.

- Giambelli formula: Let $\lambda = (\alpha_1, \dots, \alpha_p \mid \beta_1, \dots, \beta_p)$ be the Frobenius notation for λ, where $\alpha_i = \lambda_i - i$, $\beta_i = \lambda'_i - i$, and p denotes the length of the main diagonal of $D(\lambda)$. Then,

$$s_\lambda = \det \left[s_{(\alpha_i \mid \beta_j)}\right]_{1\le i,j\le p}.$$

  Here, $(\alpha \mid \beta) := (\alpha + 1, 1^\beta)$ is a partition of hook shape for $\alpha, \beta \in \mathbb{Z}_{\ge 0}$.

For the general theory of the symmetric functions, including the relations above, see [M98].

2.3 Ninth variations of Schur functions defined in [NNSY01]

Let $X = [x_{i,j}]_{1\le i,j\le N}$ be an N-by-N matrix, where each (i, j)-entry $(X)_{i,j} = x_{i,j}$ of X is assumed to be indeterminate. For subsets $I = \{i_1 < \cdots < i_r\}$ and $J = \{j_1 < \cdots < j_r\}$ of $[N]$ with cardinality r, we denote by $\xi^I_J(X) = \xi^{i_1,\dots,i_r}_{j_1,\dots,j_r}(X) := \det[x_{i_k,j_l}]_{1\le k,l\le r}$ the minor determinant corresponding to the row indices I and column indices J. Write the Gauss decomposition of X as

$$X = X_-X_0X_+,$$

where X_-, X_0 and X_+ are lower unitriangular, diagonal and upper unitriangular matrices, respectively, which are uniquely determined as matrices with entries in $\mathbb{C}(X)$, the field of rational functions over $\mathbb{C}$ in the variable $x_{i,j}$ for $1 \le i, j \le N$. It is known that their entries can be expressed in terms of minor determinants as follows:

$$(X_-)_{i,j} = \frac{\xi^{1,\dots,j-1,i}_{1,\dots,j-1,j}(X)}{\xi^{1,\dots,j}_{1,\dots,j}(X)} \qquad (i \ge j), \tag{2.1}$$

$$(X_0)_{i,i} = \frac{\xi^{1,\dots,i}_{1,\dots,i}(X)}{\xi^{1,\dots,i-1}_{1,\dots,i-1}(X)} \qquad (1 \le i \le N), \tag{2.2}$$

$$(X_+)_{i,j} = \frac{\xi^{1,\dots,i-1,i}_{1,\dots,i-1,j}(X)}{\xi^{1,\dots,i}_{1,\dots,i}(X)} \qquad (i \le j). \tag{2.3}$$

In [NNSY01], Nakagawa, Noumi, Shirakawa, and Yamada studied the ninth variation of skew Schur function $S^{(r)}_{\lambda/\mu}(X)$ defined by

$$S^{(r)}_{\lambda/\mu}(X) := \xi^I_J(X_+) \in \mathbb{C}(X), \tag{2.4}$$

where $I = \{i_1 < \dots < i_r\} \subset [N]$ with $i_a = \mu_{r+1-a} + a$ and $J = \{j_1 < \dots < j_r\} \subset [N]$ with $j_a = \lambda_{r+1-a}+a$ are the Maya diagrams of μ and λ, respectively. We remark that $S^{(r)}_{\lambda/\mu}(X)$ reduces to the classical skew Schur function $s_{\lambda/\mu}(x_1, \dots, x_n)$ of n variables when X is the Vandermonde matrix $X = \left[x_i^{j-1}\right]_{1\le i,j\le N}$ with variables $\{x_i\}_{i\in[N]}$, under the specialization $x_{n+1} = \dots = x_N = 0$, provided that r and s are sufficiently large. In what follows, for simplicity, we graphically express $S^{(r+m)}_{\lambda/\mu}(X)$ for $m \in \mathbb{Z}$ by using the Young tableau $\left(\overline{m + c(i,j)}\right)_{(i,j)\in D(\lambda/\mu)}$ of shape λ/μ. Here, $c(i,j) := j - i$ is the content of the box $(i,j) \in D(\lambda/\mu)$, and, for $n \in \mathbb{Z}$, $\overline{n} = n$ if $n \ge 0$ and $-|n|$ otherwise. For example,

$$S^{(r+1)}_{(3,3,2,1)/(1,1)}(X) = \begin{array}{|c|c|c|} \hline & 2 & 3 \\ \hline & 1 & 2 \\ \hline \overline{1} & 0 & \\ \hline \overline{2} & & \\ \hline \end{array}, \qquad S^{(r-2)}_{(4,3,3,2)/(2,1,1)}(X) = \begin{array}{|c|c|c|c|} \hline & & 0 & 1 \\ \hline & \overline{2} & \overline{1} & \\ \hline & \overline{3} & \overline{2} & \\ \hline \overline{5} & \overline{4} & & \\ \hline \end{array}.$$

Using the properties of minor determinants, one can show that $S^{(r)}_{\lambda/\mu}(X)$ satisfies the following Jacobi-Trudi and dual Jacobi-Trudi formulas.

Theorem 2.1 ([NNSY01, Theorem 1.1 and (1.25)])**.**

$$S^{(r)}_{\lambda/\mu}(X) = \det\left[h^{(r+\mu_j-j+1)}_{\lambda_i-\mu_j-i+j}(X)\right]_{1\le i,j\le \ell(\lambda)},$$
$$S^{(r)}_{\lambda/\mu}(X) = \det\left[e^{(r-\mu'_j+j-1)}_{\lambda'_i-\mu'_j-i+j}(X)\right]_{1\le i,j\le \ell(\lambda')},$$

where $h^{(r)}_d(X) := S^{(r)}_{(d)}(X) = \xi^{1,\dots,r}_{1,\dots,r-1,r+d}(X_+)$ and $e^{(r)}_d(X) := S^{(r)}_{(1^d)}(X) = \xi^{1,\dots,r}_{1,\dots,\widehat{r-d+1},\dots,r+1}(X_+)$ for $d \in \mathbb{Z}_{>0}$, with the convention that $h^{(r)}_d(X) = e^{(r)}_d(X) = 1$ and $h^{(r)}_d(X) = e^{(r)}_d(X) = 0$ for $d \in \mathbb{Z}_{<0}$. Here, $\widehat{i}$ indicates that the index i is omitted.

One can also establish a Giambelli formula for $S^{(r)}_{\lambda/\mu}(X)$; the reader is referred to [TY, Theorem 2.3] for the precise statement and proof.

For a specific choice of matrices X, $S^{(r)}_{\lambda/\mu}(X)$ admits a tableau expression, as described below.

Theorem 2.2. ([NNSY01, (2.59) and (2.64)]) For $\boldsymbol{u} = \{u_k^{(t)} \,|\, k \in [M],\, t \in [N-1]\}$, define the upper unitriangular matrx $U_M(\boldsymbol{u})$ of size N by $U_M(\boldsymbol{u}) := U_1 U_2 \cdots U_M$, where

$$U_k = \left(E + u_k^{(1)} E_{1,2}\right)\left(E + u_k^{(2)} E_{2,3}\right)\cdots\left(E + u_k^{(N-1)} E_{N-1,N}\right),$$

with E and $E_{i,j}$ for $1 \le i, j \le N$ being the unit matrix and the matrix unit of size N, respectively. Then, we have

$$S^{(r)}_{\lambda/\mu}(U_M(\boldsymbol{u})) = \sum_{(t_{i,j}) \in \mathrm{SSYT}_M(\lambda/\mu)} \prod_{(i,j) \in D(\lambda/\mu)} u_{t_{i,j}}^{(r+c(i,j))} \in \mathbb{C}[\boldsymbol{u}]. \tag{2.5}$$

2.4 Ninth variations of Schur functions defined in [FoKi21]

In [FoKi21], Foley and King introduced another type of the ninth variation of the skew Schur function, denoted by $S^{\mathrm{FK},M}_{\lambda/\mu}(\boldsymbol{w})$, for variables $\boldsymbol{w} = \{w_{k,c} \,|\, k \in [M],\, c \in \mathbb{Z}\}$:

$$S^{\mathrm{FK},M}_{\lambda/\mu}(\boldsymbol{w}) := \sum_{(t_{ij}) \in \mathrm{SSYT}_M(\lambda/\mu)} \prod_{(i,j) \in D(\lambda/\mu)} w_{t_{i,j},c(i,j)} \in \mathbb{C}[\boldsymbol{w}]. \tag{2.6}$$

This also provides a generalization of the classical Schur function: If $w_{k,c}$ does not depend on c, then $S^{\mathrm{FK},M}_{\lambda/\mu}(\boldsymbol{w}) = s_{\lambda/\mu}(w_1, \ldots, w_M)$, where $w_k = w_{k,c}$. Moreover, from (2.5), it is immediate that $S^{\mathrm{FK},M}_{\lambda/\mu}(\boldsymbol{w})$ can be obtained as a specialization of $S^{(r)}_{\lambda/\mu}(X)$.

Lemma 2.3. For $\boldsymbol{w} = \{w_{k,c} \,|\, k \in [M], c \in \mathbb{Z}\}$, define $\boldsymbol{u} = \{u_k^{(t)} \,|\, k \in [M],\, t \in [N-1]\}$ by $u_k^{(t)} = w_{k,t-r}$. Then, for $m \in \mathbb{Z}$, we have

$$S^{\mathrm{FK},M}_{\lambda/\mu}(\tau^m \boldsymbol{w}) = S^{(r+m)}_{\lambda/\mu}(U_M(\boldsymbol{u})), \tag{2.7}$$

where $\tau^m \boldsymbol{w} := \{w_{k,c}^{(m)} \,|\, k \in [M],\, c \in \mathbb{Z}\}$ with $w_{k,c}^{(m)} = w_{k,m+c}$.

As a corollary of Theorem 2.2, one obtains the following Jacobi-Trudi and dual Jacobi-Trudi formulas for $S^{\mathrm{FK},M}_{\lambda/\mu}(\boldsymbol{w})$, obtained in [FoKi21, Corollary 5.1 and Corollary 5.2].

$$S^{\mathrm{FK},M}_{\lambda/\mu}(\tau^m \boldsymbol{w}) = \det\left[h^{\mathrm{FK},M}_{\lambda_i-\mu_j-i+j}(\tau^{m+\mu_j-j+1}W)\right]_{1\le i,j\le \ell(\lambda)},$$
$$S^{\mathrm{FK},M}_{\lambda/\mu}(\tau^{-m} \boldsymbol{w}) = \det\left[e^{\mathrm{FK},M}_{\lambda'_i-\mu'_j-i+j}(\tau^{m-\mu_j+j-1}W)\right]_{1\le i,j\le \ell(\lambda')},$$

where $h^{\mathrm{FK},M}_d(\boldsymbol{w}) := S^{\mathrm{FK},M}_{(d)}(\boldsymbol{w})$ and $e^{\mathrm{FK},M}_d(\boldsymbol{w}) := S^{\mathrm{FK},M}_{(1^d)}(\boldsymbol{w})$ for $d \in \mathbb{Z}_{>0}$, with the convention that $h^{\mathrm{FK},M}_0(\boldsymbol{w}) = e^{\mathrm{FK},M}_0(\boldsymbol{w}) = 1$ and $h^{\mathrm{FK},M}_d(\boldsymbol{w}) = e^{\mathrm{FK},M}_d(\boldsymbol{w}) = 0$ for $d \in \mathbb{Z}_{<0}$. Moreover, the Giambelli formula for $S^{\mathrm{FK},M}_{\lambda/\mu}(\boldsymbol{w})$ obtained in [FoKi21, Corollary 5.3] can be derived from the corresponding formula for $S^{(r)}_{\lambda/\mu}(X)$ obtained in [TY, Theorem 2.3]. Furthermore, it was shown in [BC19] and [FoKi21] that $S^{\mathrm{FK},M}_{\lambda/\mu}(\boldsymbol{w})$ satisfies the Hamel-Goulden formula [HG95, Theorem 3.1], which simultaneously generalizes the Jacobi–Trudi, dual Jacobi–Trudi, and Giambelli formulas. More precisely, the Hamel-Goulden formula expresses $S^{\mathrm{FK},M}_{\lambda/\mu}(\boldsymbol{w})$ as the determinant of a matrix whose entries are given by $S^{\mathrm{FK},M}_{\nu}(\boldsymbol{w})$, where ν runs over the subribbons of a given ribbon R that decompose λ/μ.

2.5 Schur multiple zeta functions

The skew Schur multiple zeta function, introduced in [NPY18], is a zeta-function analogue of the skew Schur function. For an index $\boldsymbol{s} = (s_{i,j}) \in \mathrm{T}(\lambda/\mu, \mathbb{C})$, it is defined via the tableau expression

$$\zeta^M_{\lambda/\mu}(\boldsymbol{s}) := \sum_{(t_{i,j})\in \mathrm{SSYT}_M(\lambda/\mu)} \prod_{(i,j)\in D(\lambda/\mu)} t_{i,j}^{-s_{i,j}}.$$

As shown in [NPY18, Lemma 2.1], the limit $\zeta_{\lambda/\mu}(\boldsymbol{s}) := \lim_{M\to\infty} \zeta^M_{\lambda/\mu}(\boldsymbol{s})$ converges absolutely for $\boldsymbol{s} \in W_{\lambda/\mu}$, where

$$W_{\lambda/\mu} := \left\{ (s_{ij}) \in \mathrm{T}(\lambda/\mu, \mathbb{C}) \;\middle|\; \begin{array}{l} \mathrm{Re}(s_{i,j}) \geq 1 \text{ for all } (i,j) \in D(\lambda/\mu) \setminus C(\lambda/\mu) \\ \mathrm{Re}(s_{i,j}) > 1 \text{ for all } (i,j) \in C(\lambda/\mu) \end{array} \right\}. \tag{2.8}$$

The Schur multiple zeta function is a simultaneous generalization of both Euler-Zagier type multiple zeta-star function $\zeta^\star(s_1,\ldots,s_d) := \lim_{M\to\infty} \zeta^{\star,M}(s_1,\ldots,s_d)$, and the multiple zeta functions $\zeta(s_1,\ldots,s_d) := \lim_{M\to\infty} \zeta^M(s_1,\ldots,s_d)$, where

$$\zeta^{\star,M}(s_1,\ldots,s_d) := \sum_{1\leq m_1\leq\cdots\leq m_d\leq M} \frac{1}{m_1^{s_1}\cdots m_d^{s_d}}, \quad \zeta^M(s_1,\ldots,s_d) := \sum_{1\leq m_1<\cdots<m_d\leq M} \frac{1}{m_1^{s_1}\cdots m_d^{s_d}},$$

in the sense that

$$\zeta_{(d)}\left(\begin{array}{|c|c|c|}\hline s_1 & \cdots & s_d \\ \hline\end{array} \right) = \zeta^\star(s_1,\ldots,s_d), \quad \zeta_{(1^d)}\left(\begin{array}{|c|}\hline s_1 \\ \hline \vdots \\ \hline s_d \\ \hline\end{array} \right) = \zeta(s_1,\ldots,s_d). \tag{2.9}$$

If $\zeta^M_{\lambda/\mu}(\boldsymbol{s})$ is "diagonally constant", that is, $\boldsymbol{s} \in \mathrm{T}^{\mathrm{diag}}(\lambda/\mu,\mathbb{C})$, where

$$\mathrm{T}^{\mathrm{diag}}(\lambda/\mu,\mathbb{C}) := \{(t_{i,j}) \in \mathrm{T}(\lambda/\mu,\mathbb{C}) \,|\, t_{i,j} = t_{k,l} \text{ if } c(i,j) = c(k,l)\},$$

then, it is easy to see that $\zeta^M_{\lambda/\mu}(\boldsymbol{s})$ can be realized as a specialization of $S^{\mathrm{FK}}_{\lambda/\mu}(W)$ and, therefore, of $S^{(r)}_{\lambda/\mu}(X)$ via Lemma 2.3 as follows.

Lemma 2.4. For $\boldsymbol{a} = (a_c)_{c\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$, let $\boldsymbol{w} = \{w_{k,c} \,|\, k \in [M], c \in \mathbb{Z}\}$ with $w_{k,c} = k^{-a_c}$, and $\boldsymbol{u} = \{u^{(t)}_k \,|\, k \in [M], t \in [N-1]\}$ with $u^{(t)}_k = w_{k,t-r} = k^{-a_{t-r}}$. Define $\boldsymbol{a}|_{\lambda/\mu} := (a_{c(i,j)})_{(i,j)\in D(\lambda/\mu)} \in \mathrm{T}^{\mathrm{diag}}(\lambda/\mu,\mathbb{C})$. Then, for $m \in \mathbb{Z}$, we have

$$\zeta^M_{\lambda/\mu}((\tau^m\boldsymbol{a})|_{\lambda/\mu}) = S^{\mathrm{FK}}_{\lambda/\mu}(\tau^m\boldsymbol{w}) = S^{(r+m)}_{\lambda/\mu}(U_M(\boldsymbol{u})), \tag{2.10}$$

where $\tau^m\boldsymbol{a} := (a^{(m)}_c)_{c\in\mathbb{Z}}$ with $a^{(m)}_c = a_{c+m}$, and $U_M(\boldsymbol{u})$ was defined in Theorem 2.2.

For simplicity, we henceforth denote $\zeta^M_{\lambda/\mu}(\boldsymbol{a}|_{\lambda/\mu})$ by $\zeta^M_{\lambda/\mu}(\boldsymbol{a})$. Then, the above specializations are summarized as follows.

$$\begin{aligned}
S^{(r+m)}_{\lambda/\mu}(U_M(\boldsymbol{u})) &= \sum_{(t_{i,j})\in\mathrm{SSYT}_M(\lambda/\mu)}\ \prod_{(i,j)\in D(\lambda/\mu)} u^{(r+c(i,j))}_{t_{i,j}} \\
&\quad \Big\downarrow\, u^{(t)}_k = w_{k,t-r} \\
S^{\mathrm{FK},M}_{\lambda/\mu}(\tau^m\boldsymbol{w}) &= \sum_{(t_{i,j})\in\mathrm{SSYT}_M(\lambda/\mu)}\ \prod_{(i,j)\in D(\lambda/\mu)} w_{t_{i,j},c(i,j)} \\
&\quad \Big\downarrow\, w_{k,c} = k^{-a_c} \\
\zeta^{M}_{\lambda/\mu}(\tau^m\boldsymbol{a}) &= \sum_{(t_{i,j})\in\mathrm{SSYT}_M(\lambda/\mu)}\ \prod_{(i,j)\in D(\lambda/\mu)} t_{i,j}^{-a_{c(i,j)}}
\end{aligned}$$

Based on these relations, we shall investigate algebraic relations for $S^{(r)}_{\lambda/\mu}(X)$, which induce corresponding relations for $S^{\mathrm{FK},M}_{\lambda/\mu}(W)$ and $\zeta^{M}_{\lambda/\mu}(\boldsymbol{a})$.

3 Quadratic relations obtained from the Plücker relations

In this section, we derive quadratic relations for $S^{(r)}_{\lambda}(X)$ by applying the Plücker relations for determinants. In particular, we generalize the relations for Schur functions, obtained by Kleber [K01]. First, let us recall the Plücker relations.

Theorem 3.1. Let Z be a $2n$-by-n matrix whose rows are indexed by $1,\dots,n,1',\dots,n'$ and columns by $1,\dots,n$. For $1\le \ell\le n$, fix a set of row indices $\{t'_1,\dots,t'_\ell\}\subset\{1',\dots,n'\}$ with $t'_1<\cdots<t'_\ell$. Then, the Plücker relations fixing the rows $\{1',\dots,n'\}\setminus\{t'_1,\dots,t'_\ell\}$ is given by

$$\xi^{1,\dots,n}_{1,\dots,n}(Z)\xi^{1',\dots,n'}_{1,\dots,n}(Z) = \sum_{1\le s_1<\cdots<s_\ell\le n} \xi^{1,\dots,\overset{s_1}{t'_1},\dots,\overset{s_\ell}{t'_\ell},\dots,n}_{1,\dots,n}(Z)\,\xi^{1',\dots,\overset{t'_1}{s_1},\dots,\overset{t'_\ell}{s_\ell},\dots,n'}_{1,\dots,n}(Z). \tag{3.1}$$

Example 3.2. When $n=3$, $\ell=2$ and $(t'_1,t'_2)=(1',3')$, we have

$$\xi^{1,2,3}_{1,2,3}(Z)\xi^{1',2',3'}_{1,2,3}(Z) = \xi^{1',3',3}_{1,2,3}(Z)\xi^{1,2',2}_{1,2,3}(Z) + \xi^{1',2,3'}_{1,2,3}(Z)\xi^{1,2',3}_{1,2,3}(Z) + \xi^{1,1',3'}_{1,2,3}(Z)\xi^{2,2',3}_{1,2,3}(Z).$$

To state our results, we introduce some combinatorial terminology of Young diagrams. For a partition λ, the outer border of λ is the strip whose cells contain all the cells not in $D(\lambda)$ but immediately below and to the right of those in $D(\lambda)$. On the other hand, the inner border of λ is the strip whose cells contain all the right-most or the bottom-most cells in $D(\lambda)$. We denote the outer and inner borders of λ by $\mathrm{OB}(\lambda)$ and $\mathrm{IB}(\lambda)$, respectively. For $u,v\in\mathrm{OB}(\lambda)$, let $\mathrm{add}^u_v(\lambda)$ be the partition obtained by adding the substrip of $\mathrm{OB}(\lambda)$ starting from u and ending at v to $D(\lambda)$. Similarly, for $u,v\in\mathrm{IB}(\lambda)$, let $\mathrm{rem}^u_v(\lambda)$ be the partition obtained by removing the substrip of $\mathrm{IB}(\lambda)$ from u to v from $D(\lambda)$. In both cases, we provide that the resulting diagram remains a Young diagram. Assume that λ has n corners. Then, it can be written as $\lambda=(m_1^{r_1}m_2^{r_2-r_1}\cdots m_n^{r_n-r_{n-1}})$ with $m_1>m_2>\cdots>m_n>m_{n+1}=0$ and $0<r_1<r_2<\cdots<r_n$. This implies that $C(\lambda)=\{(r_1,m_1),\dots,(r_n,m_n)\}$. For $1\le p\le q\le n$, put

$$\mathrm{add}^p_q := \mathrm{add}^{(r_p+1,m_p)}_{(r_q+1,m_{q+1}+1)}, \quad \mathrm{rem}^p_q := \mathrm{rem}^{(r_p,m_p)}_{(r_q,m_{q+1}+1)}\,.$$

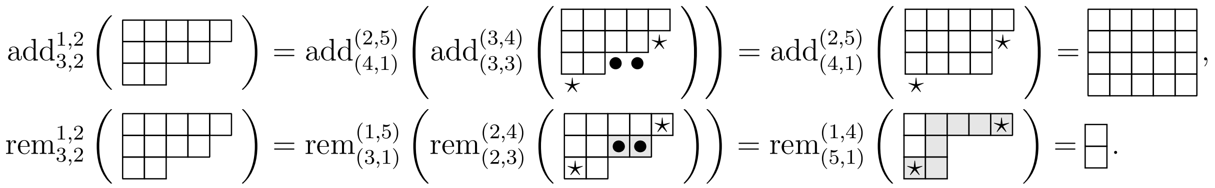
Moreover, for $1\le p_1<\cdots<p_t\le q_t<\cdots<q_1\le n$, we define the iterated operators:

$$\mathrm{add}^{p_1,\dots,p_t}_{q_1,\dots,q_t} = \mathrm{add}^{p_1}_{q_1}\circ\cdots\circ\mathrm{add}^{p_t}_{q_t}, \quad \mathrm{rem}^{p_1,\dots,p_t}_{q_1,\dots,q_t} = \mathrm{rem}^{p_1}_{q_1}\circ\cdots\circ\mathrm{rem}^{p_t}_{q_t}\,. \tag{3.2}$$

Example 3.3. Let $\lambda = (5,4,2)$. Then, we have $C(\lambda) = \{(1,5),(2,4),(3,2)\}$ and

$$\begin{aligned}\mathrm{OB}(\lambda) &= \{(1,6),(2,6),(2,5),(3,5),(3,4),(3,3),(4,3),(4,2),(4,1)\},\\ \mathrm{IB}(\lambda) &= \{(1,5),(1,4),(2,4),(2,3),(2,2),(3,2),(3,1)\}.\end{aligned}$$

For the adding and removing operators, we have, for example,



In [K01], Kleber obtained the following quadratic relations for s_λ.

Theorem 3.4 ([K01, Theorem 4.2])**.** Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition with n corners. Take $1 \le d \le n$ and denote by ℓ the height of the d-th shortest column of λ. Then, we have

$$s_\lambda s_\lambda = s_{\lambda-(1^\ell)} s_{\lambda+(1^\ell)} + \sum_{t=1}^{\min\{d,n-d+1\}} (-1)^{t-1} \sum_{\substack{1\le p_1<\cdots<p_t\le d\\ d\le q_t<\cdots<q_1\le n}} s_{\mathrm{add}^{p_1,\dots,p_t}_{q_1,\dots,q_t}(\lambda)} s_{\mathrm{rem}^{p_1,\dots,p_t}_{q_1,\dots,q_t}(\lambda)}, \tag{3.3}$$

where $\lambda \pm (1^\ell) = (\lambda_1 \pm 1, \dots, \lambda_\ell \pm 1, \lambda_{\ell+1}, \dots, \lambda_k)$.

The following theorem extends these relations to the ninth variation $S^{(r)}_\lambda(X)$.

Theorem 3.5. Let λ be a partition having n corners. Take $1 \le d \le n$.

(1) Denote the d-th shortest column height of λ as ℓ. Then, we have

$$\begin{aligned} S^{(r)}_\lambda(X) S^{(r-1)}_\lambda(X) &= S^{(r)}_{\lambda-(1^\ell)}(X) S^{(r-1)}_{\lambda+(1^\ell)}(X)\\ &+ \sum_{t=1}^{\min\{d,n-d+1\}} (-1)^{t-1} \sum_{\substack{1\le p_1<\cdots<p_t\le d\\ d\le q_t<\cdots<q_1\le n}} S^{(r)}_{\mathrm{add}^{p_1,\dots,p_t}_{q_1,\dots,q_t}(\lambda)}(X) S^{(r-1)}_{\mathrm{rem}^{p_1,\dots,p_t}_{q_1,\dots,q_t}(\lambda)}(X). \end{aligned} \tag{3.4}$$

(2) Denote the d-th shortest row length of λ as ℓ. Then, we have

$$\begin{aligned} S^{(r)}_\lambda(X) S^{(r+1)}_\lambda(X) &= S^{(r)}_{(\lambda'-(1^\ell))'}(X) S^{(r+1)}_{(\lambda'+(1^\ell))'}(X)\\ &+ \sum_{t=1}^{\min\{d,n-d+1\}} (-1)^{t-1} \sum_{\substack{1\le p_1<\cdots<p_t\le d\\ d\le q_t<\cdots<q_1\le n}} S^{(r)}_{(\mathrm{add}^{p_1,\dots,p_t}_{q_1,\dots,q_t}(\lambda'))'}(X) S^{(r+1)}_{(\mathrm{rem}^{p_1,\dots,p_t}_{q_1,\dots,q_t}(\lambda'))'}(X). \end{aligned} \tag{3.5}$$

By applying the specialization (2.10) to this theorem, we obtain the following quadratic relations for the diagonally constant Schur multiple zeta functions $\zeta^M_\lambda(\boldsymbol{a})$.

Corollary 3.6. Let $\boldsymbol{a} = (a_c)_{c\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$. Let λ be a partition having n corners. Take $1 \le d \le n$.

(1) Denote the d-th shortest column height of λ as ℓ. Then, we have

$$\begin{aligned}\zeta^M_\lambda(\boldsymbol{a})\zeta^M_\lambda(\tau^{-1}\boldsymbol{a}) &= \zeta^M_{\lambda-(1^\ell)}(\boldsymbol{a})\zeta^M_{\lambda+(1^\ell)}(\tau^{-1}\boldsymbol{a}) \\ &+ \sum_{t=1}^{\min\{d,n-d+1\}}(-1)^{t-1}\sum_{\substack{1\le p_1<\cdots<p_t\le d\\ d\le q_t<\cdots<q_1\le n}}\zeta^M_{\mathrm{add}^{p_1,\ldots,p_t}_{q_1,\ldots,q_t}(\lambda)}(\boldsymbol{a})\zeta^M_{\mathrm{rem}^{p_1,\ldots,p_t}_{q_1,\ldots,q_t}(\lambda)}(\tau^{-1}\boldsymbol{a}).\end{aligned} \tag{3.6}$$

(2) Denote the d-th shortest row length of λ as ℓ. Then, we have

$$\begin{aligned}\zeta^M_\lambda(\boldsymbol{a})\zeta^M_\lambda(\tau\boldsymbol{a}) &= \zeta^M_{(\lambda'-(1^\ell))'}(\boldsymbol{a})\zeta^M_{(\lambda'+(1^\ell))'}(\tau\boldsymbol{a}) \\ &+ \sum_{t=1}^{\min\{d,n-d+1\}}(-1)^{t-1}\sum_{\substack{1\le p_1<\cdots<p_t\le d\\ d\le q_t<\cdots<q_1\le n}}\zeta^M_{(\mathrm{add}^{p_1,\ldots,p_t}_{q_1,\ldots,q_t}(\lambda'))'}(\boldsymbol{a})\zeta^M_{(\mathrm{rem}^{p_1,\ldots,p_t}_{q_1,\ldots,q_t}(\lambda'))'}(\tau\boldsymbol{a}).\end{aligned} \tag{3.7}$$

Example 3.7. When $\lambda = (3, 2^2, 1)$ and $d = 2$, we have

$$\begin{aligned}&\begin{ytableau} 0 & 1 & 2 \\ \bar{1} & 0 \\ \bar{2} & \bar{1} \\ \bar{3} \end{ytableau}\cdot\begin{ytableau} \bar{1} & 0 & 1 \\ \bar{2} & \bar{1} \\ \bar{3} & \bar{2} \\ \bar{4} \end{ytableau} \overset{(3.4)}{=} \begin{ytableau} 0 & 1 \\ \bar{1} \\ \bar{2} \\ \bar{3} \end{ytableau}\cdot\begin{ytableau} \bar{1} & 0 & 1 & 2 \\ \bar{2} & \bar{1} & 0 \\ \bar{3} & \bar{2} & \bar{1} \\ \bar{4} \end{ytableau} + \begin{ytableau} 0 & 1 & 2 \\ \bar{1} & 0 & 1 \\ \bar{2} & \bar{1} & 0 \\ \bar{3} & \bar{2} & \bar{1} \end{ytableau}\cdot\begin{ytableau} \bar{1} \\ \bar{2} \\ \bar{3} \\ \bar{4} \end{ytableau} + \begin{ytableau} 0 & 1 & 2 \\ \bar{1} & 0 & 1 \\ \bar{2} & \bar{1} & 0 \\ \bar{3} & \bar{2} & \bar{1} \\ \bar{4} & \bar{3} \end{ytableau}\cdot\begin{ytableau} \bar{1} \\ \bar{2} \end{ytableau} \\ &+ \begin{ytableau} 0 & 1 & 2 \\ \bar{1} & 0 \\ \bar{2} & \bar{1} \\ \bar{3} & \bar{2} \end{ytableau}\cdot\begin{ytableau} \bar{1} & 0 & 1 \\ \bar{2} & \bar{1} \\ \bar{3} \\ \bar{4} \end{ytableau} + \begin{ytableau} 0 & 1 & 2 \\ \bar{1} & 0 \\ \bar{2} & \bar{1} \\ \bar{3} & \bar{2} \\ \bar{4} & \bar{3} \end{ytableau}\cdot\begin{ytableau} \bar{1} & 0 & 1 \\ \bar{2} & \bar{1} \end{ytableau} - \begin{ytableau} 0 & 1 & 2 \\ \bar{1} & 0 & 1 \\ \bar{2} & \bar{1} & 0 \\ \bar{3} & \bar{2} & \bar{1} \\ \bar{4} & \bar{3} & \bar{2} \end{ytableau}\cdot\begin{ytableau} \bar{1} \end{ytableau},\end{aligned}$$

$$\begin{aligned}&\begin{ytableau} 0 & 1 & 2 \\ \bar{1} & 0 \\ \bar{2} & \bar{1} \\ \bar{3} \end{ytableau}\cdot\begin{ytableau} 1 & 2 & 3 \\ 0 & 1 \\ \bar{1} & 0 \\ \bar{2} \end{ytableau} \overset{(3.5)}{=} \begin{ytableau} 0 & 1 & 2 \\ \bar{1} & 0 \\ \bar{2} \end{ytableau}\cdot\begin{ytableau} 1 & 2 & 3 \\ 0 & 1 \\ \bar{1} & 0 \\ \bar{2} & \bar{1} \\ \bar{3} \end{ytableau} + \begin{ytableau} 0 & 1 & 2 \\ \bar{1} & 0 & 1 \\ \bar{2} & \bar{1} & 0 \\ \bar{3} & \bar{2} & \bar{1} \end{ytableau}\cdot\begin{ytableau} 1 & 2 & 3 \\ 0 \end{ytableau} + \begin{ytableau} 0 & 1 & 2 & 3 \\ \bar{1} & 0 & 1 & 2 \\ \bar{2} & \bar{1} & 0 \\ \bar{3} & \bar{2} & \bar{1} \end{ytableau}\cdot\begin{ytableau} 1 \\ 0 \end{ytableau} \\ &+ \begin{ytableau} 0 & 1 & 2 \\ \bar{1} & 0 & 1 \\ \bar{2} & \bar{1} & 0 \\ \bar{3} \end{ytableau}\cdot\begin{ytableau} 1 & 2 & 3 \\ 0 \\ \bar{1} \\ \bar{2} \end{ytableau} + \begin{ytableau} 0 & 1 & 2 & 3 \\ \bar{1} & 0 & 1 & 2 \\ \bar{2} & \bar{1} & 0 \\ \bar{3} \end{ytableau}\cdot\begin{ytableau} 1 \\ 0 \\ \bar{1} \\ \bar{2} \end{ytableau} - \begin{ytableau} 0 & 1 & 2 & 3 \\ \bar{1} & 0 & 1 & 2 \\ \bar{2} & \bar{1} & 0 & 1 \\ \bar{3} & \bar{2} & \bar{1} & 0 \end{ytableau}.\end{aligned}$$

The proof of Theorem 3.5 follows by mimicking the argument for Theorem 3.4, in conjunction with the Plücker relations (3.1). Specifically, (3.4) is obtained by considering the $(2k+2)$-by-$(k+1)$ matrix Z of the following form:

$$Z = \begin{array}{c} \\ L \\ R \\ 1 \\ \vdots \\ k \\ 1' \\ \vdots \\ k' \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 1 & 2 & \cdots & k & k+1 \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & \cdots & 0 & 0 \\ \hline 0 & 0 & \cdots & 0 & (-1)^k \\ \hline & & & & * \\ & & H^{(r)}_{\lambda-(1^\ell)}(X) & & \vdots \\ & & & & * \\ \hline * & & & & \\ \vdots & & H^{(r-1)}_{\lambda+(1^\ell)}(X) & & \\ * & & & & \end{array}\right] \end{array},$$

where

$$H_\lambda^{(r)}(X) := \left[h_{\lambda_i - i + j}^{(r-j+1)}(X) \right]_{1 \le i,j \le k}$$

is the matrix appearing in the Jacobi-Trudi formula (Theorem 2.1). Here, the entries denoted by $*$ are naturally determined by the indices of the corresponding rows.

Furthermore, by modifying the argument proof, one obtains the following quadratic relations for $S_\lambda^{(r)}(X)$ when λ is a rectangle $[\,p\,|\,q\,] := (q^p)$. To state the result, we employ the notation $[\,p\,|\,q\,]_k^l := ((q+1)^l, q^{p-l}, k)$, $[\,p\,|\,q\,]^l := [\,p\,|\,q\,]_0^l$ and $[\,p\,|\,q\,]_k := [\,p\,|\,q\,]_k^0$ for $k, l \ge 0$.

Theorem 3.8. For $p, q \ge 1$ and $a, b \ge 0$ satisfying $a \le q$ and $a + b \le p + 1$, we have

$$\begin{aligned}
&(-1)^{a+b} S^{(r)}_{[\,p+1\,|\,q+b-1\,]}(X) S^{(r-1)}_{[\,p-1\,|\,q-a\,]^{p-a-b+1}}(X) \\
&= S^{(r)}_{[\,p\,|\,q+b-1\,]}(X) S^{(r-1)}_{[\,p\,|\,q-a+1\,]}(X) - S^{(r)}_{[\,p\,|\,q-a\,]}(X) S^{(r-1)}_{[\,p\,|\,q+b\,]}(X) \\
&\quad + \sum_{t=0}^{a+b-3} (-1)^{t-1} S^{(r)}_{[\,p\,|\,q+b-1\,]_{q-a+t+1}}(X) S^{(r-1)}_{[\,p-1\,|\,q-a\,]^{p-t-1}}(X).
\end{aligned} \tag{3.8}$$

Example 3.9. When $p = q = 3$, $(a, b) = (1, 1)$ and $(a, b) = (1, 2)$, we have

$$\begin{array}{|c|c|c|}\hline 0&1&2\\\hline \bar{1}&0&1\\\hline \bar{2}&\bar{1}&0\\\hline \bar{3}&\bar{2}&\bar{1}\\\hline\end{array} \cdot \begin{array}{|c|c|c|}\hline \bar{1}&0&1\\\hline \bar{2}&\bar{1}&0\\\hline\end{array} = \begin{array}{|c|c|c|}\hline 0&1&2\\\hline \bar{1}&0&1\\\hline \bar{2}&\bar{1}&0\\\hline\end{array} \cdot \begin{array}{|c|c|c|}\hline \bar{1}&0&1\\\hline \bar{2}&\bar{1}&0\\\hline \bar{3}&\bar{2}&\bar{1}\\\hline\end{array} - \begin{array}{|c|c|}\hline 0&1\\\hline \bar{1}&0\\\hline \bar{2}&\bar{1}\\\hline\end{array} \cdot \begin{array}{|c|c|c|c|c|}\hline \bar{1}&0&1&2&3\\\hline \bar{2}&\bar{1}&0&1&2\\\hline \bar{3}&\bar{2}&\bar{1}&0&1\\\hline\end{array},$$

$$-\begin{array}{|c|c|c|c|}\hline 0&1&2&3\\\hline \bar{1}&0&1&2\\\hline \bar{2}&\bar{1}&0&1\\\hline \bar{3}&\bar{2}&\bar{1}&0\\\hline\end{array} \cdot \begin{array}{|c|c|c}\hline \bar{1}&0&\multicolumn{1}{c|}{1}\\\hline \bar{2}&\bar{1}\\\cline{1-2}\end{array} = \begin{array}{|c|c|c|c|}\hline 0&1&2&3\\\hline \bar{1}&0&1&2\\\hline \bar{2}&\bar{1}&0&1\\\hline\end{array} \cdot \begin{array}{|c|c|c|}\hline \bar{1}&0&1\\\hline \bar{2}&\bar{1}&0\\\hline \bar{3}&\bar{2}&\bar{1}\\\hline\end{array} - \begin{array}{|c|c|}\hline 0&1\\\hline \bar{1}&0\\\hline \bar{2}&\bar{1}\\\hline\end{array} \cdot \begin{array}{|c|c|c|c|c|}\hline \bar{1}&0&1&2&3\\\hline \bar{2}&\bar{1}&0&1&2\\\hline \bar{3}&\bar{2}&\bar{1}&0&1\\\hline\end{array} - \begin{array}{|c|c|c|c}\hline 0&1&2&\multicolumn{1}{c|}{3}\\\hline \bar{1}&0&1&\multicolumn{1}{c|}{2}\\\hline \bar{2}&\bar{1}&0&\multicolumn{1}{c|}{1}\\\hline \bar{3}&\bar{2}&\bar{1}\\\cline{1-3}\end{array} \cdot \begin{array}{|c|c|c|}\hline \bar{1}&0&1\\\hline \bar{2}&\bar{1}&0\\\hline\end{array}.$$

Notice that the first equation can be also obtained by applying the Desnanot-Jacobi adjoint matrix theorem to the Jacobi-Trudi matrix $H_\lambda^{(r)}(X)$.

Acknowledgement

The author would like to thank Professors Yasuo Ohno and Shin-ichiro Seki for giving him the opportunity to talk at the RIMS conference “Various Aspects of Multiple Zeta Values”. This work was supported by JSPS KAKENHI Grant Number JP21K03206.

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