THE DOUBLE SHUFFLE LIE ALGEBRA OF CONGRUENT MZVS

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ABSTRACT. We focus on the double shuffle relations among congruent multiple zeta values as studied by Yuan and Zhao. Their relation to cyclotomic multiple zeta values enables us to construct an isomorphism over the cyclotomic field between their underlying algebraic frameworks. This enables the construction of a double shuffle Lie algebra for congruent multiple zeta values in the same fashion as Racinet's double shuffle Lie algebra for cyclotomic multiple zeta values.

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1. INTRODUCTION

Multiple zeta values (MZVs in short) are real numbers defined by the following series

$$\zeta(k_1, \dots, k_r) = \sum_{n_1 > \cdots > n_r > 0} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}},$$

where $r, k_1, \dots, k_r$ are positive integers with $k_1 \neq 1$ ([13]). These values appear in the studies of many subjects in mathematics and physics such as mixed Tate motives, knot invariant, the KZ-equation and Feynman integrals to name a few (for example, the reader may refer to [2, 4, 14] for a comprehensive presentation).

For a positive integer N, there ara mainly two level N generalizations of MZVs. The first generalization (cf. [7]), called *multiple polylogarithm values at* N^{th} *roots of unity* (N-MPVs in short), are complex numbers defined by the following series

$$\mathrm{Li}_{(k_1,\dots,k_r)}(z_1, \dots, z_r) = \sum_{n_1 > \cdots > n_r > 0} \frac{z_1^{n_1} \cdots z_r^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}}, \tag{1.1}$$

where $r, k_1, \dots, k_r$ are positive integers and $z_1, \dots, z_r$ are N^{th} roots of unity with $(k_1, z_1) \neq (1, 1)$. They are also called multiple L-values in [1], cyclotomic multiple zeta values in [11] or the colored multiple zeta values in [3]. The second generalization, called N*-congruent multiple zeta values*

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(N-CMZVs in short) are real numbers defined by the following series

$$\zeta^{\mathrm{mod}\,N}_{(\alpha_1,\dots,\alpha_r)}(k_1,\dots,k_r)=\sum_{\substack{n_1>\cdots>n_r>0\\ n_j\in\alpha_j\ 1\le j\le r}}\frac{1}{n_1^{k_1}\cdots n_r^{k_r}}, \tag{1.2}$$

where $r,k_1,\dots,k_r\in\mathbb{Z}_{>0}$ with $k_1\neq 1$ and $\alpha_1,\dots,\alpha_r\in\mathbb{Z}/N\mathbb{Z}$. They are also refered to as the *level N multiple zeta value* in [12]. Denote by $\iota:[\![1,N]\!]\to\mathbb{Z}/N\mathbb{Z}$ the substitution that identifies each element of $[\![1,N]\!]$ with its equivalence class. It has been established in [12, (8)] that N-CMZVs (1.2) are related to N-MPVs (1.1) by the following formula
(1.3)

$$\zeta^{\mathrm{mod}\,N}_{(\alpha_1,\dots,\alpha_r)}(k_1,\dots,k_r)=\frac{1}{N^r}\sum_{m_1=1}^{N}\cdots\sum_{m_r=1}^{N}e^{\frac{-2(m_1\iota^{-1}(\alpha_1)+\cdots+m_r\iota^{-1}(\alpha_r))\pi\mathrm{i}}{N}}\mathrm{Li}_{(k_1,\dots,k_r)}(e^{\frac{2m_1\pi\mathrm{i}}{N}},\dots,e^{\frac{2m_r\pi\mathrm{i}}{N}}).$$

The values (1.1) can be expressed as an iterated integral as follows (see for example [7])

$$\mathrm{Li}_{(k_1,\dots,k_r)}(z_1,\dots,z_r)=\int_0^1\Omega_0^{k_1-1}\Omega_{z_1}\Omega_0^{k_2-1}\Omega_{z_1z_2}\cdots\Omega_0^{k_r-1}\Omega_{z_1\cdots z_r}, \tag{1.4}$$

where $\Omega_0=\frac{dt}{t}$ and $\Omega_z=\frac{dt}{z^{-1}-t}$, for any root of unity z. Thanks to formula (1.3) and the iterated integral expression (1.4), one deduces the following identity (see [12, p.185])

$$\zeta^{\mathrm{mod}\,N}_{(\alpha_1,\dots,\alpha_r)}(k_1,\dots,k_r)=\frac{1}{N^r}\int_0^1\omega^{k_1-1}\omega_{\alpha_1-\alpha_2}\cdots\omega^{k_{r-1}-1}\omega_{\alpha_{r-1}-\alpha_r}\omega^{k_r-1}\omega_{\alpha_r},$$

where $\omega=\frac{dt}{t}$ and $\omega_\alpha=\frac{Nt^{\iota^{-1}(\alpha)-1}dt}{1-t^N}$ for any $\alpha\in\mathbb{Z}/N\mathbb{Z}$. One may also check that [12, p.185]

$$\omega=\Omega_0 \text{ and } \omega_\alpha=\sum_{m=1}^{N}\zeta_N^{-m\iota^{-1}(\alpha)}\Omega_{e^{\frac{2m\pi\mathrm{i}}{N}}}. \tag{1.5}$$

Using iterated sums and iterated integral expressions, one deduces linear and algebraic relations between N-MPVs called the (regularized) *double shuffle* relations. To describe these relations Hoffmann [8] introduced a word algebra setting whose letters are free noncommutative variables assimilated to the differential 1-forms Ω_0 and Ω_z ($z\in\mu_N$). This setting has been utilized by Racinet [10] to provide an algebraic frameworks for these relations using noncommutative formal power series bialgebras. The fact that N-MPVs are related to N-CMZV by identity (1.3) suggests that N-CMZVs also satisfy (regularized) double shuffle relations. In fact, this has been introduced by Yuan-Zhao [12] and written down precisely by Kanno in [9] thanks to the word algebra setting whose letters are the free noncommutative variables associated to the differential 1-forms ω and ω_α ($\alpha\in\mathbb{Z}/N\mathbb{Z}$).

In this note, we utilize identities (1.5) to construct an algebra isomorphism between the free noncommutative series algebras with coefficients in the cyclotomic field $\mathbb{Q}(\mu_N)$ associated to each set of differential 1-forms (see Proposition 4.1). This enables us to introduce a dual equivalent of the formalism of [9], which facilitates the formulation of double shuffle relations among N-CMZVs within the formalism of [10]. We then show that this algebra isomorphism is in fact a bialgebra isomorphism with respect to the newly introduced coproducts and the coproducts of [10], thus establishing the relations between both formalisms. This enables the definition of a $\mathbb{Q}$-Lie algebra $\mathfrak{dmr}_0^{[N]}$ related to the (regularized) double shuffle relations among N-CMZVs and establish the following result:

Theorem 1.1 (Theorem 5.8)**.** *There is a $\mathbb{Q}(\mu_N)$-Lie algebra isomorphism*

$$\mathbb{Q}(\mu_N)\otimes_{\mathbb{Q}}\mathfrak{dmr}_0^{[N]}\xrightarrow{\simeq}\mathbb{Q}(\mu_N)\otimes_{\mathbb{Q}}\mathfrak{dmr}_0^{\mu_N}.$$

Notation. Throughout this note, let $N \geq 3$ be a positive integer and μ_N be the group of complex roots of unity. Denote by $[\![1, N]\!] := \{1, \ldots, N\}$ and by $\iota : [\![1, N]\!] \to \mathbb{Z}/N\mathbb{Z}$ the substitution that identifies each element of $[\![1, N]\!]$ with its equivalence class.

2. The cyclotomic double shuffle $\mathbb{Q}$-Lie algebra $\mathfrak{dmr}_0^{\mu_N}$

Let $\mathbb{Q}\langle\langle X\rangle\rangle$ be the free noncommutative series $\mathbb{Q}$-algebra with unit over the alphabet $X := \{x_0, x_z \mid z \in \mu_N\}$. Let $\mathbb{Q}\langle\langle Y\rangle\rangle$ be the free noncommutative series $\mathbb{Q}$-algebra with unit over the alphabet $Y := \{y_{k,z} \mid (k, z) \in \mathbb{Z}_{>0} \times \mu_N\}$.
The $\mathbb{Q}$-algebra morphism

$$\mathbb{Q}\langle\langle Y\rangle\rangle \to \mathbb{Q}\langle\langle X\rangle\rangle, \quad y_{k,z} \mapsto x_0^{k-1}x_z, \text{ for } (k, z) \in \mathbb{Z}_{>0} \times \mu_N \tag{2.1}$$

is injective. Therefore, throughout this note, we will treat $\mathbb{Q}\langle\langle Y\rangle\rangle$ as a subalgebra of $\mathbb{Q}\langle\langle X\rangle\rangle$ by identifiying it to its image by this morphism.
Recall from [10, §2.2.5] the direct sum decomposition (of $\mathbb{Q}$-linear subspaces)

$$\mathbb{Q}\langle\langle X\rangle\rangle = \mathbb{Q}\langle\langle Y\rangle\rangle \oplus \mathbb{Q}\langle\langle X\rangle\rangle x_0.$$

Let then $\pi_Y : \mathbb{Q}\langle\langle X\rangle\rangle = \mathbb{Q}\langle\langle Y\rangle\rangle \oplus \mathbb{Q}\langle\langle X\rangle\rangle x_0 \twoheadrightarrow \mathbb{Q}\langle\langle Y\rangle\rangle$ be the projection from $\mathbb{Q}\langle\langle X\rangle\rangle$ to $\mathbb{Q}\langle\langle Y\rangle\rangle$, that is, the surjective $\mathbb{Q}$-linear map such that it is the identity on $\mathbb{Q}\langle\langle Y\rangle\rangle$ and maps any element of $\mathbb{Q}\langle\langle X\rangle\rangle x_0$ to 0.

In [10], Racinet introduces bialgebra structures on the algebras $\mathbb{Q}\langle\langle X\rangle\rangle$ and $\mathbb{Q}\langle\langle Y\rangle\rangle$. Namely, the $\mathbb{Q}$-algebra $\mathbb{Q}\langle\langle X\rangle\rangle$ is equipped with a bialgebra structure with respect to the *shuffle coproduct*, which is the algebra morphism $\widehat{\Delta}_ш : \mathbb{Q}\langle\langle X\rangle\rangle \to \mathbb{Q}\langle\langle X\rangle\rangle^{\otimes 2}$ given by

$$x_0 \mapsto x_0 \otimes 1 + 1 \otimes x_0 \text{ and } x_z \mapsto x_z \otimes 1 + 1 \otimes x_z, \text{ for } z \in \mu_N.$$

The $\mathbb{Q}$-algebra $\mathbb{Q}\langle\langle Y\rangle\rangle$ is equipped with a bialgebra structure with respect to the *harmonic coproduct*, which is the algebra morphism $\widehat{\Delta}_* : \mathbb{Q}\langle\langle Y\rangle\rangle \to \mathbb{Q}\langle\langle Y\rangle\rangle^{\otimes 2}$ given by

$$y_{k,z} \mapsto y_{k,z} \otimes 1 + 1 \otimes y_{k,z} + \sum_{\substack{k_1+k_2=k \\ z_1 z_2 = z}} y_{k_1,z_1} \otimes y_{k_2,z_2}, \text{ for } (k, z) \in \mathbb{Z}_{>0} \times \mu_N.$$

Definition 2.1 ([10, Definitions 3.3.1 and 3.3.8])**.** We define $\mathfrak{dmr}_0^{\mu_N}$ to be the $\mathbb{Q}$-linear subspace of elements $\psi \in \mathbb{Q}\langle\langle X\rangle\rangle$ such that

(i) $(\psi \mid x_0) = (\psi \mid x_1) = 0$;
(ii) $\widehat{\Delta}_ш(\psi) = \psi \otimes 1 + 1 \otimes \psi$;
(iii) [1] $(\psi \mid x_z - x_{z^{-1}}) = 0$ for any $z \in \mu_N$;
(iv) $\widehat{\Delta}_*(\psi_*) = \psi_* \otimes 1 + 1 \otimes \psi_*$;

where the notation $(\psi \mid w)$ means "the coefficient of the word w in ψ"[2], and

$$\psi_* = \pi_Y \circ \mathbf{p}^{-1}(\psi) + \sum_{n \geq 2} \frac{(-1)^{n-1}}{n} (\psi \mid x_0^{n-1}x_1) y_{1,1}^n \in \mathbb{Q}\langle\langle Y\rangle\rangle,$$

with $\mathbf{p}$ being the $\mathbb{Q}$-linear automorphism of $\mathbb{Q}\langle\langle X\rangle\rangle$ given by

$$\mathbf{p}(x_0^{k_1-1}x_{z_1}x_0^{k_2-1}x_{z_2} \cdots x_0^{k_r-1}x_{z_r}x_0^{k_{r+1}-1}) = x_0^{k_1-1}x_{z_1}x_0^{k_2-1}x_{z_1 z_2} \cdots x_0^{k_r-1}x_{z_1 \cdots z_r}x_0^{k_{r+1}-1},$$

for $r \in \mathbb{Z}_{\geq 0}$, $k_1, \ldots, k_{r+1} \in \mathbb{Z}_{>0}$ and $z_1, \ldots, z_r \in \mu_N$.

[1] In [10], this condition is given as $\left(\psi_* \mid x_0^{k-1}x_z\right) = (-1)^{k-1}\left(\psi_* \mid x_0^{k-1}x_{z^{-1}}\right)$ for $(k, z) \in \mathbb{Z}_{>0} \times \mu_N$, but thanks to [10, Propositions 3.3.3 and 3.3.7], it is enough to have (iii) for $k = 1$ and any $z \in \mu_N$ since this identity is always true for all the other cases. By definition of ψ_* this is equivalent to the stated identity.

[2] and extend by linearity.

In order to equip the space $\mathfrak{dmr}_0^{\mu_N}$ with a Lie algebra structure, Racinet [10, §3.1.10.2] considers on $\mathbb{Q}\langle\langle X\rangle\rangle$ the bracket:

$$\langle \psi_1, \psi_2\rangle := d_{\psi_1}(\psi_2) - d_{\psi_2}(\psi_1) + [\psi_1, \psi_2],$$

where, for $\psi \in \mathbb{Q}\langle\langle X\rangle\rangle$, d_ψ is the derivation of $\mathbb{Q}\langle\langle X\rangle\rangle$ given by

$$x_0 \mapsto 0, \text{ and } x_z \mapsto t_z([x_1, \psi]) \text{ for } z \in \mu_N,$$

with t_z being the $\mathbb{Q}$-algebra automorphism of $\mathbb{Q}\langle\langle X\rangle\rangle$ given by

$$x_0 \mapsto x_0, \text{ and } x_\zeta \mapsto x_{z\zeta}, \text{ for } \zeta \in \mu_N, \tag{2.2}$$

and $[-,-]$ being the usual bracket on $\mathbb{Q}\langle\langle X\rangle\rangle$. We then have

Proposition 2.2 ([10, Proposition 4.A.i)]). *The pair* $(\mathfrak{dmr}_0^{\mu_N}, \langle\cdot,\cdot\rangle)$ *is a Lie subalgebra of* $(\mathbb{Q}\langle\langle X\rangle\rangle, \langle\cdot,\cdot\rangle)$.

3. The congruent double shuffle $\mathbb{Q}$-space $\mathfrak{dmr}_0^{[N]}$

Let $\mathbb{Q}\langle\langle \widetilde{X}\rangle\rangle$ be the free noncommutative series $\mathbb{Q}$-algebra with unit over the alphabet $\widetilde{X} := \{\tilde{x}, \tilde{x}_\alpha \mid \alpha \in \mathbb{Z}/N\mathbb{Z}\}$. Let $\mathbb{Q}\langle\langle \widetilde{Y}\rangle\rangle$ be the free noncommutative series $\mathbb{Q}$-algebra with unit over the alphabet $\widetilde{Y} := \{\tilde{y}_{k,\alpha} \mid (k,\alpha) \in \mathbb{Z}_{>0} \times \mathbb{Z}/N\mathbb{Z}\}$.
The $\mathbb{Q}$-algebra morphism

$$\mathbb{Q}\langle\langle \widetilde{Y}\rangle\rangle \to \mathbb{Q}\langle\langle \widetilde{X}\rangle\rangle, \quad \tilde{y}_{k,\alpha} \mapsto \tilde{x}^{k-1}\tilde{x}_\alpha, \text{ for } (k,\alpha) \in \mathbb{Z}_{>0} \times \mathbb{Z}/N\mathbb{Z} \tag{3.1}$$

is injective. Therefore, throughout this notes, we will treat $\mathbb{Q}\langle\langle \widetilde{Y}\rangle\rangle$ as a subalgebra of $\mathbb{Q}\langle\langle \widetilde{X}\rangle\rangle$ by identifying it to its image by the morphism.
One checks that we have the direct sum decomposition (of $\mathbb{Q}$-linear subspaces)

$$\mathbb{Q}\langle\langle \widetilde{X}\rangle\rangle = \mathbb{Q}\langle\langle \widetilde{Y}\rangle\rangle \oplus \mathbb{Q}\langle\langle \widetilde{X}\rangle\rangle\tilde{x}.$$

Let then $\pi_{\widetilde{Y}} : \mathbb{Q}\langle\langle \widetilde{X}\rangle\rangle = \mathbb{Q}\langle\langle \widetilde{Y}\rangle\rangle \oplus \mathbb{Q}\langle\langle \widetilde{X}\rangle\rangle\tilde{x} \twoheadrightarrow \mathbb{Q}\langle\langle \widetilde{Y}\rangle\rangle$ be the projection from $\mathbb{Q}\langle\langle \widetilde{X}\rangle\rangle$ to $\mathbb{Q}\langle\langle \widetilde{Y}\rangle\rangle$, that is, the surjective $\mathbb{Q}$-linear map such that it is the identity on $\mathbb{Q}\langle\langle \widetilde{Y}\rangle\rangle$ and maps any element of $\mathbb{Q}\langle\langle \widetilde{X}\rangle\rangle\tilde{x}$ to 0.

Lemma 3.1. *(a) The $\mathbb{Q}$-algebra $\mathbb{Q}\langle\langle \widetilde{X}\rangle\rangle$ is equipped with a bialgebra structure with respect to the* shuffle coproduct*, which is the algebra morphism $\widehat{\Delta}_{\tilde{ш}} : \mathbb{Q}\langle\langle \widetilde{X}\rangle\rangle \to \mathbb{Q}\langle\langle \widetilde{X}\rangle\rangle^{\otimes 2}$ given by*

$$\tilde{x} \mapsto \tilde{x} \otimes 1 + 1 \otimes \tilde{x} \text{ and } \tilde{x}_\alpha \mapsto \tilde{x}_\alpha \otimes 1 + 1 \otimes \tilde{x}_\alpha, \text{ for } \alpha \in \mathbb{Z}/N\mathbb{Z}.$$

(b) The $\mathbb{Q}$-algebra $\mathbb{Q}\langle\langle \widetilde{Y}\rangle\rangle$ is equipped with a bialgebra structure with respect to the harmonic coproduct*, which is the algebra morphism $\widehat{\Delta}_{\tilde{*}} : \mathbb{Q}\langle\langle \widetilde{Y}\rangle\rangle \to \mathbb{Q}\langle\langle \widetilde{Y}\rangle\rangle^{\otimes 2}$ given by*

$$\tilde{y}_{k,\alpha} \mapsto \tilde{y}_{k,\alpha} \otimes 1 + 1 \otimes \tilde{y}_{k,\alpha} + \sum_{k_1+k_2=k} \tilde{y}_{k_1,\alpha} \otimes \tilde{y}_{k_2,\alpha}, \text{ for } (k,\alpha) \in \mathbb{Z}_{>0} \times \mathbb{Z}/N\mathbb{Z}.$$

Definition 3.2. We define $\mathfrak{dmr}_0^{[N]}$ to be the $\mathbb{Q}$-linear subspace of elements $\widetilde{\psi} \in \mathbb{Q}\langle\langle \widetilde{X}\rangle\rangle$ such that

(i) $(\widetilde{\psi} \mid \tilde{x}) = \sum_{\alpha\in\mathbb{Z}/N\mathbb{Z}} (\widetilde{\psi} \mid \tilde{x}_\alpha) = 0$;

(ii) $\widehat{\Delta}_{\tilde{ш}}(\widetilde{\psi}) = \widetilde{\psi} \otimes 1 + 1 \otimes \widetilde{\psi}$;

(iii) $(\widetilde{\psi} \mid \tilde{x}_\alpha - \tilde{x}_{-\alpha}) = 0$ for any $\alpha \in \mathbb{Z}/N\mathbb{Z}$;

(iv) $\widehat{\Delta}_{\tilde{*}}(\widetilde{\psi}_{\tilde{*}}) = \widetilde{\psi}_{\tilde{*}} \otimes 1 + 1 \otimes \widetilde{\psi}_{\tilde{*}}$;

where the notation $(\widetilde{\psi} \mid \widetilde{w})$ means "the coefficient of the word $\widetilde{w}$ in $\widetilde{\psi}$"[3], and

$$\widetilde{\psi}_{\tilde{*}} = \pi_{\widetilde{Y}} \circ \widetilde{\mathbf{q}}^{-1}(\widetilde{\psi}) + \sum_{n\geq 2}\sum_{a=1}^{N}\sum_{b_1=1}^{N}\cdots\sum_{b_n=1}^{N} \frac{(-1)^{n-1}}{n\ N^{n+1}} \left(\widetilde{\psi} \mid \tilde{x}^{n-1}\tilde{x}_{\iota(a)}\right) \tilde{y}_{1,\iota(b_1)} \cdots \tilde{y}_{1,\iota(b_n)} \in \mathbb{Q}\langle\langle \widetilde{Y}\rangle\rangle,$$

[3] and extend by linearity.

with $\widetilde{\mathbf{q}}$ being the $\mathbb{Q}$-linear automorphism of $\mathbb{Q}\langle\langle \widetilde{X}\rangle\rangle$ given by

$$\widetilde{\mathbf{q}}(\tilde{x}^{k_1-1}\tilde{x}_{\alpha_1}\cdots\tilde{x}^{k_{r-1}-1}\tilde{x}_{\alpha_{r-1}}\tilde{x}^{k_r-1}\tilde{x}_{\alpha_r}\tilde{x}^{k_{r+1}-1}) = \tilde{x}^{k_1-1}\tilde{x}_{\alpha_1-\alpha_2}\cdots\tilde{x}^{k_{r-1}-1}\tilde{x}_{\alpha_{r-1}-\alpha_r}\tilde{x}^{k_r-1}\tilde{x}_{\alpha_r}\tilde{x}^{k_{r+1}-1},$$

for $r\in\mathbb{Z}_{\geq 0}$, $k_1,\dots,k_{r+1}\in\mathbb{Z}_{>0}$ and $\alpha_1,\dots,\alpha_r\in\mathbb{Z}/N\mathbb{Z}$.

4. Comparison of algebraic setups

Let $\mathbb{Q}(\mu_N)$ be the cyclotomic field, that is, the smallest field containing $\mathbb{Q}$ and μ_N. Set $\underline{e}_N(k) := e^{\frac{i2k\pi}{N}}$, for any $k\in[\![1,N]\!]$.

Proposition 4.1. *The $\mathbb{Q}(\mu_N)$-algebra morphism $\mathcal{F}:\mathbb{Q}(\mu_N)\langle\langle\widetilde{X}\rangle\rangle\to\mathbb{Q}(\mu_N)\langle\langle X\rangle\rangle$ given by*

$$\tilde{x}\mapsto x_0 \text{ and } \tilde{x}_\alpha\mapsto\sum_{m=1}^{N}\underline{e}_N\left(-m\ \iota^{-1}(\alpha)\right)x_{\underline{e}_N(m)} \text{ for } \alpha\in\mathbb{Z}/N\mathbb{Z},$$

is an isomorphism whose inverse $\mathcal{F}^{-1}$is given by

$$x_0\mapsto\tilde{x} \text{ and } x_{\underline{e}_N(m)}\mapsto\frac{1}{N}\sum_{a=1}^{N}\underline{e}_N(am)\tilde{x}_{\iota(a)} \text{ for } m\in[\![1,N]\!].$$

Proof. It is enough to check that the composition of both maps is the identity of generators. It is immediate that the image of the generators x_0 and $\tilde{x}$ by both compositions gives the identity. Let us compute the image of the other generators by the composition. For $m\in[\![1,N]\!]$, we have

$$\begin{aligned}
x_{\underline{e}_N(m)} &\mapsto \frac{1}{N}\sum_{a=1}^{N}\underline{e}_N(am)\tilde{x}_{\iota(a)} \mapsto x_{\underline{e}_N(m)}\mapsto\frac{1}{N}\sum_{a=1}^{N}\underline{e}_N(am)\sum_{k=1}^{N}\underline{e}_N\left(-k\ \iota^{-1}(\iota(a))\right)x_{\underline{e}_N(k)}\\
&=\frac{1}{N}\sum_{a=1}^{N}\underline{e}_N(am)\sum_{k=1}^{N}\underline{e}_N(-ka)x_{\underline{e}_N(k)}=\frac{1}{N}\sum_{k=1}^{N}\left(\sum_{a=1}^{N}\underline{e}_N(a(m-k))\right)x_{\underline{e}_N(k)}\\
&=\frac{1}{N}Nx_{\underline{e}_N(m)}=x_{\underline{e}_N(m)},
\end{aligned}$$

where the first equality comes from the identity $\underline{e}_N\left(-k\ \iota^{-1}(\iota(a))\right)=\underline{e}_N(-ka)$; the second one from the identity $\underline{e}_N(am)\underline{e}_N(-ka)=\underline{e}_N(a(m-k))$; and the last one from the identity

$$\sum_{j=1}^{N}\underline{e}_N(Mj)=\begin{cases}N & \text{if } N\mid M\\ 0 & \text{otherwise}\end{cases} \tag{4.1}$$

applied to $M=m-k$, where in this case, $N\mid M$ if and only if $k=m$.
On the other hand, for $\alpha\in\mathbb{Z}/N\mathbb{Z}$, we have

$$\begin{aligned}
\tilde{x}_\alpha &\mapsto\sum_{m=1}^{N}\underline{e}_N\left(-m\ \iota^{-1}(\alpha)\right)x_{\underline{e}_N(m)}\mapsto\sum_{m=1}^{N}\underline{e}_N\left(-m\ \iota^{-1}(\alpha)\right)\frac{1}{N}\sum_{a=1}^{N}\underline{e}_N(am)\tilde{x}_{\iota(a)}\\
&=\frac{1}{N}\sum_{a=1}^{N}\left(\sum_{m=1}^{N}\underline{e}_N\left(m(a-\iota^{-1}(\alpha)\right)\right)\tilde{x}_{\iota(a)}=\frac{1}{N}N\tilde{x}_{\iota(\iota^{-1}(\alpha))}=\tilde{x}_\alpha
\end{aligned}$$

where the first equality comes from the identity $\underline{e}_N(-m\iota^{-1}(\alpha))\underline{e}_N(am)=\underline{e}_N\left(m(a-\iota^{-1}(\alpha)\right)$; and the second one from identity (4.1) applied to $M=a-\iota^{-1}(\alpha)$, where in this case, $N\mid M$ if and only if $a=\iota^{-1}(\alpha)$. □

Proposition 4.2. *The $\mathbb{Q}(\mu_N)$-algebra isomorphism $\mathcal{F} : \mathbb{Q}(\mu_N)\langle\langle \widetilde{X}\rangle\rangle \to \mathbb{Q}(\mu_N)\langle\langle X\rangle\rangle$ restricts to the $\mathbb{Q}(\mu_N)$-algebra isomorphism $\mathcal{F}_Y : \mathbb{Q}(\mu_N)\langle\langle \widetilde{Y}\rangle\rangle \to \mathbb{Q}(\mu_N)\langle\langle Y\rangle\rangle$ given by*

$$\tilde{y}_{k,\alpha} \mapsto \sum_{m=1}^{N} \underline{e}_N\left(-m\iota^{-1}(\alpha)\right) y_{k,\underline{e}_N(m)} \text{ for } (k,\alpha) \in \mathbb{Z}_{>0} \times \mathbb{Z}/N\mathbb{Z}.$$

Its inverse $\mathcal{F}_Y^{-1} : \mathbb{Q}(\mu_N)\langle\langle Y\rangle\rangle \to \mathbb{Q}(\mu_N)\langle\langle \widetilde{Y}\rangle\rangle$ is given by

$$y_{k,\underline{e}_N(m)} \mapsto \frac{1}{N}\sum_{a=1}^{N} \underline{e}_N(am)\tilde{y}_{k,\iota(a)} \text{ for } (k,m) \in \mathbb{Z}_{>0} \times [\![1,N]\!].$$

Proof. Let $(k,\alpha) \in \mathbb{Z}_{>0} \times \mathbb{Z}/N\mathbb{Z}$. Thanks to the injection (3.1), the element $\tilde{y}_{k,\alpha}$ is identified with $\tilde{x}^{k-1}\tilde{x}_\alpha$. Moreover,

$$\mathcal{F}(\tilde{x}^{k-1}\tilde{x}_\alpha) = \mathcal{F}(\tilde{x})^{k-1}\mathcal{F}(\tilde{x}_\alpha) = x_0^{k-1}\sum_{m=1}^{N} \underline{e}_N\left(-m\,\iota^{-1}(\alpha)\right) x_{\underline{e}_N(m)} = \sum_{m=1}^{N} \underline{e}_N\left(-m\,\iota^{-1}(\alpha)\right) x_0^{k-1} x_{\underline{e}_N(m)}.$$

Finally, thanks to the injection (2.1), the element $x_0^{k-1}x_{\underline{e}_N(m)}$ is identified with $y_{k,\underline{e}_N(m)}$, thus proving that the $\mathbb{Q}(\mu_N)$-algebra morphism $\mathcal{F} : \mathbb{Q}(\mu_N)\langle\langle \widetilde{X}\rangle\rangle \to \mathbb{Q}(\mu_N)\langle\langle X\rangle\rangle$ restricts to the morphism $\mathcal{F}_Y : \mathbb{Q}(\mu_N)\langle\langle \widetilde{Y}\rangle\rangle \to \mathbb{Q}(\mu_N)\langle\langle Y\rangle\rangle$.
An analogous argument proves that $\mathcal{F}^{-1} : \mathbb{Q}(\mu_N)\langle\langle X\rangle\rangle \to \mathbb{Q}(\mu_N)\langle\langle \widetilde{X}\rangle\rangle$ restricts to the given $\mathcal{F}_Y^{-1} : \mathbb{Q}(\mu_N)\langle\langle Y\rangle\rangle \to \mathbb{Q}(\mu_N)\langle\langle \widetilde{Y}\rangle\rangle$. □

Recall that for any set S, one has

$$\mathbb{Q}(\mu_N)\langle\langle S\rangle\rangle \simeq \mathbb{Q}(\mu_N) \otimes \mathbb{Q}\langle\langle S\rangle\rangle. \tag{4.2}$$

For any sets S and T and a $\mathbb{Q}$-linear map $f : \mathbb{Q}\langle\langle S\rangle\rangle \to \mathbb{Q}\langle\langle T\rangle\rangle$, when there is no ambiguity, we shall abusively denote by f the $\mathbb{Q}(\mu_N)$-linear morphism $\mathbb{Q}(\mu_N)\langle\langle S\rangle\rangle \to \mathbb{Q}(\mu_N)\langle\langle T\rangle\rangle$ induced by this map.

Lemma 4.3. *(a) We have (equality of $\mathbb{Q}(\mu_N)$-linear maps $\mathbb{Q}(\mu_N)\langle\langle \widetilde{X}\rangle\rangle \to \mathbb{Q}(\mu_N)\langle\langle Y\rangle\rangle$)*

$$\pi_Y \circ \mathcal{F} = \mathcal{F}_Y \circ \pi_{\widetilde{Y}}.$$

(b) We have (equality of $\mathbb{Q}(\mu_N)$-linear maps $\mathbb{Q}(\mu_N)\langle\langle \widetilde{X}\rangle\rangle \to \mathbb{Q}(\mu_N)\langle\langle X\rangle\rangle$)

$$\mathbf{p} \circ \mathcal{F} = \mathcal{F} \circ \tilde{\mathbf{q}}.$$

Proof. Since all the maps are $\mathbb{Q}(\mu_N)$-linear maps, it is enough to check the claimed equalities on a generators of $\mathbb{Q}(\mu_N)\langle\langle \widetilde{X}\rangle\rangle$. Let $\tilde{x}^{k_1-1}\tilde{x}_{\alpha_1}\tilde{x}^{k_2-1}\tilde{x}_{\alpha_2}\cdots\tilde{x}^{k_r-1}\tilde{x}_{\alpha_r}\tilde{x}^{k_{r+1}-1}$ with $r \in \mathbb{Z}_{\geq 0}$, $k_1,\dots,k_{r+1} \in \mathbb{Z}_{>0}$ and $\alpha_1,\dots,\alpha_r \in \mathbb{Z}/N\mathbb{Z}$, be such an element.

(a) If $k_{r+1} \neq 1$, we have

$$\mathcal{F}_Y \circ \pi_{\widetilde{Y}}(\tilde{x}^{k_1-1}\tilde{x}_{\alpha_1}\tilde{x}^{k_2-1}\tilde{x}_{\alpha_2}\cdots\tilde{x}^{k_r-1}\tilde{x}_{\alpha_r}\tilde{x}^{k_{r+1}-1}) = 0.$$

On the other hand,

$$\begin{aligned}&\pi_Y \circ \mathcal{F}(\tilde{x}^{k_1-1}\tilde{x}_{\alpha_1}\tilde{x}^{k_2-1}\tilde{x}_{\alpha_2}\cdots\tilde{x}^{k_r-1}\tilde{x}_{\alpha_r}\tilde{x}^{k_{r+1}-1})\\ &\qquad = \pi_Y\left(\mathcal{F}(\tilde{x}^{k_1-1}\tilde{x}_{\alpha_1}\tilde{x}^{k_2-1}\tilde{x}_{\alpha_2}\cdots\tilde{x}^{k_r-1}\tilde{x}_{\alpha_r})x_0^{k_{r+1}-1}\right) = 0,\end{aligned}$$

where the second equality follows from the fact that $\mathcal{F}$ is an algebra morphism.
If $k_{r+1} = 1$ then

$$\mathcal{F}_Y \circ \pi_{\widetilde{Y}}(\tilde{x}^{k_1-1}\tilde{x}_{\alpha_1}\tilde{x}^{k_2-1}\tilde{x}_{\alpha_2}\cdots\tilde{x}^{k_r-1}\tilde{x}_{\alpha_r}) = \mathcal{F}_Y(\tilde{y}_{k_1,\alpha_1}\cdots\tilde{y}_{k_r,\alpha_r})$$

$$= \sum_{m_1=1}^{N} \cdots \sum_{m_r=1}^{N} \underline{e}_N \left(-m_1\iota^{-1}(\alpha_1) - \cdots - m_r\iota^{-1}(\alpha_r)\right) y_{k_1,\underline{e}_N(m_1)} \cdots y_{k_r,\underline{e}_N(m_r)}$$

On the other hand,

$$\pi_Y \circ \mathcal{F}(\tilde{x}^{k_1-1}\tilde{x}_{\alpha_1}\tilde{x}^{k_2-1}\tilde{x}_{\alpha_2}\cdots\tilde{x}^{k_r-1}\tilde{x}_{\alpha_r}) = \pi_Y\left(\mathcal{F}(\tilde{x}^{k_1-1}\tilde{x}_{\alpha_1}\tilde{x}^{k_2-1}\tilde{x}_{\alpha_2}\cdots\tilde{x}^{k_r-1}\tilde{x}_{\alpha_r})\right)$$

$$= \pi_Y\left(\sum_{m_1=1}^{N} \cdots \sum_{m_r=1}^{N} \underline{e}_N\left(-m_1\iota^{-1}(\alpha_1) - \cdots - m_r\iota^{-1}(\alpha_r)\right) x_0^{k_1-1}x_{\underline{e}_N(m_1)}\cdots x_0^{k_r-1}x_{\underline{e}_N(m_r)}\right)$$

$$= \sum_{m_1=1}^{N} \cdots \sum_{m_r=1}^{N} \underline{e}_N\left(-m_1\iota^{-1}(\alpha_1) - \cdots - m_r\iota^{-1}(\alpha_r)\right) y_{k_1,\underline{e}_N(m_1)}\cdots y_{k_r,\underline{e}_N(m_r)},$$

thus proving the wanted identity.

(b) We have

$$\mathcal{F}\circ\widetilde{\mathbf{q}}(\tilde{x}^{k_1-1}\tilde{x}_{\alpha_1}\cdots\tilde{x}^{k_{r-1}-1}\tilde{x}_{\alpha_{r-1}}\tilde{x}^{k_r-1}\tilde{x}_{\alpha_r}\tilde{x}^{k_{r+1}-1})$$
$$= \mathcal{F}(\tilde{x}^{k_1-1}\tilde{x}_{\alpha_1-\alpha_2}\cdots\tilde{x}^{k_{r-1}-1}\tilde{x}_{\alpha_{r-1}-\alpha_r}\tilde{x}^{k_r-1}\tilde{x}_{\alpha_r}\tilde{x}^{k_{r+1}-1})$$
$$= \sum_{m'_1=1}^{N}\sum_{m'_2=1}^{N}\cdots\sum_{m'_r=1}^{N} \underline{e}_N(-m'_1\iota^{-1}(\alpha_1) + (m'_1 - m'_2)\iota^{-1}(\alpha_2) + \cdots + (m'_{r-1} - m'_r)\iota^{-1}(\alpha_r))$$
$$x_0^{k_1-1}x_{\underline{e}_N(m'_1)}x_0^{k_2-1}x_{\underline{e}_N(m'_2)}\cdots x_0^{k_r-1}x_{\underline{e}_N(m'_r)}x_0^{k_{r+1}-1}$$
$$= \sum_{m_1=1}^{N}\sum_{m_2=1}^{N}\cdots\sum_{m_r=1}^{N} \underline{e}_N(-m_1\iota^{-1}(\alpha_1) - m_2\iota^{-1}(\alpha_2) - \cdots - m_r\iota^{-1}(\alpha_r))$$
$$x_0^{k_1-1}x_{\underline{e}_N(m_1)}x_0^{k_2-1}x_{\underline{e}_N(m_1+m_2)}\cdots x_0^{k_r-1}x_{\underline{e}_N(m_1+\cdots+m_r)}x_0^{k_{r+1}-1},$$

where the last equality follows from the change of variable $m_1 = m'_1$, $m_2 = m'_2 - m'_1, \ldots$, $m_r = m'_r - m'_{r-1}$. On the other hand,

$$\mathbf{p}\circ\mathcal{F}(\tilde{x}^{k_1-1}\tilde{x}_{\alpha_1}\tilde{x}^{k_2-1}\tilde{x}_{\alpha_2}\cdots\tilde{x}^{k_r-1}\tilde{x}_{\alpha_r}\tilde{x}^{k_{r+1}-1})$$
$$= \sum_{m_1=1}^{N}\sum_{m_2=1}^{N}\cdots\sum_{m_r=1}^{N} \underline{e}_N(-m_1\iota^{-1}(\alpha_1) - m_2\iota^{-1}(\alpha_2) - \cdots - m_r\iota^{-1}(\alpha_r))$$
$$x_0^{k_1-1}x_{\underline{e}_N(m_1)}x_0^{k_2-1}x_{\underline{e}_N(m_1+m_2)}\cdots x_0^{k_r-1}x_{\underline{e}_N(m_1+\cdots+m_r)}x_0^{k_{r+1}-1},$$

thus proving the wanted identity.

□

Thanks to the isomorphism (4.2), for any set S and a $\mathbb{Q}$-algebra morphism $\Delta : \mathbb{Q}\langle\langle S\rangle\rangle \to \mathbb{Q}\langle\langle S\rangle\rangle^{\otimes 2}$, when there is no ambiguity, we shall abusively denote by Δ the $\mathbb{Q}(\mu_N)$-algebra morphism $\mathbb{Q}(\mu_N)\langle\langle S\rangle\rangle \to \mathbb{Q}(\mu_N)\langle\langle S\rangle\rangle^{\otimes 2}$ induced by this map.

Lemma 4.4. *(a) We have (equality of $\mathbb{Q}(\mu_N)$-algebra morphisms $\mathbb{Q}(\mu_N)\langle\langle \widetilde{X}\rangle\rangle \to \mathbb{Q}(\mu_N)\langle\langle X\rangle\rangle^{\otimes 2}$)*

$$\widehat{\Delta}_{ш} \circ \mathcal{F} = \mathcal{F}^{\otimes 2} \circ \widehat{\Delta}_{\tilde{ш}}.$$

(b) We have (equality of $\mathbb{Q}(\mu_N)$-algebra morphisms $\mathbb{Q}(\mu_N)\langle\langle \widetilde{Y}\rangle\rangle \to \mathbb{Q}(\mu_N)\langle\langle Y\rangle\rangle^{\otimes 2}$)

$$\widehat{\Delta}_{*} \circ \mathcal{F}_Y = \mathcal{F}_Y^{\otimes 2} \circ \widehat{\Delta}_{\tilde{*}}.$$

Proof. (a) Since all the maps are $\mathbb{Q}(\mu_N)$-algebra morphisms, it suffices to check the identity on generators of the algebra $\mathbb{Q}(\mu_N)\langle\langle\widetilde{X}\rangle\rangle$. The equality is immediate for $\tilde{x}$. For $\alpha \in \mathbb{Z}/N\mathbb{Z}$,

$$\begin{aligned}
\widehat{\Delta}_{Ш} \circ \mathcal{F}(\tilde{x}_\alpha) &= \sum_{m=1}^{N} \underline{e}_N\left(-m\,\iota^{-1}(\alpha)\right) \widehat{\Delta}_{Ш}(x_{\underline{e}_N(m)}) \\
&= \sum_{m=1}^{N} \underline{e}_N\left(-m\,\iota^{-1}(\alpha)\right) (x_{\underline{e}_N(m)} \otimes 1 + 1 \otimes x_{\underline{e}_N(m)}) \\
&= \sum_{m=1}^{N} \underline{e}_N\left(-m\,\iota^{-1}(\alpha)\right) x_{\underline{e}_N(m)} \otimes 1 + 1 \otimes \sum_{m=1}^{N} \underline{e}_N\left(-m\,\iota^{-1}(\alpha)\right) x_{\underline{e}_N(m)} \\
&= \mathcal{F}^{\otimes 2}(\tilde{x}_\alpha \otimes 1 + 1 \otimes \tilde{x}_\alpha) = \mathcal{F}^{\otimes 2} \circ \widehat{\Delta}_{\tilde{Ш}}(\tilde{x}_\alpha).
\end{aligned}$$

(b) Since all arrows are $\mathbb{Q}(\mu_N)$-algebra morphisms, it suffices to check the identity on generators of the algebra $\mathbb{Q}(\mu_N)\langle\langle\widetilde{Y}\rangle\rangle$. For $(k,\alpha) \in \mathbb{Z}_{>0} \times \mathbb{Z}/N\mathbb{Z}$ we have

$$\begin{aligned}
&\widehat{\Delta}_* \circ \mathcal{F}_Y(\tilde{y}_{k,\alpha}) = \sum_{m=1}^{N} \underline{e}_N\left(-m\iota^{-1}(\alpha)\right) \widehat{\Delta}_*(y_{k,\underline{e}_N(m)}) \\
&= \sum_{m=1}^{N} \underline{e}_N\left(-m\,\iota^{-1}(\alpha)\right) \Big(y_{k,\underline{e}_N(m)} \otimes 1 + 1 \otimes y_{k,\underline{e}_N(m)} + \sum_{\substack{k_1+k_2=k \\ m_1+m_2=m}} y_{k_1,\underline{e}_N(m_1)} \otimes y_{k_2,\underline{e}_N(m_2)}\Big) \\
&= \sum_{m=1}^{N} \underline{e}_N\left(-m\,\iota^{-1}(\alpha)\right) y_{k,\underline{e}_N(m)} \otimes 1 + 1 \otimes \sum_{m=1}^{N} \underline{e}_N\left(-m\,\iota^{-1}(\alpha)\right) y_{k,\underline{e}_N(m)} \\
&+ \sum_{k_1+k_2=k} \sum_{m_1=1}^{N} \sum_{m_2=1}^{N} \underline{e}_N\left(-(m_1+m_2)\,\iota^{-1}(\alpha)\right) y_{k,\underline{e}_N(m_1)} \otimes y_{k,\underline{e}_N(m_2)} \\
&= \sum_{m=1}^{N} \underline{e}_N\left(-m\,\iota^{-1}(\alpha)\right) y_{k,\underline{e}_N(m)} \otimes 1 + 1 \otimes \sum_{m=1}^{N} \underline{e}_N\left(-m\,\iota^{-1}(\alpha)\right) y_{k,\underline{e}_N(m)} \\
&+ \sum_{k_1+k_2=k} \sum_{m_1=1}^{N} \underline{e}_N\left(-m_1\,\iota^{-1}(\alpha)\right) y_{k,\underline{e}_N(m_1)} \otimes \sum_{m_2=1}^{N} \underline{e}_N\left(-m_2\,\iota^{-1}(\alpha)\right) y_{k,\underline{e}_N(m_2)} \\
&= \mathcal{F}^{\otimes 2}(\tilde{y}_{k,\alpha} \otimes 1 + 1 \otimes \tilde{y}_{k,\alpha} + \sum_{k_1+k_2=k} \tilde{y}_{k_1,\alpha} \otimes \tilde{y}_{k_2,\alpha}) = \mathcal{F}_Y^{\otimes 2} \circ \widehat{\Delta}_{\tilde{*}}(\tilde{y}_{k,\alpha}).
\end{aligned}$$

□

5. A $\mathbb{Q}$-Lie bracket on $\mathfrak{dmr}_0^{[N]}$

Let $a \in [\![1, N]\!]$. Define $\tilde{t}_a$ to be the $\mathbb{Q}(\mu_N)$-algebra automorphism of $\mathbb{Q}(\mu_N)\langle\langle\widetilde{X}\rangle\rangle$ given by

$$\tilde{x} \mapsto \tilde{x} \text{ and } \tilde{x}_\alpha \mapsto \underline{e}_N(a\,\iota^{-1}(\alpha))\tilde{x}_\alpha \text{ for } \alpha \in \mathbb{Z}/N\mathbb{Z}.$$

Lemma 5.1. *For $a \in [\![1, N]\!]$, we have (equality of $\mathbb{Q}(\mu_N)$-algebra morphisms $\mathbb{Q}(\mu_N)\langle\langle\widetilde{X}\rangle\rangle \to \mathbb{Q}(\mu_N)\langle\langle X\rangle\rangle$)*

$$t_{\underline{e}_N(a)} \circ \mathcal{F} = \mathcal{F} \circ \tilde{t}_a.$$

where $t_{\underline{e}_N(a)}$ is the $\mathbb{Q}(\mu_N)$-algebra automorphism of $\mathbb{Q}(\mu_N)\langle\langle X\rangle\rangle$ induced by the automorphism of $\mathbb{Q}\langle\langle X\rangle\rangle$ given in (2.2).

Proof. Since all the maps are $\mathbb{Q}(\mu_N)$-algebra morphisms, it suffices to check the identity on generators of the algebra $\mathbb{Q}(\mu_N)\langle\langle \widetilde{X}\rangle\rangle$. The equality is immediate for $\tilde{x}$. For $\alpha \in \mathbb{Z}/N\mathbb{Z}$,

$$\begin{aligned}\mathcal{F}\circ \tilde{t}_a(\tilde{x}_\alpha) &= \mathcal{F}(\underline{e}_N(a\iota^{-1}(\alpha))\tilde{x}_\alpha) = \underline{e}_N(a\iota^{-1}(\alpha)\sum_{m=1}^{N}\underline{e}_N(-m\iota^{-1}(\alpha))x_{\underline{e}_N(m)} \\ &= \sum_{m=1}^{N}\underline{e}_N(-(m-a)\iota^{-1}(\alpha))x_{\underline{e}_N(m)} = \sum_{m'=1}^{N}\underline{e}_N(-m'\iota^{-1}(\alpha))x_{\underline{e}_N(m'+a)} \\ &= t_{\underline{e}_N(a)}\left(\sum_{m'=1}^{N}\underline{e}_N(-m'\iota^{-1}(\alpha))x_{\underline{e}_N(m')}\right) = t_{\underline{e}_N(a)}\circ\mathcal{F}(\tilde{x}_\alpha),\end{aligned}$$

thus proving the wanted identity. □

Recall that the Galois group $\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$ is canonically isomorphic to the multiplicative group $(\mathbb{Z}/N\mathbb{Z})^\times$, with an automorphism σ_k corresponding to $\iota(k)$ where $k \in [\![1, N-1]\!]$ such that $\gcd(k, N) = 1$.

Lemma 5.2. *The group G_N acts on the $\mathbb{Q}(\mu_N)$-linear span of $\mathrm{id}_{\mathbb{Q}(\mu_N)\langle\langle \widetilde{X}\rangle\rangle}$ and $(\tilde{t}_a)_{a\in[\![1,N]\!]}$ by*

$$\sigma\cdot r\,\mathrm{id}_{\mathbb{Q}(\mu_N)\langle\langle \widetilde{X}\rangle\rangle} = \sigma(r)\,\mathrm{id}_{\mathbb{Q}(\mu_N)\langle\langle \widetilde{X}\rangle\rangle},\ \textit{for } \sigma\in G_N \textit{ and } r\in\mathbb{Q}(\mu_N) \tag{5.1}$$

and [4]

$$\sigma_k\cdot\tilde{t}_a = \tilde{t}_{\iota^{-1}(\overline{ak})}\ \textit{for } k\in[\![1,N-1]\!] \textit{ with } \gcd(k,N)=1 \textit{ and } a\in[\![1,N]\!]. \tag{5.2}$$

Proof. It is immediate that the mapping given in (5.1) is an action. For $a \in [\![1, N]\!]$, the mapping (5.2) is acts thanks to the identity

$$\overline{\iota^{-1}(\overline{al})k} = \overline{akl}, \quad \forall k,l\in[\![1,N]\!] \text{ with } \gcd(k,N)=\gcd(l,N)=1.$$

□

Proposition-Definition 5.3. *Let $a \in [\![1, N]\!]$. Define $\widetilde{T}_a$ to be the $\mathbb{Q}(\mu_N)$-linear automorphism of $\mathbb{Q}(\mu_N)\langle\langle \widetilde{X}\rangle\rangle$ given by*

$$\widetilde{T}_a := \frac{1}{N}\sum_{m=1}^{N}\underline{e}_N(-ma)\ \tilde{t}_m.$$

Then $\widetilde{T}_a \in \mathrm{Aut}_{\mathbb{Q}\text{-lin}}(\mathbb{Q}\langle\langle \widetilde{X}\rangle\rangle)$.

Proof. Let $\sigma_k \in G_N$. For any $\widetilde{\psi} \in \mathbb{Q}\langle\langle \widetilde{X}\rangle\rangle$, we have

$$\begin{aligned}\sigma_k\cdot\widetilde{T}_a(\widetilde{\psi}) &= \frac{1}{N}\sum_{m'=1}^{N}\sigma_k(\underline{e}_N(-m'a))\ \sigma_k\cdot\tilde{t}_{m'}(\widetilde{\psi}) = \frac{1}{N}\sum_{m'=1}^{N}\underline{e}_N\left(-\iota^{-1}(\overline{km'})a\right)\ \tilde{t}_{\iota^{-1}(\overline{km'})}(\widetilde{\psi}) \\ &= \frac{1}{N}\sum_{m=1}^{N}\underline{e}_N(-ma)\ \tilde{t}_m(\widetilde{\psi}) = \widetilde{T}_a(\widetilde{\psi}),\end{aligned}$$

where the second equality follows from the identity $\iota^{-1}(\overline{km'}) = km'$ and from identities (5.1) and (5.2) of Lemma 5.2. Finally from this equality we conclude that $\widetilde{T}_a(\widetilde{\psi}) \in \mathbb{Q}\langle\langle \widetilde{X}\rangle\rangle$. □

[4] Recall that ak is not necessarily an element of $[\![1, N]\!]$, there we use the notation $\overline{ak}$ for the class in $\mathbb{Z}/N\mathbb{Z}$ instead of using the map ι.

Proposition-Definition 5.4. *For $\widetilde{\psi} \in \mathbb{Q}\langle\langle \widetilde{X} \rangle\rangle$, there exists a derivation $\tilde{d}_{\widetilde{\psi}}$ of $\mathbb{Q}\langle\langle \widetilde{X} \rangle\rangle$ uniquely defined by*

$$\tilde{x} \mapsto 0, \text{ and } \tilde{x}_\alpha \mapsto \widetilde{T}_{\iota^{-1}(\alpha)}\left(\left[\textstyle\sum_{\beta \in \mathbb{Z}/N\mathbb{Z}} \tilde{x}_\beta,\ \widetilde{\psi}\right]\right) \text{ for } \alpha \in \mathbb{Z}/N\mathbb{Z}.$$

Proof. This follows from Proposition-Definition 5.3. □

Proposition 5.5. *For $\widetilde{\psi} \in \mathbb{Q}\langle\langle \widetilde{X} \rangle\rangle$, we have (equality of compositions of $\mathbb{Q}(\mu_N)$-algebra morphisms and $\mathbb{Q}(\mu_N)$-derivations $\mathbb{Q}(\mu_N)\langle\langle \widetilde{X} \rangle\rangle \to \mathbb{Q}(\mu_N)\langle\langle X \rangle\rangle$)*

$$\mathcal{F} \circ \tilde{d}_{\widetilde{\psi}} = d_{\mathcal{F}(\widetilde{\psi})} \circ \mathcal{F}.$$

Proof. We prove this equality by applying both sides to generators $\tilde{x}$ and $\tilde{x}_\alpha$ ($\alpha \in \mathbb{Z}/N\mathbb{Z}$) and showing that they yield the same result. The computation for $\tilde{x}$ being immediate, we will focus on $\tilde{x}_\alpha$. We have

$$\begin{aligned}
\mathcal{F} \circ \tilde{d}_{\widetilde{\psi}}(\tilde{x}_a) &= \mathcal{F} \circ \widetilde{T}_{\iota^{-1}(\alpha)}\left(\left[\textstyle\sum_{\beta \in \mathbb{Z}/N\mathbb{Z}} \tilde{x}_\beta,\ \widetilde{\psi}\right]\right) \\
&= \frac{1}{N} \sum_{m=1}^{N} \underline{e}_N(-m\iota^{-1}(\alpha))\ \mathcal{F} \circ \tilde{t}_m \left(\left[\textstyle\sum_{\beta \in \mathbb{Z}/N\mathbb{Z}} \tilde{x}_\beta,\ \widetilde{\psi}\right]\right) \\
&= \sum_{m=1}^{N} \underline{e}_N(-m\iota^{-1}(\alpha))\ t_{\underline{e}_N(m)} \circ \mathcal{F}\left(\left[\tfrac{1}{N}\textstyle\sum_{\beta \in \mathbb{Z}/N\mathbb{Z}} \tilde{x}_\beta,\ \widetilde{\psi}\right]\right) \\
&= \sum_{m=1}^{N} \underline{e}_N(-m\iota^{-1}(\alpha))\ t_{\underline{e}_N(m)}\left(\left[\mathcal{F}\left(\tfrac{1}{N}\textstyle\sum_{\beta \in \mathbb{Z}/N\mathbb{Z}} \tilde{x}_\beta\right),\ \mathcal{F}(\widetilde{\psi})\right]\right) \\
&= \sum_{m=1}^{N} \underline{e}_N(-m\iota^{-1}(\alpha))\ t_{\underline{e}_N(m)}\left(\left[x_1,\ \mathcal{F}(\widetilde{\psi})\right]\right) = \sum_{m=1}^{N} \underline{e}_N(-m\iota^{-1}(\alpha))\ d_{\mathcal{F}(\widetilde{\psi})}(x_{\underline{e}_N(m)}) \\
&= d_{\mathcal{F}(\widetilde{\psi})}\left(\sum_{m=1}^{N} \underline{e}_N(-m\iota^{-1}(\alpha))\ x_{\underline{e}_N(m)}\right) = d_{\mathcal{F}(\widetilde{\psi})} \circ \mathcal{F}(\tilde{x}_\alpha),
\end{aligned}$$

where the third equality comes from Lemma 5.1 and the fourth one from the fact that $\mathcal{F}$ is an algebra morphism. □

Lemma 5.6. *The $\mathbb{Q}$-bilinear map given by*

$$(\widetilde{\psi}_1, \widetilde{\psi}_2) \mapsto \langle \widetilde{\psi}_1, \widetilde{\psi}_2 \rangle^{\sim} := \tilde{d}_{\widetilde{\psi}_1}(\widetilde{\psi}_2) - \tilde{d}_{\widetilde{\psi}_2}(\widetilde{\psi}_1) + [\widetilde{\psi}_1, \widetilde{\psi}_2], \tag{5.3}$$

is a Lie bracket on $\mathbb{Q}\langle\langle \widetilde{X} \rangle\rangle$.

Proof. By definition, the triple $(\mathbb{Q}\langle\langle \widetilde{X} \rangle\rangle, \langle -, - \tilde{\rangle}, \tilde{d})$ is a post-Lie algebra. Therefore, the constructed bracket (5.3) is a Lie algebra bracket thanks to [5, Proposition 2.2]. □

Lemma 5.7. *We have (equality of bilinear maps $\mathbb{Q}(\mu_N)\langle\langle \widetilde{X} \rangle\rangle \times \mathbb{Q}(\mu_N)\langle\langle \widetilde{X} \rangle\rangle \to \mathbb{Q}(\mu_N)\langle\langle X \rangle\rangle$)*

$$\mathcal{F} \circ \langle -, - \rangle^{\sim} = \langle \mathcal{F}(-), \mathcal{F}(-) \rangle.$$

Proof. For $\widetilde{\psi}_1, \widetilde{\psi}_2 \in \mathbb{Q}(\mu_N)\langle\langle \widetilde{X} \rangle\rangle$, we have

$$\begin{aligned}
\mathcal{F}\left(\langle \widetilde{\psi}_1, \widetilde{\psi}_2 \rangle^{\sim}\right) &= \mathcal{F}\left(\tilde{d}_{\widetilde{\psi}_1}(\widetilde{\psi}_2) - \tilde{d}_{\widetilde{\psi}_2}(\widetilde{\psi}_1) + [\widetilde{\psi}_1, \widetilde{\psi}_2]\right) \\
&= d_{\mathcal{F}(\widetilde{\psi}_1)}\left(\mathcal{F}(\widetilde{\psi}_2)\right) - d_{\mathcal{F}(\widetilde{\psi}_2)}\left(\mathcal{F}(\widetilde{\psi}_1)\right) + [\mathcal{F}(\widetilde{\psi}_1), \mathcal{F}(\widetilde{\psi}_2)] = \langle \mathcal{F}(\widetilde{\psi}_1), \mathcal{F}(\widetilde{\psi}_2) \rangle,
\end{aligned}$$

where the second equality follows from Proposition 5.5. □

Theorem 5.8. *(a) The map $\mathcal{F}$ induces a $\mathbb{Q}(\mu_N)$-Lie algebra isomorphism*

$$\left(\mathbb{Q}(\mu_N)\ \otimes_{\mathbb{Q}}\ \mathfrak{dmr}_0^{[N]},\ \langle -,-\rangle^{\sim}\right) \xrightarrow{\simeq} \left(\mathbb{Q}(\mu_N)\ \otimes_{\mathbb{Q}}\ \mathfrak{dmr}_0^{\mu_N},\ \langle -,-\rangle\right). \tag{5.4}$$

(b) The $\mathbb{Q}$-linear space $\mathfrak{dmr}_0^{[N]}$ forms a Lie algebra under the bracket $\langle -,-\rangle^{\sim}$.

Proof. (a) Let $\widetilde{\psi} \in \mathbb{Q}(\mu_N)\ \otimes_{\mathbb{Q}}\ \mathfrak{dmr}_0^{[N]}$. Let us show that $\mathcal{F}(\widetilde{\psi}) \in \mathbb{Q}(\mu_N)\ \otimes_{\mathbb{Q}}\ \mathfrak{dmr}_0^{\mu_N}$.
First, we have $(\mathcal{F}(\widetilde{\psi}) \mid x_0) = (\mathcal{F}(\widetilde{\psi}) \mid \mathcal{F}(\tilde{x})) = (\widetilde{\psi} \mid \tilde{x}) = 0$, where the last equality follows from (i) of Definition 3.2. Next,

$$(\mathcal{F}(\widetilde{\psi}) \mid x_1) = (\mathcal{F}(\widetilde{\psi}) \mid \mathcal{F}(\textstyle\sum_{\alpha\in\mathbb{Z}/N\mathbb{Z}} \tilde{x}_\alpha)) = \displaystyle\sum_{\alpha\in\mathbb{Z}/N\mathbb{Z}} (\widetilde{\psi} \mid \tilde{x}_\alpha) = 0,$$

where the last equality follows from (i) of Definition 3.2. Therefore $\mathcal{F}(\widetilde{\psi})$ satisfies condition (i) of Definition 2.1. Next, for $m \in [\![1, N]\!]$, we have

$$\begin{aligned}
\left(\mathcal{F}(\widetilde{\psi}) \mid x_{\underline{e}_N(m)}\right) &= \left(\mathcal{F}(\widetilde{\psi}) \mid \mathcal{F}\left(\tfrac{1}{N}\textstyle\sum_{a=1}^N \underline{e}_N(am)\tilde{x}_{\iota(a)}\right)\right) = \frac{1}{N}\sum_{a=1}^N \underline{e}_N(am)\left(\widetilde{\psi} \mid \tilde{x}_{\iota(a)}\right) \\
&= \frac{1}{N}\sum_{a=1}^N \underline{e}_N(am)\left(\widetilde{\psi} \mid \tilde{x}_{-\iota(a)}\right) = \frac{1}{N}\sum_{a=1}^N \underline{e}_N(-am)\left(\widetilde{\psi} \mid \tilde{x}_{\iota(a)}\right) \\
&= \left(\mathcal{F}(\widetilde{\psi}) \mid \mathcal{F}\left(\tfrac{1}{N}\textstyle\sum_{a=1}^N \underline{e}_N(-am)\tilde{x}_{\iota(a)}\right)\right) = \left(\mathcal{F}(\widetilde{\psi}) \mid x_{\underline{e}_N(-m)}\right),
\end{aligned}$$

where the third equality follows from (iii) of Definition 3.2. Therefore $\mathcal{F}(\widetilde{\psi})$ satisfies condition (iii) of Definition 2.1. Next, we have

$$\widehat{\Delta}_{ш}\left(\mathcal{F}(\widetilde{\psi})\right) = \mathcal{F}^{\otimes 2}\left(\widehat{\Delta}_{ш}(\widetilde{\psi})\right) = \mathcal{F}^{\otimes 2}\left(\widetilde{\psi}\otimes 1 + 1\otimes\widetilde{\psi}\right) = \mathcal{F}^{\otimes 2}(\widetilde{\psi})\otimes 1 + 1\otimes\mathcal{F}^{\otimes 2}(\widetilde{\psi}),$$

where the first equality follows from Lemma 4.4 (a); and the second one from (ii) of Definition 3.2. Therefore $\mathcal{F}(\widetilde{\psi})$ satisfies condition (ii) of Definition 2.1. Finally, we have

$$\begin{aligned}
\mathcal{F}_Y(\widetilde{\psi}_*) &= \mathcal{F}_Y\Big(\pi_{\widetilde{Y}}\circ\widetilde{\mathbf{q}}^{-1}(\widetilde{\psi}) + \sum_{n\geq 2}\sum_{a=1}^N\sum_{b_1=1}^N\cdots\sum_{b_n=1}^N \frac{(-1)^{n-1}}{n\ N^{n+1}}\left(\widetilde{\psi} \mid \tilde{x}^{n-1}\tilde{x}_{\iota(a)}\right)\ \tilde{y}_{1,\iota(b_1)}\cdots\tilde{y}_{1,\iota(b_n)}\Big) \\
&= \pi_Y\circ\mathbf{p}^{-1}(\psi) + \sum_{n\geq 2}\frac{(-1)^{n-1}}{n}\Big(\widetilde{\psi} \mid \frac{1}{N}\sum_{a=1}^N \tilde{x}^{n-1}\tilde{x}_{\iota(a)}\Big)y_{1,1}^n \\
&= \pi_Y\circ\mathbf{p}^{-1}(\psi) + \sum_{n\geq 2}\frac{(-1)^{n-1}}{n}\left(\widetilde{\psi} \mid \mathcal{F}^{-1}(x_0^{n-1}x_1)\right)y_{1,1}^n \\
&= \pi_Y\circ\mathbf{p}^{-1}(\psi) + \sum_{n\geq 2}\frac{(-1)^{n-1}}{n}\left(\mathcal{F}(\widetilde{\psi}) \mid x_0^{n-1}x_1\right)y_{1,1}^n = \psi_*,
\end{aligned}$$

where the second equality follows from Lemma 4.3. By Lemma 4.4 (b), and condition (iv) of Definition 3.2, a computation analogous to the one on the shuffle coproducts enables to show that $\mathcal{F}(\widetilde{\psi})$ satisfies condition (iv) of Definition 2.1.
With analogous arguments, one shows that

$$\mathcal{F}^{-1}(\psi) \in \mathbb{Q}(\mu_N)\ \otimes_{\mathbb{Q}}\ \mathfrak{dmr}_0^{[N]},$$

for $\psi \in \mathbb{Q}(\mu_N)\otimes_{\mathbb{Q}}\mathfrak{dmr}_0^{\mu_N}$. Thus proving that the map (5.4) is a $\mathbb{Q}(\mu_N)$-linear isomorphism. Finally, Lemma 5.7 establishes that it is in fact a $\mathbb{Q}(\mu_N)$-Lie algebra isomorphism.

(b) This follows from the claim (a) since for any $\widetilde{\psi} \in \mathbb{Q}\langle\langle \widetilde{X} \rangle\rangle$, the map $\tilde{d}_{\widetilde{\psi}}$ is, by definition, a derivation of $\mathbb{Q}\langle\langle \widetilde{X} \rangle\rangle \subset \mathbb{Q}(\mu_N)\langle\langle \widetilde{X} \rangle\rangle$.

□

In conclusion, Theorem 5.8 implies that so far we only have a $\mathbb{Q}(\mu_N)$ isomorphism between the two structures. The best relization over $\mathbb{Q}$ is provided by [6, Theorem 3.16]. This observation leads us to the following problem:

Problem 5.9. Are the Lie algebras $\mathfrak{dmr}_0^{[N]}$ and $\mathfrak{dmr}_0^{\mu_N}$ isomorphic over $\mathbb{Q}$?

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