TRANSLATION FORMULAE: A METHOD OF MEROMORPHIC CONTINUATION AND OTHER ANALYTIC QUESTIONS

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Abstract. In this report, we collate various translation formulae satisfied by some of the important Dirichlet series, such as the Riemann zeta function, Dirichlet L-functions, and their multivariable avatars. These translation formulae are often very useful in further analytic study of the concerned functions.

1. Introduction

For a complex number s with $\Re(s) > 1$, the Riemann zeta function is defined by

$$\zeta(s) := \sum_{n \geq 1} \frac{1}{n^s}.$$

The above series defines a holomorphic function on the complex half plane $\Re(s) > 1$. In 1859, Riemann [22] showed that $\zeta(s)$ admits a meromorphic continuation to the entire complex plane $\mathbb{C}$, with a simple pole at $s = 1$ of residue 1. He established the celebrated functional equation for $\zeta(s)$, which is given by

$$\zeta(s) = 2^s \pi^{s-1} \sin\Big(\frac{\pi s}{2}\Big) \Gamma(1-s)\zeta(1-s),$$

and it is valid as equality between meromorphic functions of $\mathbb{C}$. This functional equation allows one to determine the values of $\zeta(s)$ at negative integers.

After Riemann's work, several alternative approaches were developed to obtain the meromorphic continuation of $\zeta(s)$ and his method has been generalised for other Dirichlet series to find the corresponding functional equations. However, there exists another elegant, though less well-known, method of analytic continuation for the Riemann zeta function due to Ramanujan. In 1917, Ramanujan [20] proved that for $\Re(s) > 1$, the Riemann zeta function satisfies the translation formula

$$1 = \sum_{k \geq 0} \frac{(s-1)_{k+1}}{(k+1)!} \big(\zeta(s+k) - 1\big), \tag{1}$$

where for a complex number s and an integer $k \geq 1$, $(s)_k$ denotes the Pochhammer symbol defined by

$$(s)_k := s(s+1)\cdots(s+k-1).$$

Possibly Ramanujan's motivation was to obtain a series representation for the Euler's constant γ. Beside that, his translation formula can also be used to establish the meromorphic continuation

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of $\zeta(s)$ to the entire complex plane $\mathbb{C}$. Motivated by Ramanujan's work, in 1934, Ramaswami [21] obtained some interesting translation formulae for $\zeta(s)$, which he called the *recurrence* formulae for the ζ-function. For $\Re(s) > 1$, he derived the following identities satisfied by $\zeta(s)$:

$$\zeta(s)\left(1-2^{1-s}\right) = \sum_{k\geq 1} \frac{(s)_k \zeta(s+k)}{k! 2^{s+k}}, \tag{2}$$

$$\zeta(s)\left(1-3^{1-s}\right) = 1 + 2\sum_{k\geq 1} \frac{(s)_{2k} \zeta(s+2k)}{(2k)! 3^{s+2k}}, \tag{3}$$

$$\zeta(s)\left(1-2^{1-s}-3^{1-s}-6^{1-s}\right) = 1 + 2\sum_{k\geq 1} \frac{(s)_{2k}(s) \zeta(s+2k)}{(2k)! 6^{s+2k}}. \tag{4}$$

As an application of these identities, he established the meromorphic continuation of $\zeta(s)$ to the entire complex plane $\mathbb{C}$. He further showed that $\zeta(s)$ does not vanish in the region $\Re(s) \geq 1/2$ and $|s| \leq 8$. Many other extensions of Ramanujan and Ramaswami's translation formulae for the Riemann zeta function have been studied (see [3], [6], [10] and the references therein). In §2, we discuss some translation formulae for other important Dirichlet series and their meromorphic continuation using these translation formulae. In §3, we explore the translation formulae satisfied by the several variable avatars of Dirichlet series (which we call the multiple Dirichlet series) and their meromorphic continuation using these translation formulae. In §4, we discuss some more applications of these translation formulae to study further analytic properties of the multiple Dirichlet series.

2. Meromorphic continuation of $\zeta(s)$ and translation formulae for other Dirichlet series

We first give a brief idea to get the meromorphic continuation for $\zeta(s)$ on the entire complex plane $\mathbb{C}$ using Ramanujan's translation formula (1).

2.1. Meromorphic continuation of $\zeta(s)$: We need the following lemma.

Lemma 1. *Let $U \subseteq \mathbb{C}$ and k_0 be the smallest non-negative integer such that $\Re(s+k_0) > 1$ for all $s \in U$. Then the family of holomorphic functions*

$$\left(\frac{(s-1)_{k+1}}{(k+1)!} n^{-s-k}\right)_{n\geq 2, k\geq k_0},$$

is normally summable on every compact subset K of U.

Proof. Note that K is a compact subset of the complex half plane $\Re(s) > -k_0 + 1$. For every integer $n \geq 2$ and $k \geq k_0$, we have

$$\left\| \frac{1}{n^{s+k}} \right\|_K \leq \frac{1}{2^{k-k_0}} \left\| \frac{1}{n^{s+k_0}} \right\|_K.$$

Also, if $A := \sup_{s\in K} |s-1|$, then we complete the proof of the lemma by noticing that

$$\sum_{k\geq k_0} \left\| \frac{(s-1)_{k+1}}{(k+1)! 2^{k-k_0}} \right\|_K \leq \sum_{k\geq k_0} \frac{(A)_{k+1}}{(k+1)! 2^{k-k_0}} < \infty.$$

□

Note that by using the above lemma, we get that

$$\sum_{k\geq k_0} \frac{(s-1)_{k+1}}{(k+1)!}(\zeta(s+k)-1)$$

is a homolorphic function on the complex half plane $\Re(s) > -k_0 + 1$. We first prove the meromorphic continuation of $\zeta(s)$ on $\Re(s) > 0$, i.e., for the case $k_0 = 1$. From the above discussion, we see that the function

$$1 - \sum_{k\geq 1} \frac{(s-1)_{k+1}}{(k+1)!}(\zeta(s+k)-1)$$

is a holomorphic function on $\Re(s) > 0$. In view of the translation formula (1), if we set

$$(s-1)(\zeta(s)-1) = 1 - \sum_{k\geq 1} \frac{(s-1)_{k+1}}{(k+1)!}(\zeta(s+k)-1),$$

this gives a meromorphic continuation of $\zeta(s)$ on $\Re(s) > 0$ with a simple pole at $s = 1$ having residue 1. Now we use induction on k_0 to show that $\zeta(s)$ can be extended to a meromorphic function on $\Re(s) > -k_0+1$ with only a simple pole at $s = 1$ with residue 1. Assume that $k_0 \geq 2$. From Lemma 1, we have the function

$$1 - \sum_{k\geq k_0} \frac{(s-1)_{k+1}}{(k+1)!}(\zeta(s+k)-1)$$

is a holomorphic function on $\Re(s) > -k_0 + 1$. In view of the translation formula (1), we set

$$\sum_{k=0}^{k_0-1} \frac{(s-1)_{k+1}}{(k+1)!}(\zeta(s+k)-1) = 1 - \sum_{k\geq k_0} \frac{(s-1)_{k+1}}{(k+1)!}(\zeta(s+k)-1).$$

Note that by the induction hypothesis on k_0, for all $1 \leq k \leq k_0 - 1$, $(s+k-1)\zeta(s+k)$ are holomorphic functions on $\Re(s) > -k_0 + 1$ and this implies that $\zeta(s)$ has a meromorphic continuation on $\Re(s) > -k_0 + 1$ with only a simple pole at $s = 1$ with residue 1. Since the complex half planes of the form $\Re(s) > -k$ cover the entire complex plane $\mathbb{C}$, we get the meromorphic continuation of $\zeta(s)$ on $\mathbb{C}$.

2.2. **Translation formulae for other Dirichlet series.** This method of meromorphic continuation of the Riemann zeta function by using the translation formula can be generalised to other important Dirichlet series. Murty and Reece [17] considered the Dirichlet series attached to an arithmetical function f, defined by

$$D(s,f) := \sum_{n\geq 1} \frac{f(n)}{n^s},$$

and the associated Hurwitz series

$$D(s,x,f) := \sum_{n\geq 0} \frac{f(n)}{(n+x)^s},$$

for $0 < x < 1$. They proved the following theorem to get the meromorphic continuation of $D(s,x,f)$ from that of $D(s,f)$.

Theorem 1. [17, Theorem 2.4] *Assume that the Dirichlet series $D(s,f)$ has a meromorphic extension to the entire complex plane $\mathbb{C}$. Then $D(s,x,f)$ extends as a meromorphic function on $\mathbb{C}$, and we have*

$$D(s,x,f)-\frac{f(1)}{x^s}=D(s,f)+\sum_{r\geq 1}(-1)^k\frac{(s)_r}{r!}D(s+r,f)x^r.$$

Moreover, for any integer $k\geq 1$,

$$D(1-k,x,f)-\frac{f(1)}{x^{1-k}}=D(1-k,f)+\sum_{r=1}^{k-1}\frac{(k-1)_r}{r!}D(1-k+r,f)x^r.$$

As a special case, this gives the meromorphic continuation of the Hurwitz zeta function $\zeta(s,x)$. Saha [23] considered the Dirichlet series $D(s,f)$ attached to some special type of arithmetical function f. He derived some Ramanujan-type translation formulae for the related Dirichlet series and established the meromorphic continuation on the entire complex plane $\mathbb{C}$. He first considered the case of periodic arithmetical function f, i.e., $f:\mathbb{N}\to\mathbb{C}$ such that there exists an integer $q\geq 1$ and $f(n+q)=f(n)$ for all $n\in\mathbb{N}$. Then he proved the following theorem.

Theorem 2. [23] *Let f be a q-periodic arithmetical function. Then for every complex number s with $\Re(s)>1$, $D(s,f)$ and $D(s,x,f)$ satisfy the following translation formulae:*

$$\sum_{n=1}^{q}\frac{f(n)}{n^{s-1}}=\sum_{k\geq 0}\frac{(s-1)_{k+1}}{(k+1)!}q^{k+1}\left(D(s+k,f)-\sum_{n=1}^{q}\frac{f(n)}{n^{s+k}}\right),$$

$$\sum_{n=1}^{q}\frac{f(n)}{(n+x)^{s-1}}=\sum_{k\geq 0}\frac{(s-1)_{k+1}}{(k+1)!}q^{k+1}\left(D(s+k,x,f)-\sum_{n=1}^{q}\frac{f(n)}{(n+x)^{s+k}}\right).$$

Moreover, $D(s,f)$ and $D(s,x,f)$ can be extended to meromorphic functions on $\mathbb{C}$, with only possible simple pole at $s=1$ having residue $\left(\sum_{n=1}^{q}f(n)\right)/q$.

As a special case, the above theorem gives the meromorphic continuation of the Dirichlet L-function $L(s,\chi)$ and the twisted Hurwitz zeta function $D(s,x,\chi)$ associated with a Dirichlet character χ mod q. When $\chi=\chi_0$ is the trivial character mod q, both $L(s,\chi)$ and $D(s,\chi,x)$ have a simple pole at $s=1$ with residue $\phi(q)/q$. Otherwise, these functions can be extended to entire functions on $\mathbb{C}$.

Next, consider the Dirichlet series $D(s,f)$ associated to additive characters f. So we have,

$$f(n+m)=f(n)f(m),\qquad \text{for all } n,m\in\mathbb{N},$$

with $f(1)\neq 0$. To guarantee the convergence of $D(s,f)$ for some complex number s, we need to impose the condition $|f(1)|\leq 1$. Under this assumption, the Dirichlet series $D(s,f)$ converges absolutely for all $s\in\mathbb{C}$ with $\Re(s)>1$. Then, Saha in [23] proved the following theorem for such Dirichlet series.

Theorem 3. [23] *Let f be an additive character with $|f(1)|\leq 1$ and $f(1)\neq 1$. Then for every complex number s with $\Re(s)>1$, $D(s,f)$ and $D(s,x,f)$ satisfy the following translation* *formulae:*

$$f(1)=(1-f(1))D(s,f)+\sum_{k\geq 0}\frac{(s)_{k+1}}{(k+1)!}\left(D(s+k+1,f)-f(1)\right),$$

$$\frac{1}{x^s}=(1-f(1))D(s,x,f)+\sum_{k\geq 0}\frac{(s)_{k+1}}{(k+1)!}\left(D(s+k+1,x,f)-f(1)\right).$$

Moreover, $D(s,f)$ and $D(s,x,f)$ can be extended to entire functions on $\mathbb{C}$.

To prove Theorem 2, one first needs the binomial theorem, to get the identity

$$\frac{f(n)}{(n-q)^{s-1}}-\frac{f(n)}{n^{s-1}}=\sum_{k\geq 0}\frac{(s-1)_{k+1}}{(k+1)!}q^{k+1}\frac{f(n)}{n^{s+k}}.$$

Now for a complex number s with $\Re(s)>1$, one sums for $n>q$ on both sides of the above equation and apply a suitable variant of Lemma 1 to get the translation formula for $D(s,f)$ as in Theorem 2. A similar argument with some modification yields the translation formulae for $D(s,f)$ for an additive character f as in Theorem 3 and for the associated Hurwitz series $D(s,x,f)$. The proofs of the meromorphic continuation of these Dirichlet series follow analogously to the proof of the meromorphic continuation of the Riemann zeta function $\zeta(s)$, using Ramanujan's translation formula.

Recently, translation formulae extending the work of Ramaswami [21] and Apostol [3] on Dirichlet series associated with certain additive characters, have been established in [14].

3. Multiple Dirichlet series and their translation formulae

Let $r\geq 1$ be an integer. One of the most studied several variable avatar of the Riemann zeta function is the multiple zeta function, also known as the Euler-Zagier multiple zeta function, which is defined as

$$\zeta(s_1,\ldots,s_r):=\sum_{n_1>\cdots>n_r>0}\frac{1}{n_1^{s_1}\cdots n_r^{s_r}},$$

for an integer integer $r\geq 1$ and $(s_1,\ldots,s_r)\in U_r$, where

$$U_r:=\{(s_1,\ldots,s_r)\in\mathbb{C}^r:\Re(s_1+\cdots+s_i)>i\text{ for all }1\leq i\leq r\}.$$

The above series converges uniformly on every compact subset of the open set U_r, hence it defines a holomorphic function there. Its meromorphic extension to $\mathbb{C}^r$, the possible singularities and its special values have been studied by several mathematicians. Atkinson [5] considered $\zeta(s_1,s_2)$ and used the Poisson summation formula. Arakawa and Kaneko [4] consider the one-variable case by fixing $s_2,\ldots,s_r$ to the set of positive integers and obtained its special values in terms of the poly-Bernoulli numbers. For the meromorphic continuation to $\mathbb{C}^r$, Zhao [27] applied the generalised function theory, Akiyama, Egami and Tanigawa [1] used the Euler-Maclaurin summation formula and Matsumoto [11] employed the Mellin-Barnes integrals.

Let $r\geq 1$ be an integer and $f_1,\ldots,f_r$ be arithmetical functions. The multiple Dirichlet series (of depth r) associated with $f_1,\ldots,f_r$ is defined by

$$D_r\left(s_1,\ldots,s_r;f_1,\ldots,f_r\right):=\sum_{n_1>\cdots>n_r>0}\frac{f_1(n_1)\cdots f_r(n_r)}{n_1^{s_1}\cdots n_r^{s_r}}.$$

For the convergence of the above multiple Dirichlet series for some $(s_1, \ldots, s_r) \in \mathbb{C}^r$, we put the condition that $f_1, \ldots, f_r$ are bounded arithematical functions. With this condition, the above series defines a holomorphic function on U_r. When for all $1 \le i \le r$ and $n \in \mathbb{N}$, $f_i(n) = 1$, then we get back the multiple zeta function (of depth r). The moromorphic continuation for the case, $f_i = \chi_i$, where χ_i's are the Dirichlet characters mod q for all $1 \le i \le r$ was studied by Akiyama and Ishikawa in [2]. A more general multiple Dirichlet series, its extension to $\mathbb{C}^r$, and the possible singularities were studied by Furusho, Komori, Matsumoto and Tsumura in [8]. Now, we first collate the translation formulae satisfied by these multiple Dirichlet series.

3.1. **Translational formulae for the multiple zeta functions.** The first known translation formula for the multiple zeta function can be found in the work of Mehta, Saha and Viswanadham [13] (also indicated by Ecalle [7]).

Theorem 4. [13, Theorem 1] *For any integer $r \ge 2$ and for all $(s_1, \ldots, s_r) \in U_r$, we have*

$$\zeta(s_1 + s_2 - 1, s_3, \ldots, s_r) = \sum_{k \ge 0} \frac{(s_1 - 1)_{k+1}}{(k+1)!} \zeta(s_1 + k, s_2, \ldots, s_r), \tag{5}$$

where the series on the right-hand side converges normally on any compact subset of U_r.

Sketch of the proof. Let $s_1 \in \mathbb{C}$ be any complex number and $n_1 \ge 2$ be an integer. Then we have

$$\frac{1}{(n_1 - 1)^{s_1 - 1}} - \frac{1}{n_1^{s_1 - 1}} = \sum_{k \ge 0} \frac{(s_1 - 1)_{k+1}}{(k+1)!} \frac{1}{(n_1 - 1)^{s_1 + k}}.$$

Now we multiply on both sides of the above equation by $1/(n_2^{s_2} \cdots n_r^{s_r})$ and then take the sum over positive integers $n_1, \ldots, n_r$ with $n_1 > \cdots > n_r > 0$. For $(s_1, \ldots, s_r) \in U_r$, we apply the following lemma to get the translation formula (5). □

Lemma 2. *Let $r \ge 1$ be an integer. Let X be any open subset of $\mathbb{C}^r$ and k_0 be the smallest positive integer such that $(s_1 + k_0, s_2, \ldots, s_r) \in U_r$ for all $(s_1, \ldots, s_r) \in X$. Then the family of holomorphic functions*

$$\left(\frac{(s_1 - 1)_{k+1}}{(k+1)!} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}} \right)_{n_1 > \cdots > n_r > 0, k \ge k_0}$$

is normally summable on every compact subset of X.

3.2. **Meromorphic continuation of the multiple zeta functions.** We give a brief sketch of the meromorphic continuation of the multiple zeta functions on $\mathbb{C}^r$ using the translation formula (5).

Theorem 5. [13, Theorem 2] *Let $r \ge 2$ be an integer. Then the multiple zeta function of depth r can be extended to a meromorphic function on $\mathbb{C}^r$ and it satisfies the translation formula (5) as a meromorphic function of $\mathbb{C}^r$.*

Sketch of the proof. The idea of the proof is similar to that of the Riemann zeta function, using Ramanujan's translation formula. For any integer $N \ge 1$, we prove that $\zeta(s_1, \ldots, s_r)$ can be extended to a meromorphic function on $U_r(N)$, where

$$U_r(N) := \{(s_1, \ldots, s_r) \in \mathbb{C}^r : (s_1 + N, s_2, \ldots, s_r) \in U_r\}.$$

We prove this by induction on r and then induction on N.

First, assume that $r = 2$. Then $\zeta(s_1 + s_2 - 1)$ can be extended to a meromorphic function on $\mathbb{C}^2$. Now, if $N = 0$ then we know that $\zeta(s_1, s_2)$ is holomorphic on U_r. Assume that $N \geq 1$. In view of (5), we set

$$\sum_{k=0}^{N-1} \frac{(s_1 - 1)_{k+1}}{(k+1)!} \zeta(s_1 + k, s_2) = \zeta(s_1 + s_2 - 1) - \sum_{k \geq N} \frac{(s_1 - 1)_{k+1}}{(k+1)!} \zeta(s_1 + k, s_2).$$

By the induction hypothesis on N, for every integer $1 \leq k \leq N - 1$, $\zeta(s_1 + k, s_2)$ can be extended to a meromorphic function on $U_2(N)$. This implies that $\zeta(s_1, s_2)$, can be extended to a meromorphic function on $U_2(N)$, due to Lemma 2. Since the collection of the open subsets of the form $U_2(N)$ covers the whole space $\mathbb{C}^2$, we get that $\zeta(s_1, s_2)$ can be extended to a meromorphic function on $\mathbb{C}^2$.

Now assume that $r \geq 3$. By the induction hypothesis on r, we get $\zeta_{r-1}(s_1 + s_2 - 1, s_3, \ldots, s_r)$ can be extended to a meromorphic function on $\mathbb{C}^2$. Now, one needs to use induction on N as in the case of $r = 2$ to complete the proof of the theorem. Therefore, we skip the details here. □

Observe that, when $r = 2$ and $N = 1$, we have $(s_1 + s_2 - 2)(s_1 - 1)\zeta(s_1, s_2)$ is a holomorphic function on $U_2(1)$, i.e., $s_1 = 1, s_1 + s_2 = 2$ are the only possible simple polar hyperplans for $\zeta(s_1, s_2)$ intersecting with the open set $U_2(1)$. Similarly, one can calculate the possible polar hyperplane for $\zeta(s_1, s_2)$ passing through the open set $U_2(N)$ by using the translation formula (5), recursively. In general, if for each integer $1 \leq i \leq r$ and $k \geq 0$, $H_{i,k}$ denotes the hyperplane of $\mathbb{C}^r$ given by the equation

$$s_1 + \cdots + s_i = i - k,$$

then one gets the following theorem (see [13, Theorem 3]).

Theorem 6. *The multiple zeta function of depth r is holomorphic outside the union of the hyperplanes $H_{1,0}$ and $H_{i,k}$, where $2 \leq i \leq r$ and $k \geq 0$. It has at most a simple pole along each of these hyperplanes.*

This theorem has been proved in [27], [1], [19], [8] by various different methods. In [13], the authors also calculated the residue along each of these polar hyperplanes and as a consequence, they obtained the exact list of singularities for the multiple zeta functions (see [13, Theorems 4, 5], also [1]). In [1], the proof uses the Euler-Maclaurin summation formula, and as shown in [13], this process is complementary to the process of meromorphic continuation via the translation formula.

3.3. **Translation formulae for other multiple Dirichlet series.** In this section, we explore translation formulae for certain Dirichlet series. Using these translation formulae, the meromorphic continuation of these multiple Dirichlet series can be obtained by following the analogous approach to that for the multiple zeta functions.

Gun and Saha in [9] considered the multiple Lerch zeta function defined by

$$D_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r; e(\lambda_1), \ldots, e(\lambda_r)) := \sum_{n_1 > \cdots > n_r > 0} \frac{e(\lambda_1 n_1) \cdots e(\lambda_r n_r)}{(n_1 + \alpha_1)^{s_1} \cdots (n_r + \alpha_r)^{s_r}},$$

where $\lambda_i, \alpha_i \in (0,1]$ for all $1 \leq i \leq r$ and $e(\lambda) = \exp(2\lambda\pi\iota)$ for $\lambda \in \mathbb{R}$. Following the analogous method above, they proved the following:

Theorem 7. [9, Theorem 4] *Let $r \geq 2$ be an integer. Then for all $(s_1, \ldots, s_r) \in U_r$, we have*

$$\begin{aligned} e(\lambda_1) &\sum_{k\geq -1} \frac{(s_1)_{k+1}}{(k+1)!}(\alpha_2-\alpha_1)^{k+1} \\ &\times D_{r-1}(s_1+s_2+k+1, s_3, \ldots, s_r; \alpha_2, \ldots, \alpha_r; e(\lambda_1+\lambda_2), e(\lambda_3), \ldots, e(\lambda_r)) \\ =&(1-e(\lambda_1))D_r(s_1, \ldots, s_r; \alpha_1, \ldots, \alpha_r; e(\lambda_1), \ldots, e(\lambda_r)) \\ &+ \sum_{k\geq 0} \frac{(s_1)_{k+1}}{(k+1)!} D_r(s_1+k+1, s_2, \ldots, s_r; \alpha_1, \ldots, \alpha_r; e(\lambda_1), \ldots, e(\lambda_r)), \end{aligned}$$

where $(s_1)_0 := 1$. Moreover, the series on both sides converges normally on every compact subset of U_r.

Saha in [24] established translation formulae for the multiple Dirichlet series associated with additive and Dirichlet characters. Note that one can write multiple Dirichlet series associated with Dirichlet characters in the finite $\mathbb{C}$-linear combinations of certain multiple Dirichlet series associated with additive characters. So, it is enough to derive a translation formula for the multiple Dirichlet series associated with additive characters.

Theorem 8. [24, Theorem 6,7] *Let $r \geq 2$ be an integer. Let $f_1, \ldots, f_r$ be the additive characters such that $|g_i(1)| \leq 1$ for all $1 \leq i \leq r$, where $g_i(n) := f_1(n) \cdots f_i(n)$ for all $n \in \mathbb{N}$. Then the associated multiple Dirichlet series satisfy the following translation formulae in U_r depending on the value of $f_1(1)$:*

(a) If $f_1(1) = 1$; we have

$$D_{r-1}(s_1+s_2-1, s_3, \ldots, s_r; f_2, \ldots, f_r) = \sum_{k\geq 0} \frac{(s_1-1)_{k+1}}{(k+1)!} D_r(s_1+k, s_2, \ldots, s_r; f_1, \ldots, f_r).$$

(b) If $f_1(1) \neq 1$; we have

$$\begin{aligned} &f_1(1)D_{r-1}(s_1+s_2, s_3, \ldots, s_r; g_2, f_3, \ldots, f_r) \\ &= (1-f_1(1))D_r(s_1, \ldots, s_r; f_1, \ldots, f_r) + \sum_{k\geq 0} \frac{(s_1)_{k+1}}{(k+1)!} D_r(s_1+k+1, s_2, \ldots, s_r; f_1, \ldots, f_r). \end{aligned}$$

Moreover, the series on the right-hand sides of the above translation formulae converge normally on every compact subset of U_r.

The proofs of Theorem 7 and 8 can be given following the proof of Theorem 4. Murty and Sinha in [17] derived a certain translation formula for the poly-Hurwitz zeta functions and for the multiple Hurwitz zeta functions. Recently, Sahoo in [26] obtained a Ramasawami-type translation formula for the multiple zeta functions.

4. Other applications of the translation formulae for multiple Dirichlet series

In earlier sections, we have seen that the translation formulae for certain multiple Dirichlet series provide a more straightforward approach to establish their meromorphic continuation, to determine their singularities, and to calculate the residues along these singularities. In some cases, finding analogous translation formulae also yields much deeper insights about the (multiple) Dirichlet series. In this section, we present some applications of these translation formulae.

4.1. **Laurent type expansion of the Multiple Dirichlet series.** For functions of several complex variable, there is no unified notion of Laurent series expansion. Matsumoto, Onozuka and Wakabayashi in [12] introduced an algorithm to obtain such expressions for the multiple zeta functions using the Mellin-Barnes integral formula. Around the same time, Saha in [25] also devised a natural method to write down the Laurent-type expansion for the multiple zeta functions around integer points of $\mathbb{C}^r$. For example, he proved the following theorem.

Theorem 9. [25, Theorem 2] *Let $r \geq 1$ be an integer. Then for each integer $1 \leq i \leq r$, there exist meromorphic functions $\zeta^{\mathrm{Reg}}_{(1,\ldots,1)}(s_{i+1},\ldots,s_r)$ on $\mathbb{C}^{r-i}$ such that $\zeta^{\mathrm{Reg}}_{(1,\ldots,1)}(s_{i+1},\ldots,s_r)$ is holomorphic in a neighbourhood of $(1,\ldots,1) \in \mathbb{C}^r$ and the following identity holds between the meromorphic functions:*

$$\zeta(s_1,\ldots,s_r) = \sum_{i=0}^{r} \frac{\zeta^{\mathrm{Reg}}_{(1,\ldots,1)}(s_{i+1},\ldots,s_r)}{(s_1-1)\cdots(s_1+\cdots+s_i-i)}.$$

In fact, $\zeta^{\mathrm{Reg}}_{(1,\ldots,1)}(s_{i+1},\ldots,s_r)$ is defined by means of some convergent power series around the point $(1,\ldots,1)$, where the coefficients of the power series are certain *regularised values* of the multiple zeta functions $\zeta(s_{i+1},\ldots,s_r)$ and its formal derivatives. For example, if we take $r = 1$, then around the point $s = 1$,

$$\zeta^{\mathrm{Reg}}_{(1)}(s) = \sum_{k\geq 0} \frac{(-1)^k \gamma_k}{k!}(s-1)^k,$$

where for $k \geq 0$, γ_k are the classical Stieltjes constants defined by

$$\gamma_k := \lim_{N\to\infty} \left(\sum_{N>n\geq 1} \frac{\log^k n}{n} - \frac{\log^{k+1} N}{k+1} \right).$$

Therefore, around the point $s = 1$, he recovers the well-known Laurent series expansion of $\zeta(s)$ given by

$$\zeta(s) = \frac{1}{s-1} + \sum_{k\geq 0} \frac{(-1)^k \gamma_k}{k!}(s-1)^k.$$

To study the behaviour of $\zeta(s_1,\ldots,s_r)$ at the general integer points of $\mathbb{C}^r$, he used certain translation formulae for the cognate multiple zeta-star functions. Namely, for an integer $N \geq 1$, he first considered the following variant of the multiple zeta function

$$\zeta^\star(s_1,\ldots,s_r)_{\geq N} := \sum_{n_1\geq\cdots\geq n_r\geq N} \frac{1}{n_1^{s_1}\cdots n_r^{s_r}}.$$

He derived the following translation formula for $\zeta^\star(s_1,\ldots,s_r)_{\geq N}$ which was the key to his proof (see [25, eq. (28), (29)]). For $r=1$,

$$N^{1-s}=\sum_{k\geq 0}(-1)^k\frac{(s-1)_{k+1}}{(k+1)!}\zeta^\star(s+k)_{\geq N},$$

and for $r>1$,

$$\zeta^\star(s_1+s_{r-1}-1,s_3,\ldots,s_r)_{\geq N}=\sum_{k\geq 0}(-1)^k\frac{(s_1-1)_{k+1}}{(k+1)!}\zeta^\star(s_1+k,s_2,\ldots,s_r)_{\geq N}.$$

Using a more general framework, recently in a joint work with Saha in [16], we devised an algorithm to write down the Laurent type expansion for the multiple polylogarithm functions around the integer points of $\mathbb{C}^r$.

4.2. **Determining an open domain of convergence of the multiple polylogarithms.** For integer $r\geq 1$ and for each integer $1\leq i\leq r$, consider the arithematical functions f_i such that

$$f_i(n)=z_i^n \quad \text{for all} \quad n\in\mathbb{N},$$

where z_i is some complex number with $|z_i|\leq 1$. The multiple polylogarithm function of depth r is defined by

$$D(s_1,\ldots,s_r;z_1,\ldots,z_r)=D(s_1,\ldots,s_r;f_1,\ldots,f_r):=\sum_{n_1>\cdots>n_r>0}\frac{z_1^{n_1}\cdots z_r^{n_r}}{n_1^{s_1}\cdots n_r^{s_r}}.$$

The above series converges absolutely for $(s_1,\ldots,s_r)\in U_r$. But the above series make sense in a larger open domain of $\mathbb{C}^r$ compared to U_r in the sense of the limit

$$\lim_{N\to\infty}\sum_{N>n_1>\cdots>n_r>0}\frac{z_1^{n_1}\cdots z_r^{n_r}}{n_1^{s_1}\cdots n_r^{s_r}}. \tag{6}$$

This determines the open domain of (conditional) convergence of the multiple polylogarithm functions. Note that when $r=1$ and $z\neq 1$, the domain of convergence for the series $D(s,z)$ is $\Re(s)>0$, which is well-known and can be obtained using Abel's partial summation technique. The above limit can be studied by using the translation formulae satisfied by

$$\sum_{\substack{M>n_1>\cdots>n_r>0\\ n_1\geq N}}\frac{z_1^{n_1}\cdots z_r^{n_r}}{n_1^{s_1}\cdots n_r^{s_r}}.$$

For example, if we consider $r=1$ and integers $M>N\geq 2$, then we first need translation formulae for the finite sum

$$D(s,z)_{M,N}:=\sum_{M>n\geq N}\frac{z^n}{n^s}.$$

One can easily derive the following translation formula for all $s\in\mathbb{C}$:

$$(z-1)D(s,z)_{M,N}+\frac{z^N}{(N-1)^s}-\frac{z^M}{(M-1)^s}=\sum_{k\geq 0}\frac{(s)_{k+1}}{(k+1)!}D(s+k+1,z)_{M,N}.$$

Now for fixed $z\neq 1$ and $s\in\mathbb{C}$ such that $\Re(s)>0$, with a more careful analysis and following the proof of Lemma 1, we can prove that there exists an open set U containing s such that the series in right-hand side of the above equation converges normally on U and as $N\to\infty$, the sum is $O(N^{-\epsilon})$ for some $\epsilon > 0$. This implies that as $N \to \infty$, $D(s,z)_{M,N} \to 0$. Therefore, the partial sum

$$D(s,z)_{<N} := \sum_{N>n>0} \frac{z^n}{n^s},$$

converges as $N \to \infty$ for $\Re(s) > 0$. A similar method can be employed to obtain an open domain of convergence for certain multiple Dirichlet series.

Recently, in a joint work with Saha in [15], we have proved a more general result for the existence of the limit (6) to determine an open domain of (conditional) convergence for the multiple polylogarithms (see [15, Theorems 2, 3]). This open domain depends on the parameters $z_1, \ldots, z_r$. To achieve optimality in this domain, we have established a regularisation process for the multiple polylogarithms and their formal derivatives. This regularisation process is also key to study the local behaviour of the multiple polylogarithms by writing the Laurent-type expansions around integer points in $\mathbb{C}^r$ (see [16]).

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