Generalized q-Stirling numbers and q-multiple zeta values

Takao Komatsu
Institute of Mathematics, Henan Academy of Sciences
and
Department of Mathematics, Institute of Science Tokyo

1 q-generalized (r,s)-Stirling numbers

Let $[n]_q$ denote the q-number, defined by

$$[n]_q = \frac{q^n - 1}{q - 1} \quad (q \neq 1)\,.$$

Its q-factorial is given by $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$ with $[0]_q! = 1$. Let r be a positive integer. The q-version of r-Stirling numbers of the first kind with higher level (level s) are denoted by $\left[\!\left[n \atop k \right]\!\right]_q^{(r,s)}$, and appear in the coefficients in the expansion

$$(x)_{n,q}^{(r,s)} = \sum_{k=0}^{n} (-1)^{n-k} \left[\!\!\left[n \atop k \right]\!\!\right]_q^{(r,s)} x^k\,, \tag{1}$$

where for $r, s \geq 1$, $(x)_{n,q}^{(r,s)}$ is defined by

$$(x)_{n,q}^{(r,s)} := x^r \prod_{i=r}^{n-1} \bigl(x - ([i]_q)^s \bigr) \quad (n > r)$$

with $(x)_{r,q}^{(r,s)} = x^r$. When $r = s = 1$, $s_q(n,k) = (-1)^{n-k} \left[\!\left[n \atop k \right]\!\right]_q^{(1,1)}$ are the signed q-Stirling numbers of the first kind (see, e.g., [8]), and $\left[\!\left[n \atop k \right]\!\right]_q = \left[\!\left[n \atop k \right]\!\right]_q^{(1,1)}$ are the unsigned q-Stirling numbers of the first kind. When $r = 1$ and $q \to 1$, $\left[\!\left[n \atop k \right]\!\right]^{(s)} = \left[\!\left[n \atop k \right]\!\right]_1^{(1,s)}$ are the (unsigned) Stirling numbers of the first kind with higher level ([18]). When $s = 1$ and $q \to 1$, $\left[n \atop k \right] = \left[\!\left[n \atop k \right]\!\right]_1^{(r,1)}$ are the unsigned r-Stirling numbers of the first kind ([5]). When $r = s = 1$ and $q \to 1$, $\left[n \atop k \right] = \left[\!\left[n \atop k \right]\!\right]_1^{(1,1)}$ are the unsigned Stirling numbers of the first kind.

The q-version of r-Stirling numbers of the second kind with higher level are denoted by $\left\{\!\left\{ n \atop k \right\}\!\right\}_q^{(r,s)}$, and appear in the coefficients in the expansion

$$x^n = \sum_{k=0}^{n} \left\{\!\!\left\{ n \atop k \right\}\!\!\right\}_q^{(r,s)} (x)_{k,q}^{(r,s)}\,. \tag{2}$$

When $r=1$ and $q\to 1$, $\left\{\!\!\left\{ {n \atop k} \right\}\!\!\right\}^{(s)} = \left\{\!\!\left\{ {n \atop k} \right\}\!\!\right\}_q^{(1,s)}$ are the Stirling numbers of the second kind with higher level, studied in [19]. When $r=s=1$, $S_q(n,k) = \left\{\!\!\left\{ {n \atop k} \right\}\!\!\right\}_q = (-1)^{n-k} \left\{\!\!\left\{ {n \atop k} \right\}\!\!\right\}_q^{(1,1)}$ are the signed q-Stirling numbers of the second kind (see, e.g., [8]). When $s=1$ and $q\to 1$, $\left\{ {n \atop k} \right\} = \left\{\!\!\left\{ {n \atop k} \right\}\!\!\right\}_1^{(r,1)}$ are the r-Stirling numbers of the second kind. When $r=s=1$ and $q\to 1$, $\left\{ {n \atop k} \right\} = \left\{\!\!\left\{ {n \atop k} \right\}\!\!\right\}_1^{(1,1)}$ are the classical Stirling numbers of the second kind.

From the definitions in (1), the recurrence relation is given by

$$\left[\!\!\left[{n \atop k} \right]\!\!\right]_q^{(r,s)} = \left[\!\!\left[{n-1 \atop k-1} \right]\!\!\right]_q^{(r,s)} + \bigl([n-1]_q\bigr)^s \left[\!\!\left[{n-1 \atop k} \right]\!\!\right]_q^{(r,s)} \tag{3}$$

with

$$\left[\!\!\left[{n \atop k} \right]\!\!\right]_q^{(r,s)} = 0 \quad (0\le k\le r,\, n\ge k),$$
$$\left[\!\!\left[{n \atop k} \right]\!\!\right]_q^{(r,s)} = 0 \quad (n<k), \qquad \left[\!\!\left[{0 \atop 0} \right]\!\!\right]_q^{(r,s)} = 1\,.$$

By using the recurrence relation (3), we have expressions with combinatorial summations ([10]).

Lemma 1. *For $r\le m\le n-1$ and $r\ge 1$, we have*

$$\left[\!\!\left[{n \atop m} \right]\!\!\right]_q^{(r,s)} = \left(\frac{[n-1]_q!}{[r-1]_q!}\right)^s \sum_{r\le i_1<\dots<i_{m-r}\le n-1} \frac{1}{\bigl([i_1]_q\dots[i_{m-r}]_q\bigr)^s}\,.$$

For $n-m\ge r$ and $r\ge 1$, we have

$$\begin{aligned}\left[\!\!\left[{n \atop n-m} \right]\!\!\right]_q^{(r,s)} &= \sum_{r\le i_1<\dots<i_m\le n-1} \bigl([i_1]_q\dots[i_m]_q\bigr)^s\\ &= \sum_{r\le i_1\le i_2\le\dots\le i_m\le n-m} \bigl([i_1]_q[i_2+1]_q\cdots[i_m+m-1]_q\bigr)^s\\ &= \sum_{i_m=r}^{n-m}\bigl([i_m+m-1]_q\bigr)^s \sum_{i_{m-1}=r}^{i_m}\bigl([i_{m-1}+m-2]_q\bigr)^s\cdots\sum_{i_2=r}^{i_3}\bigl([i_2+1]_q\bigr)^s\sum_{i_1=r}^{i_2}\bigl([i_1]_q\bigr)^s\,.\end{aligned}$$

From the definitions in (2), the recurrence relation is given by

$$\left\{\!\!\left\{ {n \atop k} \right\}\!\!\right\}_q^{(r,s)} = \left\{\!\!\left\{ {n-1 \atop k-1} \right\}\!\!\right\}_q^{(r,s)} + \bigl([k]_q\bigr)^s \left\{\!\!\left\{ {n-1 \atop k} \right\}\!\!\right\}_q^{(r,s)} \tag{4}$$

with

$$\left\{\!\!\left\{ {n \atop k} \right\}\!\!\right\}_q^{(r,s)} = 0 \quad (0\le k\le r-1,\ n\ge k), \qquad \left\{\!\!\left\{ {0 \atop 0} \right\}\!\!\right\}_q^{(r,s)} = 1,$$

$$\left\{\!\!\left\{ {n \atop k} \right\}\!\!\right\}_q^{(r,s)} = 0 \quad (n \le k)\,.$$

The q-Stirling numbers of the second kind with higher level can be expressed in terms of iterated summations.

Lemma 2. *For $r+1 \le k \le n$ and $r \ge 1$,*

$$\left\{\!\!\left\{ {n \atop k} \right\}\!\!\right\}_q^{(r,s)} = \sum_{i_{k-r}=0}^{n-k} \big([k]_q\big)^{(n-k-i_{k-r})s} \sum_{i_{k-r-1}=0}^{i_{k-r}} \big([k-1]_q\big)^{(i_{k-r}-i_{k-r-1})s}$$
$$\cdots \sum_{i_2=0}^{i_3} \big([r+2]_q\big)^{(i_3-i_2)s} \sum_{i_1=0}^{i_2} \big([r+1]_q\big)^{(i_2-i_1)s} \big([r]_q\big)^{i_1 s}\,.$$

For $n-k \ge r \ge 1$,

$$\left\{\!\!\left\{ {n \atop n-k} \right\}\!\!\right\}_q^{(r,s)} = \sum_{r \le i_1 \le i_2 \le \cdots \le i_k \le n-k} \big([i_1]_q [i_2]_q \cdots [i_k]_q\big)^s\,.$$

2 q-Stirling numbers and q-multiple zeta values

Several different types of q-multiple zeta functions have been studied by many researchers (see, e.g., [4, 23, 27, 28]). In [24] the function

$$\sum_{1 \le i_1 < i_2 < \cdots < i_m} \frac{1}{(1-q^{i_1})^{s_1}(1-q^{i_2})^{s_2}\cdots(1-q^{i_m})^{s_m}}$$

is proposed. In [10], in the course of studying generalizations of the classically famous Stirling numbers and their transformations, we naturally came to consider the following finite multiple zeta values. This is the case where the multiple zeta values mentioned above are finite and with $s = s_1 = s_2 = \cdots = s_m$. Namely,

$$\mathfrak{z}_n(q; m, s) := \sum_{1 \le i_1 < i_2 < \cdots < i_m \le n-1} \frac{1}{(1-q^{i_1})^{s}(1-q^{i_2})^{s}\cdots(1-q^{i_m})^{s}} \tag{5}$$

is considered.

By introducing q-generalized (r,s)-Stirling numbers $\left[\!\!\left[{n \atop k} \right]\!\!\right]_q^{(r,s)}$ and applying them to the values of the q-multiple zeta functions in (5), it is shown that

$$\mathfrak{z}_n(\zeta_n; m, 1) = \frac{1}{m+1}\binom{n-1}{m} \tag{6}$$

together with some more specific values. Here, $\zeta_n = e^{2\pi\sqrt{-1}/n}$ is the n-th primitive root of unity. It is noticed that from the first identity of Lemma 1 we have the relation

$$\sum_{r \le i_1 < i_2 < \cdots < i_m \le n-1} \frac{1}{(1-q^{i_1})^{s}(1-q^{i_2})^{s}\cdots(1-q^{i_m})^{s}} = \frac{([r-1]_q!)^s}{(1-q)^{sm}([n-1]_q!)^s} \left[\!\!\left[{n \atop m+1} \right]\!\!\right]_q^{(r,s)}\,.$$

3 Bell polynomials

We shall use the following expression (see, e.g., [7, p.247,(6e)]).

Lemma 3. *Let n, M and K be positive integers with $M \ge K$. For $g_n := a_1^n + a_2^n + \cdots + a_M^n$ we have*

$$\sum_{1 \le j_1 < j_2 < \cdots < j_K \le M} a_{j_1} a_{j_2} \cdots a_{j_K} = \frac{1}{K!} \mathbf{Y}_K(g_1, -1! g_2, 2! g_3, -3! g_4, \dots),$$

where $\mathbf{Y}_n(x_1, x_2, \dots, x_n)$ is the (complete exponential) Bell polynomial, defined by

$$\exp\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right) = 1 + \sum_{n=1}^{\infty} \mathbf{Y}_n(x_1, x_2, \dots, x_n) \frac{t^n}{n!}.$$

That is,

$$\mathbf{Y}_n(x_1, x_2, x_3, \dots, x_n) = \sum_{k=1}^{n} \sum_{\substack{i_1 + 2i_2 + \cdots + (n-k+1) i_{n-k+1} = n \\ i_1 + i_2 + i_3 + \cdots = k}} \frac{n!}{i_1! i_2! \cdots i_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{i_1} \left(\frac{x_2}{2!}\right)^{i_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{i_{n-k+1}}$$

with $\mathbf{Y}_0 = 1$.

For two sequences $a_0 = 1, a_1, a_2, \dots$ and $b_0 = 1, b_1, b_2, \dots$ we have the equivalent expressions (see [11, 17]).

Lemma 4. *The following expressions are equivalent.*

(1) $$b_m = \sum_{\substack{i_1 + 2i_2 + \cdots + m i_m = m \\ i_1, i_2, \dots, i_m \ge 0}} \frac{1}{i_1! i_2! \cdots i_m!} \left(\frac{a_1}{1}\right)^{i_1} \left(\frac{-a_2}{2}\right)^{i_2} \cdots \left(\frac{(-1)^{m-1} a_m}{m}\right)^{i_m}$$

(2) $$b_m = \frac{1}{m!} \begin{vmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & a_1 & 2 & & \vdots \\ \vdots & & \ddots & & 0 \\ a_{m-1} & a_{m-2} & \cdots & a_1 & m-1 \\ a_m & a_{m-1} & \cdots & a_2 & a_1 \end{vmatrix}$$

(3) $$a_n = \begin{vmatrix} b_1 & 1 & 0 & \cdots & 0 \\ 2b_2 & b_1 & 1 & & \vdots \\ \vdots & & \ddots & & 0 \\ (n-1) b_{n-1} & b_{n-2} & \cdots & b_1 & 1 \\ n b_n & b_{n-1} & \cdots & b_2 & b_1 \end{vmatrix}$$

(4) $$m b_m = \sum_{i=1}^{m} (-1)^{i-1} a_i b_{m-i}$$

(5) $a_n = \sum_{j=1}^{n-1} (-1)^{j-1} b_j a_{n-j} + (-1)^{n+1} n b_n$

4 The value $\mathfrak{z}_n(\zeta_n; 1, s)$

If g_ν is replaced by $\mathfrak{z}_n(\zeta_n; 1, \nu)$, then we have

$$\begin{aligned}\mathfrak{z}_n(\zeta_n; m, 1) &= \frac{1}{m+1}\binom{n-1}{m}\\ &= \sum_{i_1+2i_2+\cdots=m} \frac{1}{i_1! i_2! \cdots} \left(\frac{\mathfrak{z}_n(\zeta_n; 1, 1)}{1}\right)^{i_1} \left(-\frac{\mathfrak{z}_n(\zeta_n; 1, 2)}{2}\right)^{i_2}\\ &\quad \cdot \left(\frac{\mathfrak{z}_n(\zeta_n; 1, 3)}{3}\right)^{i_3} \left(-\frac{\mathfrak{z}_n(\zeta_n; 1, 4)}{4}\right)^{i_4} \cdots .\end{aligned}$$

In general, we have

$$\begin{aligned}&\mathfrak{z}_n(\zeta_n; m, s)\\ &= \sum_{i_1+2i_2+\cdots=m} \frac{1}{i_1! i_2! \cdots} \left(\frac{\mathfrak{z}_n(\zeta_n; 1, s)}{1}\right)^{i_1} \left(-\frac{\mathfrak{z}_n(\zeta_n; 1, 2s)}{2}\right)^{i_2}\\ &\quad \cdot \left(\frac{\mathfrak{z}_n(\zeta_n; 1, 3s)}{3}\right)^{i_3} \left(-\frac{\mathfrak{z}_n(\zeta_n; 1, 4s)}{4}\right)^{i_4} \cdots .\end{aligned}$$

By using the Bell polynomials, we have

Theorem 1.

$$\begin{aligned}&\mathfrak{z}_n(\zeta_n; m, s)\\ &= \frac{1}{m!}\mathbf{Y}_m\big(\mathfrak{z}_n(\zeta_n; 1, s), -1!\mathfrak{z}_n(\zeta_n; 1, 2s), 2!\mathfrak{z}_n(\zeta_n; 1, 3s), -3!\mathfrak{z}_n(\zeta_n; 1, 4s), \dots\big) .\end{aligned}$$

By using the equivalent relations in Lemma 4 below, we also have the following determinant expression.

Theorem 2. *For integers n, m with $n \ge 2$ and $m \ge 1$, we have*

$$\mathfrak{z}_n(\zeta_n; m, s) = \frac{1}{m!}\begin{vmatrix} \mathfrak{z}_n(\zeta_n; 1, s) & 1 & 0 & \cdots & \\ \mathfrak{z}_n(\zeta_n; 1, 2s) & \mathfrak{z}_n(\zeta_n; 1, s) & 2 & & \vdots \\ \vdots & & \ddots & & 0 \\ \mathfrak{z}_n\big(\zeta_n; 1, (m-1)s\big) & \mathfrak{z}_n\big(\zeta_n; 1, (m-2)s\big) & \cdots & \mathfrak{z}_n(\zeta_n; 1, s) & m-1 \\ \mathfrak{z}_n(\zeta_n; 1, ms) & \mathfrak{z}_n\big(\zeta_n; 1, (m-1)s\big) & \cdots & \mathfrak{z}_n(\zeta_n; 1, 2s) & \mathfrak{z}_n(\zeta_n; 1, s) \end{vmatrix} .$$

Theorem 3. *For integers n, m with $n \ge 2$ and $m \ge 1$, we have*

$$\mathfrak{z}_n(\zeta_n; 1, ms) = \begin{vmatrix} \mathfrak{z}_n(\zeta_n;1,s) & 1 & 0 & \cdots & \\ 2\mathfrak{z}_n(\zeta_n;2,s) & \mathfrak{z}_n(\zeta_n;1,s) & 1 & & \vdots \\ \vdots & & \ddots & & 0 \\ (m-1)\mathfrak{z}_n(\zeta_n;m-1,s) & \mathfrak{z}_n(\zeta_n;m-2,s) & \cdots & \mathfrak{z}_n(\zeta_n;1,s) & 1 \\ m\mathfrak{z}_n(\zeta_n;m,s) & \mathfrak{z}_n(\zeta_n;m-1,s) & \cdots & \mathfrak{z}_n(\zeta_n;2,s) & \mathfrak{z}_n(\zeta_n;1,s) \end{vmatrix} .$$

If we put $s = 1$ and m is replaced by s in Theorem 3, by using (6) we have the determinant formula for $\mathfrak{z}_n(\zeta_n; 1, s)$.

Corollary 1. *For integers n, s with $n, s \ge 2$, we have*

$$\mathfrak{z}_n(\zeta_n; 1, s) = \begin{vmatrix} \frac{n-1}{2} & 1 & 0 & \cdots & \\ \frac{2}{3}\binom{n-1}{2} & \frac{n-1}{2} & 1 & & \vdots \\ \vdots & & \ddots & & 0 \\ \frac{s-1}{s}\binom{n-1}{s-1} & \frac{1}{s-1}\binom{n-1}{s-2} & \cdots & \frac{n-1}{2} & 1 \\ \frac{s}{s+1}\binom{n-1}{s} & \frac{1}{s}\binom{n-1}{s-1} & \cdots & \frac{1}{3}\binom{n-1}{2} & \frac{n-1}{2} \end{vmatrix} .$$

5 The value $\mathfrak{z}_n(\zeta_n; m, s)$ for $s = 2, 3$

We can show an explicit formula for $\mathfrak{z}_n(\zeta_n; m, 2)$.

Theorem 4. *For a positive integer m, we have*

$$\mathfrak{z}_n(\zeta_n; m, 2) = \frac{1}{n(m+1)} \left(\binom{n-1}{m} + (-1)^m \binom{n-1}{2m+1} \right) .$$

Remark. For $m = 1, 2, 3, 4$ we have

$$\mathfrak{z}_n(\zeta_n; 1, 2) = -\frac{2(n-1)(n-5)}{4!},$$
$$\mathfrak{z}_n(\zeta_n; 2, 2) = \frac{2(n-1)(n-2)(n^2-12n+47)}{6!},$$
$$\mathfrak{z}_n(\zeta_n; 3, 2) = -\frac{2(n-1)(n-2)(n-3)(n^3-22n^2+179n-638)}{8!},$$
$$\mathfrak{z}_n(\zeta_n; 4, 2) = \frac{2(n-1)(n-2)(n-3)(n-4)(n^4-35n^3+485n^2-3325n+11274)}{10!},$$

It is easy to predict the form of $\mathfrak{z}_n(\zeta_n; m, 2)$ for general m. However, what are the final factors of the molecule, $n-5$, $n^2-12n+47$, $n^3-22n^2+179n-638$, $n^4-35n^3+485n^2-3325n+11274$, ...? In fact, they are included in the sequences [22, A177938,A054655], which can be written in terms of the r-Stirling numbers of the first kind. See Corollary 2.

Corollary 2. *For a positive integer m, we have*

$$\mathfrak{z}_n(\zeta_n; m, 2) = \frac{2\cdot m!}{(2m+2)!}\binom{n-1}{m}\sum_{k=0}^{m}\left[\!\left[\begin{matrix}2m+2\\ m+k+2\end{matrix}\right]\!\right]_1^{(m+1,1)}(-n)^k\,.$$

We can also obtain an explicit formula of $\mathfrak{z}_n(\zeta_n; m, 3)$.

Theorem 5. *For a positive integer m, we have*

$$\begin{aligned}&\mathfrak{z}_n(\zeta_n; m, 3)\\ &= \frac{1}{n^2(m+1)}\left(\binom{n-1}{m} + \binom{n-1}{3m+2}\right)\\ &\quad - \frac{1}{n^2}\sum_{k=0}^{\lfloor\frac{m+1}{2}\rfloor}\sum_{i=0}^{m-2k+1}\frac{2^i(-3)^{m-2k-i+1}}{m-k+1}\binom{m-k+1}{k}\binom{m-2k+1}{i}\binom{n+m-2k-i}{3m-3k+2}.\end{aligned}$$

For the moment, we cannot find any simplified form of the second term.

By a similar technique as in the proof of Theorem 4, it seems possible to obtain an explicit expression of $\mathfrak{z}_n(\zeta_n; m, 4)$ too. However, the calculation to obtain a general formula is more difficult than one might imagine.

6 Some generalized functions

The function in (5) can be modified or generalized in several directions. When generalizing, it is also important to note that there exist corresponding generalized Stirling numbers.

In [12], we considered the function

$$\mathfrak{z}_n^\star(q; m, s) := \sum_{1\le i_1\le i_2\le\cdots\le i_m\le n-1}\frac{1}{(1-q^{i_1})^s(1-q^{i_2})^s\cdots(1-q^{i_m})^s}$$

and obtained explicit expressions of $\mathfrak{z}_n^\star(\zeta_n; m, s)$ for some m and s. We used some properties of complete homogeneous symmetric functions.

In [20], we studied the function

$$\mathfrak{t}_n(q; m, s) := \sum_{1\le i_1< i_2<\cdots< i_m\le n}\frac{1}{(1-q^{2i_1-1})^s(1-q^{2i_2-1})^s\cdots(1-q^{2i_m-1})^s}$$

and obtained the explicit polynomial forms when $q = \zeta_{2n}$. This function has some close relations with B-type q-generalized Stirling numbers ([3, 14]). In [6], this function was further generalized as

$$\mathfrak{t}_{n,N,a}(q; m, s) := \sum_{1\le i_1< i_2<\cdots< i_m\le n}\frac{1}{(1-q^{Ni_1-N+a})^s(1-q^{Ni_2-N+a})^s\cdots(1-q^{Ni_m-N+a})^s}$$

and obtained the values when $q = \zeta_{Nn}$. Here, N and a are integers with $N \ge 2$ and $0 \le a < N$.

Another approach is to consider the alternating sum. In [16], we studied the function

$$\widetilde{\mathfrak{Z}}_n(q; m, s) := \sum_{1 \le i_1 < i_2 < \cdots < i_m \le n-1} \frac{(-1)^{i_1 + i_2 + \cdots + i_m}}{(1 - q^{i_1})^s (1 - q^{i_2})^s \cdots (1 - q^{i_m})^s}\,.$$

This function has some close relations with alternating Stirling numbers. See [22, A140956] though the plus and minus are reversed. Then, in [15], we studied alternating q-multiple t-function of general level

$$\widetilde{\mathfrak{t}}_{n,N,a}(q; m, s) := \sum_{1 \le i_1 < i_2 < \cdots < i_m \le n} \frac{(-1)^{i_1 + i_2 + \cdots + i_m}}{(1 - q^{Ni_1 - N + a})^s (1 - q^{2i_2 - N + a})^s \cdots (1 - q^{2i_m - N + a})^s}\,.$$

7 Some q-multiple zeta functions whose denominator powers are not uniform

We can consider a more general function

$$\mathfrak{Z}_n(q; ; s_1, s_2, \dots, s_m) := \sum_{1 \le i_1 < i_2 < \cdots < i_m \le n-1} \frac{1}{(1 - q^{i_1})^{s_1} (1 - q^{i_2})^{s_2} \cdots (1 - q^{i_m})^{s_m}}\,. \tag{7}$$

In previous papers [6, 12, 13, 15, 16, 20], we have treated the function when $s = s_1 = \cdots = s_m$ as $\mathfrak{Z}_n(q; m, s) = \mathfrak{Z}_n(q; ; \underbrace{s, \dots, s}_{m})$. One reason for doing so was the background correspondence with Stirling numbers and their generalizations.

In fact, when the values of $s_1, \dots, s_m$ are not uniform, it is difficult to study because we cannot directly use the properties of Stirling numbers or symmetric functions. Furthermore, since we are taking q as a root of unity, it is rare for the values to be purely real, and an imaginary part appears, which is very cumbersome to handle. There are also several related conjectures in [2], but they do not seem to have been proven yet.

In fact, it seems quite difficult to provide explicit polynomials for individual multiple zeta values, but it appears that explicit formulas can be obtained for their sum or average.

For convenience, put

$$\mathcal{Y}_n(s_1, \dots, s_m) := \sum_{\sigma \in \langle s_1, s_2, \dots, s_m \rangle} \mathfrak{Z}_n(\zeta_n; ; \sigma)\,,$$

where the sum is taken over all distinct permutations of $\{s_1, s_2, \dots, s_m\}$.

Then, one of several new results is as follows.

Theorem 6. *For an integer r with $0 \le r \le m$, we have*

$$\mathcal{Y}_n(\underbrace{2, \dots, 2}_{r}, \underbrace{1, \dots, 1}_{m-r}) = \frac{1}{(r+1)n} \binom{m}{r} \left(\binom{n-1}{m} + (-1)^r \binom{n-1}{m+r+1} \right)\,.$$

Remark. When $r=0$ in Theorem 6, we have

$$\mathcal{Y}_n(\underbrace{1,\dots,1}_m)=\frac{1}{n}\binom{n}{m+1}=\frac{1}{m+1}\binom{n-1}{m}=\mathfrak{Z}_n(\zeta_n;m,1)\,,$$

which is (6), that is, [10, Theorem 9].

When $r=m$ in Theorem 6, we have

$$\mathcal{Y}_n(\underbrace{2,\dots,2}_m)=\frac{1}{(m+1)n}\left(\binom{n-1}{m}+(-1)^m\binom{n-1}{2m+1}\right)\,,$$

which is the identity in Theorem 4, that is, [13, Theorem 5].

Examples. When $n=7$, $m=5$ and $r=3$, we have

$$\begin{aligned}&\mathcal{Y}_n(2,2,2,1,1)\\&=\mathfrak{Z}_n(\zeta_n;;2,2,2,1,1)+\mathfrak{Z}_n(\zeta_n;;2,2,1,2,1)+\mathfrak{Z}_n(\zeta_n;;2,1,2,2,1)\\&\quad+\mathfrak{Z}_n(\zeta_n;;1,2,2,2,1)+\mathfrak{Z}_n(\zeta_n;;2,2,1,1,2)+\mathfrak{Z}_n(\zeta_n;;2,1,2,1,2)\\&\quad+\mathfrak{Z}_n(\zeta_n;;1,2,2,1,2)+\mathfrak{Z}_n(\zeta_n;;2,1,1,2,2)+\mathfrak{Z}_n(\zeta_n;;1,2,1,2,2)\\&\quad+\mathfrak{Z}_n(\zeta_n;;1,1,2,2,2)\\&=(-0.0853489+0.241918\sqrt{-1})+(0.116855+0.296646\sqrt{-1})\\&\quad+(0.214286+0.115717\sqrt{-1})+(0.112724)\\&\quad+(0.51392+0.406544\sqrt{-1})+(0.510709)\\&\quad+(0.214286-0.115717\sqrt{-1})+(0.51392-0.406544\sqrt{-1})\\&\quad+(0.116855-0.296646\sqrt{-1})+(-0.0853489-0.241918\sqrt{-1})\\&=2.14286\,.\end{aligned}$$

On the other hand, by $\binom{6}{9}=0$

$$\frac{1}{4\cdot 7}\binom{5}{3}\left(\binom{6}{5}+(-1)^3\binom{6}{9}\right)=\frac{10\cdot 6}{4\cdot 7}=\frac{15}{7}=2.14286\,.$$

When $n=17$, $m=5$ and $r=3$, we have

$$\begin{aligned}&\mathcal{Y}_{17}(2,2,2,1,1)\\&=(238.473-467.725\sqrt{-1})+(101.801+878.455\sqrt{-1})\\&\quad+(-104.-313.494\sqrt{-1})+(20.7385)\\&\quad+(-446.473+3423.97\sqrt{-1})+(-640.34)\\&\quad+(-104.+313.494\sqrt{-1})+(-446.473-3423.97\sqrt{-1})\\&\quad+(101.801-878.455\sqrt{-1})+(238.473+467.725\sqrt{-1})\\&=-1040\,.\end{aligned}$$

On the other hand,

$$\frac{1}{4\cdot 17}\binom{5}{3}\left(\binom{16}{5}+(-1)^3\binom{16}{9}\right)=\frac{10\cdot(4368-11440)}{4\cdot 17}=-\frac{28585}{7}=-1040\,.$$

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Institute of Mathematics
Henan Academy of Sciences
Zhengzhou 450046
CHINA
and
Department of Mathematics
Institute of Science Tokyo

Tokyo 152-8551
JAPAN
E-mail address: komatsu@zstu.edu.cn; komatsu.t.al@m.titech.ac.jp