AN INTRODUCTION TO ELLIPTIC MULTIPLE ZETA VALUES AND THEIR REGULARIZATION

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0. Introduction

This note is a survey of [5], which studies the regularization of elliptic multiple zeta values. Elliptic multiple zeta values, introduced by Enriquez [3], are elliptic analogues of multiple zeta values and are defined by iterated integrals on the family of punctured elliptic curves. Fix τ in the upper half-plane $\mathfrak{h}$. Elliptic multiple zeta values are given by iterated integrals of the form

$$I^A(k_1,\dots,k_r;\tau)=\int_{0<z_1<\cdots<z_r<1} f_\tau^{(k_1)}(z_1)\,dz_1\cdots f_\tau^{(k_r)}(z_r)\,dz_r,$$

where the functions $f_\tau^{(n)}(z)$ arise from the Eisenstein–Kronecker series. The integral converges when $k_1\neq 1$ and $k_r\neq 1$; such indices are called *admissible*. If $k_1=1$ or $k_r=1$, the integral diverges and one defines $I^A(k_1,\dots,k_r;\tau)$ by a suitable regularization.

Elliptic multiple zeta values appear naturally as coefficients of the elliptic associator, which is defined from the monodromy of the Knizhnik–Zamolodchikov–Bernard (KZB) connection on elliptic curves [1, 3]. In genus zero, the corresponding KZ associator has coefficients given by regularized iterated integrals, and it was shown in [4, 6] that they can be expressed in terms of convergent iterated integrals only. This motivates the study of an analogous phenomenon in the elliptic setting.

The purpose of this note is to explain the regularization problem for elliptic multiple zeta values and to describe structural results relating regularized values to those with admissible indices. Full details and proofs can be found in [5].

1. Definition of elliptic multiple zeta values

Definition 1.1 (cf. [8])**.** Fix $\tau\in\mathfrak{h}$. The **Eisenstein–Kronecker series** is the two-variable complex function $F_\tau\binom{\alpha}{z}$ $(\alpha,z\in\mathbb{C})$ defined by

$$F_\tau\binom{\alpha}{z}:=\frac{\theta_\tau(z+\alpha)\theta'_\tau(0)}{\theta_\tau(z)\theta_\tau(\alpha)}$$

where θ_τ is the odd Jacobi theta function given by

$$\theta_\tau(z):=\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2}w^{n+\frac{1}{2}}\quad(q:=e^{2\pi i\tau},w:=e^{2\pi iz}).$$

The Laurent expansion of $F_\tau\binom{\alpha}{z}$ at $\alpha=0$ is given by

$$F_\tau\binom{\alpha}{z}=\sum_{n\geq 0}f_\tau^{(n)}(z)\,\alpha^{n-1},$$

Date: February 2, 2026.

which defines meromorphic functions $f_\tau^{(n)}(z)$ on $\mathbb{C}$. Restricting to the real interval $0 \le z \le 1$, the only singularities arise from $f_\tau^{(1)}(z)$, which has simple poles at $z=0$ and $z=1$. For $n \neq 1$, the function $f_\tau^{(n)}(z)$ is holomorphic.

Lemma 1.2 ([5])**.** *When $k_1 \neq 1$ and $k_r \neq 1$, the iterated integral $\int_{0<z_1<\cdots<z_r<1} f_\tau^{(k_1)}(z_1)dz_1 \cdots f_\tau^{(k_r)}(z_r)dz_r$ converges.*

Definition 1.3 ([3])**.** For $r \geq 0$, $\boldsymbol{k} = (k_1, \dots, k_r) \in \mathbb{N}_0^r$ with $k_1, k_r \neq 1$ and $\tau \in \mathfrak{h}$, the **elliptic multiple zeta value** is defined by the following iterated integral:

$$I^A(\boldsymbol{k};\tau) = \int_{0<z_1<\cdots<z_r<1} f_\tau^{(k_1)}(z_1)dz_1 \cdots f_\tau^{(k_r)}(z_r)dz_r \tag{1.1}$$

We refer to $k_1 + \cdots + k_r$ as the weight of $\boldsymbol{k}$, and r as the length of $\boldsymbol{k}$. We treat $\emptyset$ as the index of length 0 and weight 0, and define $I^A(\emptyset;\tau) := 1$.

Remark 1.4. In [3], another type of eMZV $I^B(\boldsymbol{k};\tau)$ is defined by the same iterated integral (1.1) with the interval of integration $[0,1]$ replaced with $[0,\tau]$. However, it can be verified from [3, (26)] that

$$I^B(\boldsymbol{k};\tau) = \tau^{k_1+\cdots+k_r-r} I^A(\boldsymbol{k};-1/\tau).$$

As mentioned in [7, Remark 2.3.4], the definition of $I^A(\boldsymbol{k};\tau)$ agrees with one of the regularizations of iterated integrals introduced in [2] for the tangential base points $(0, (-2\pi i)^{-1}\frac{\partial}{\partial z})$ and $(1, -(-2\pi i)^{-1}\frac{\partial}{\partial z})$. These base points correspond to $(1, -\frac{\partial}{\partial w})$ and $(1, \frac{\partial}{\partial w})$ on the Tate curve $\mathbb{C}^\times/q^{\mathbb{Z}}$ where $w = e^{2\pi i z}$ and $q = e^{2\pi i \tau}$.

Definition 1.5. Let $f : (0,1) \to \mathbb{C}$ be an analytic function. Assume f has an asymptotic behavior of the form

$$f(\epsilon) = \sum_{n=0}^{N} c_n (\log(-2\pi i \epsilon))^n + O(\epsilon^\delta). \tag{1.2}$$

for sufficiently small $\epsilon > 0$, where $\delta > 0$, $N \in \mathbb{N}_0$, and $c_0, \dots, c_N \in \mathbb{C}$ are constants. Then, the regularized value as $\epsilon \to 0$ is defined by

$$\mathrm{Reg}_{\epsilon\to 0}^{-2\pi i} f(\epsilon) := c_0 \in \mathbb{C}.$$

Here, the branch of the logarithm is taken such that $\log(-i) = -\frac{\pi i}{2}$.

For any $k_1, \dots, k_r \geq 0$, the iterated integrals

$$\int_{\epsilon<z_1<\cdots<z_r<1-\epsilon} f_\tau^{(k_1)}(z_1)dz_1 \cdots f_\tau^{(k_r)}(z_r)dz_r$$

have asymptotic behaviors of the form (1.2).

Definition 1.6 ([3])**.** For any $\boldsymbol{k} = (k_1, \dots, k_r) \in \mathbb{N}_0^r$, we define $I^A(\boldsymbol{k};\tau)$ by

$$I^A(\boldsymbol{k};\tau) := \mathrm{Reg}_{\epsilon\to 0}^{-2\pi i} \int_{\epsilon<z_1<\cdots<z_r<1-\epsilon} f_\tau^{(k_1)}(z_1)dz_1 \cdots f_\tau^{(k_r)}(z_r)dz_r$$

From now on, we write $I^A(\boldsymbol{k})$ for $I^A(\boldsymbol{k};\tau)$.

Remark 1.7. When $k_1, k_r \neq 1$, since the limit

$$\lim_{\epsilon\to 0} \int_{\epsilon<z_1<\cdots<z_r<1-\epsilon} f_\tau^{(k_1)}(z_1)dz_1 \cdots f_\tau^{(k_r)}(z_r)dz_r$$

converges, we have

$$\begin{aligned}&\operatorname{Reg}^{-2\pi i}_{\epsilon\to 0}\int_{\epsilon<z_1<\cdots<z_r<1-\epsilon} f_\tau^{(k_1)}(z_1)dz_1\cdots f_\tau^{(k_r)}(z_r)dz_r\\ &=\lim_{\epsilon\to 0}\int_{\epsilon<z_1<\cdots<z_r<1-\epsilon} f_\tau^{(k_1)}(z_1)dz_1\cdots f_\tau^{(k_r)}(z_r)dz_r.\end{aligned}$$

Therefore, Definition 1.6 agrees with Definition 1.3 in the case with $k_1, k_r \neq 1$.

Example 1.8 ([7])**.** The elliptic multiple zeta value $I^A(k)\,(k\geq 0)$ of length 1 is given by

$$I^A(k)=\begin{cases}-2\zeta(k) & (k \text{ is even}),\\ 0 & (k \text{ is odd})\end{cases}$$

and it is independent of τ. Here, we put $\zeta(0)=-\frac{1}{2}$.

2. Some relations among elliptic multiple zeta values

In this section, we collect some basic relations satisfied by elliptic multiple zeta values.

Proposition 2.1 ([3])**.**

$$I^A(i_1,\ldots,i_n)I^A(i_{n+1},\ldots,i_{n+m})=\sum_{\sigma\in\mathcal{S}_{n,m}} I^A(i_{\sigma(1)},\ldots,i_{\sigma(n+m)}) \tag{2.1}$$

where $\mathcal{S}_{n,m}$ denotes the set of all permutations of $\mathfrak{S}_{n+m}$ such that

$$\sigma^{-1}(1)<\cdots<\sigma^{-1}(n), \sigma^{-1}(n+1)<\cdots<\sigma^{-1}(n+m).$$

Remark 2.2. Equation (2.1) is called the **shuffle relation** for elliptic multiple zeta values.

Proposition 2.3 ([7])**.**

$$I^A(i_n,\ldots,i_1)=(-1)^{i_1+\cdots+i_n}I^A(i_1,\ldots,i_n) \tag{2.2}$$

Remark 2.4. Equation (2.2) is called the **reflection relation** for elliptic multiple zeta values.

Finally, elliptic multiple zeta values satisfy, in addition to shuffle and reflection relations, further relations known as Fay relations, which are derived from the Fay identity of the Eisenstein–Kronecker series. As an illustration, an explicit formula in length two was established in [7].

Proposition 2.5 ([7])**.** *For any $r, s\in\mathbb{N}_0$ with $(r,s)\neq(1,1)$, we have:*

$$\begin{aligned}I^A(r,s)=&-(-1)^sI^A(0,r+s)+\sum_{n=0}^{s}(-1)^{s-n}\binom{r-1+n}{r-1}I^A(r+n,s-n)\\ &+\sum_{n=0}^{r}(-1)^{s+n}\binom{s-1+n}{s-1}I^A(s+n,r-n).\end{aligned}$$

More generally, Fay relations can be extended to higher length. To describe their explicit form, we introduce the following combinatorial data. For $\boldsymbol{l}=(l_1,\ldots,l_r)\in\mathbb{N}_0^r$, we define the rational function $P_{\boldsymbol{l}}(u_1,\ldots,u_r)\in\mathbb{Q}(u_1,\ldots,u_r)$ as follows:

$$P_{\boldsymbol{l}}(u_1,\ldots,u_r):=\sum_{i=0}^{r-1}u_1^{l_1-1}\cdots u_{i-1}^{l_{i-1}-1}(u_i+\cdots+u_r)^{l_i-1}(-u_{i+1}-\cdots-u_r)^{l_{i+1}-1}u_{i+1}^{l_{i+2}-1}\cdots u_{r-1}^{l_r-1}.$$

Lemma 2.6 ([5]). *For any* $\boldsymbol{l} = (l_1, \ldots, l_r) \in \mathbb{N}_0^r$, $u_1 \cdots u_r P_{\boldsymbol{l}}(u_1, \ldots, u_r)$ *is a* $\mathbb{Z}$*-polynomial (i.e.* $u_1 \cdots u_r P_{\boldsymbol{l}}(u_1, \ldots, u_r) \in \mathbb{Z}[u_1, \ldots, u_r]$*), and furthermore, it is a homogeneous polynomial of degree* $l_1 + \cdots + l_r$*.*

For $\boldsymbol{k} = (k_1, \ldots, k_r), \boldsymbol{l} = (l_1, \ldots, l_r) \in \mathbb{N}_0^r$, let $c\langle \boldsymbol{l} \mid \boldsymbol{k} \rangle$ denote the coefficient of $u_1^{k_1} \cdots u_r^{k_r}$ in $u_1 \cdots u_r P_{\boldsymbol{l}}(u_1, \ldots, u_r)$. In other words,

$$u_1 \cdots u_r P_{\boldsymbol{l}}(u_1, \ldots, u_r) = \sum_{\boldsymbol{k} \in \mathbb{N}_0^r} c\langle \boldsymbol{l} \mid \boldsymbol{k} \rangle u_1^{k_1} \cdots u_r^{k_r}.$$

Theorem 2.7 ([5]). *For any* $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{N}_0^r$ *where* $r = 1$ *or* $r > 1, k_r \neq 1$*, we have*

$$\begin{aligned} I^A(\boldsymbol{k}) = &\sum_{i=2}^{r} (-1)^i \delta_{1,k_1} \cdots \delta_{1,k_{i-1}} \delta_{1,k_r} \zeta(i) I^A(k_i, \ldots, k_{r-1}) \\ &- \sum_{\boldsymbol{l} \in \mathbb{N}_0^r} c\langle \boldsymbol{l} \mid \boldsymbol{k} \rangle I^A(l_1, \ldots, l_r). \end{aligned}$$

3. Regularization

In this section, we introduce some notation and state the main regularization result for elliptic multiple zeta values.

Let

$$\mathcal{E}^A\mathcal{Z} := \langle I^A(k_1, \ldots, k_r) \mid r \geq 0,\ k_1, \ldots, k_r \in \mathbb{N}_0 \rangle_{\mathbb{Q}}$$

be the $\mathbb{Q}$-linear space spanned by regularized elliptic multiple zeta values. The $\mathbb{Q}$-linear subspace $\mathcal{E}^A\mathcal{Z}_{\mathrm{adm}} \subset \mathcal{E}^A\mathcal{Z}$ is defined by

$$\mathcal{E}^A\mathcal{Z}_{\mathrm{adm}} := \langle I^A(k_1, \ldots, k_r) \mid r \geq 0,\ k_1, \ldots, k_r \in \mathbb{N}_0,\ k_1, k_r \neq 1 \rangle_{\mathbb{Q}}.$$

For $w \geq 0$, we also define the weight-w linear subspaces $(\mathcal{E}^A\mathcal{Z})_w$ and $(\mathcal{E}^A\mathcal{Z}_{\mathrm{adm}})_w$ by

$$\begin{aligned} (\mathcal{E}^A\mathcal{Z})_w &:= \langle I^A(k_1, \ldots, k_r) \mid r \geq 0,\ k_1, \ldots, k_r \in \mathbb{N}_0,\ k_1 + \cdots + k_r = w \rangle_{\mathbb{Q}}, \\ (\mathcal{E}^A\mathcal{Z}_{\mathrm{adm}})_w &:= \langle I^A(k_1, \ldots, k_r) \mid r \geq 0,\ k_1, \ldots, k_r \in \mathbb{N}_0,\ k_1, k_r \neq 1,\ k_1 + \cdots + k_r = w \rangle_{\mathbb{Q}}. \end{aligned}$$

It follows from the shuffle relations that both $\mathcal{E}^A\mathcal{Z}$ and $\mathcal{E}^A\mathcal{Z}_{\mathrm{adm}}$ are closed under multiplication, and hence form $\mathbb{Q}$-algebras.

Theorem 3.1 ([5]). *As a* $\mathbb{Q}$*-algebra, the following equality holds:*

$$\mathcal{E}^A\mathcal{Z} = \mathcal{E}^A\mathcal{Z}_{\mathrm{adm}}\big[I^A(k_1, \ldots, k_r) \mid r \in \mathbb{N}_0,\ k_1, \ldots, k_r \in \{0, 1\}\big].$$

The proof of Theorem 3.1 relies on the Fay relations given in Theorem 2.7. For full details, we refer to [5].

Acknowledgements

The author is grateful to the organizers, Yasuo Ohno and Shin-ichiro Seki, for the opportunity to present this work.

References

1. Damien Calaque, Benjamin Enriquez, and Pavel Etingof, *Universal KZB equations: the elliptic case*, Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I, Progr. Math., vol. 269, Birkhäuser Boston, Boston, MA, 2009, pp. 165–266. MR 2641173
2. P. Deligne, *Le groupe fondamental de la droite projective moins trois points*, Galois groups over **Q** (Berkeley, CA, 1987), Math. Sci. Res. Inst. Publ., vol. 16, Springer, New York, 1989, pp. 79–297. MR 1012168
3. Benjamin Enriquez, *Analogues elliptiques des nombres multizétas*, Bull. Soc. Math. France **144** (2016), no. 3, 395–427. MR 3558428
4. Hidekazu Furusho, *The multiple zeta value algebra and the stable derivation algebra*, Publ. Res. Inst. Math. Sci. **39** (2003), no. 4, 695–720. MR 2025460
5. Taichi Katayama, *Regularization of elliptic multiple zeta values*, arXiv preprint arXiv:2504.15621 (2025).
6. Thang Tu Quoc Le and Jun Murakami, *Kontsevich's integral for the Kauffman polynomial*, Nagoya Math. J. **142** (1996), 39–65. MR 1399467
7. Nils Matthes, *Elliptic multiple zeta values*, Ph.D. thesis, Staats-und Universitätsbibliothek Hamburg Carl von Ossietzky, 2016.
8. Don Zagier, *Periods of modular forms and Jacobi theta functions*, Invent. Math. **104** (1991), no. 3, 449–465. MR 1106744

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