THE PRODUCT FORMULA AND THE SKEW TYPE FORMULA FOR SCHUR P-, Q-MULTIPLE ZETA FUNCTIONS

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ABSTRACT. The Schur P-, Q-multiple zeta functions were defined by Nakasuji and Takeda inspired by the tableau representation of Schur P-, Q-functions. While a product of Schur P-functions expands as a linear combination of Schur P-functions, we obtain an analogous formula for the product of Schur P-multiple zeta functions by taking the summation over all permutations of the variables. Similarly, we obtain a formula for skew Schur Q-multiple zeta functions by taking the summation over permutations of the variables. This formula is more refined than the product formula of Schur P-multiple zeta functions, as it restricts the summation over the symmetric group to its subgroup.

1. SEMI-STANDARD MARKED SHIFTED TABLEAU

The *Schur P-, Q-multiple zeta functions* were introduced by Nakasuji-Takeda [NT]. These are defined as sums over combinatorial objects called semi-standard marked shifted tableau, similar to usual Schur P-, Q-functions. We now review the detailed definitions.

A *strict partition* is a finite sequence $\lambda = (\lambda_1, \dots, \lambda_r)$ of non-negative integers in strictly decreasing order: $\lambda_1 > \cdots > \lambda_r \geq 0$. The sum of λ_i is the *weight* of λ, denoted by $|\lambda|$. Throughout this article, let λ, μ, ν be strict partitions. For $\lambda = (\lambda_1, \dots, \lambda_r)$, let $SD(\lambda)$ be the subset of $\mathbb{Z}^2$:

$$SD(\lambda) = \{(i,j) \in \mathbb{Z}^2 | 1 \leq i \leq r, i \leq j \leq \lambda_i + i - 1\}$$

and depicted as a collection of square boxes arranged with the i-th row having λ_i boxes and indented by $i-1$ spaces. We call this the *shifted diagram* of shape λ. We identify λ with this collection of boxes, and we call it *shifted shape* λ and denote it by the same symbol λ as in Figure 1. We say that $(i,j) \in SD(\lambda)$ is a *corner* of λ if $(i+1,j) \notin SD(\lambda)$ and $(i,j+1) \notin SD(\lambda)$ and denote by $SC(\lambda) \subset SD(\lambda)$ the set of all corners of λ.

A *skew strict partition* is a pair of strict partitions λ/μ such that $\mu_i \leq \lambda_i$ for all i. The shifted diagram of skew strict partition λ/μ denoted by $SD(\lambda/\mu)$, is defined by $SD(\lambda/\mu) = SD(\lambda) \setminus SD(\mu)$ and is depicted as the diagram obtained by removing that of μ from that of λ. For clarity, we depict the diagram of λ/μ by coloring the boxes of μ in black as in Figure 1. When $\mu = \emptyset$ (i.e., $\mu_i = 0$ for all i), we identify skew strict partition λ/μ with λ. Unless otherwise stated, partitions are assumed to be non-empty. The corners of λ/μ are defined in the same way as in the case of normal partition.

$\lambda = (6,3)$ $\qquad$ $\lambda/\mu = (6,3,1)/(3,1)$

FIGURE 1. Example of diagram of non skew or skew strict partition

Let X be a set. A *shifted tableau* $T = (t_{ij})$ of shifted shape λ over X is a filling of $SD(\lambda)$ obtained by putting $t_{ij} \in X$ into the (i,j) box of $SD(\lambda)$. We denote by $ST(\lambda, X)$ the set of all shifted tableaux of shape λ over X. A *skew shifted tableau* is defined analogously.

Let $\mathbb{N}'$ be the set $\{1', 1, 2', 2, \ldots\}$ with the total ordering $1' < 1 < 2' < 2 < \cdots$. The symbols $1', 2', \ldots$ are said to be *marked*, and we denote $|k| = |k'| = k$ for each $k \in \mathbb{N}$.

Definition 1.1 (Semi-standard marked shifted tableau)**.** Let $\delta = \lambda/\mu$ (now, including $\mu = \emptyset$). Then, a *semi-standard marked shifted tableau* $M = (m_{ij}) \in ST(\delta, \mathbb{N}')$ is a shifted tableau satisfying the following conditions.

PST1: The entries of M are weakly increasing along each column and row of M.
PST2: For each $k = 1, 2, \ldots$, there is at most one k' per row.
PST3: For each $k = 1, 2, \ldots$, there is at most one k per column.

We denote by $\mathrm{QSST}(\delta)$ the set of semi-standard marked shifted tableaux of shape δ. We denote by $\mathrm{PSST}(\delta)$ the subset of $\mathrm{QSST}(\delta)$ with the following condition PST4:

PST4: There is no k' on the main diagonal.

We remark that sometimes the shifted tableau in $\mathrm{PSST}(\delta)$ is called *semi-standard shifted Young tableau* or *shifted Young tableau* (cf. [Ser]).

2. Schur P-, Q-multiple zeta function

Schur P-, Q-functions are symmetric functions expressed in terms of shifted tableaux as follows: for a normal or skew strict partition δ,

$$P_\delta = \sum_{(m_{ij}) \in \mathrm{PSST}(\delta)} \prod_{(i,j) \in SD(\delta)} x_{m_{ij}},$$

and

$$Q_\delta = \sum_{(m_{ij}) \in \mathrm{QSST}(\delta)} \prod_{(i,j) \in SD(\delta)} x_{m_{ij}}.$$

Schur P-, Q-multiple zeta function (Schur P-, Q-MZF) are defined similarly as sums over semi-standard shifted tableaux.

Definition 2.1. For a given shifted tableau $\boldsymbol{v} = (v_{ij}) \in ST(\delta, \mathbb{C})$ of variables, *the Schur P-multiple zeta function* and *the Schur Q-multiple zeta function* of shape δ are defined as

$$\zeta_\delta^P(\boldsymbol{v}) = \sum_{M \in \mathrm{PSST}(\delta)} \frac{1}{M^{\boldsymbol{v}}},$$

and

$$\zeta_\delta^Q(\boldsymbol{v}) = \sum_{M \in \mathrm{QSST}(\delta)} \frac{1}{M^{\boldsymbol{v}}},$$

respectively, where $M^{\boldsymbol{v}} = \prod_{(i,j) \in SD(\delta)} |m_{ij}|^{v_{ij}}$ for $M = (m_{ij}) \in \mathrm{QSST}(\delta)$.

Both functions $\zeta_\delta^P(\boldsymbol{v})$ and $\zeta_\delta^Q(\boldsymbol{v})$ converge absolutely in

$$W_\delta^Q = \left\{ \boldsymbol{v} = (v_{ij}) \in ST(\delta, \mathbb{C}) \;\middle|\; \begin{array}{l} \Re(v_{ij}) \geq 1 \text{ for all } (i,j) \in SD(\delta) \setminus SC(\delta), \\ \Re(v_{ij}) > 1 \text{ for all } (i,j) \in SC(\delta) \end{array} \right\}.$$

3. The Littlewood-Richardson rule for Schur P-, Q-MZFs

Since the Schur P-, Q-multiple zeta functions have a structure similar to that of the Schur P-, Q-functions, they are expected to have similar properties to those for Schur P-, Q-functions. It is known that Schur P-functions have the following expansion:

$$P_\mu P_\nu = \sum_{\lambda:\text{strict partition}} f_{\mu\nu}^\lambda P_\lambda. \tag{1}$$

It is known that the coefficients $f^{\lambda}_{\mu\nu}$ satisfying the equality (1) are unique for μ, ν, λ, and then $f^{\lambda}_{\mu\nu}$ are defined by this expansion and sometimes called *shifted Littlewood-Richardson coefficients* (cf. [Ser]). For these coefficients, it also holds that

$$Q_{\lambda/\mu} = \sum_{\nu:\text{strict partition}} f^{\lambda}_{\mu\nu} Q_{\nu}. \tag{2}$$

We obtain the following theorem which is analogous to the equality (1) for Schur P-multiple zeta functions by taking the summation over the symmetric group permuting all the variables.

Main Theorem 1. *Let μ, ν be strict partitions. Let $\boldsymbol{s} = (s_{ij}) \in ST(\mu, \mathbb{C}), \boldsymbol{t} = (t_{ij}) \in ST(\nu, \mathbb{C})$ be variables. Assume that the real parts of all variables $\{s_{ij}\}, \{t_{ij}\}$ are greater than 1. For $\lambda \in \mathcal{G}^P(\mu, \nu)$, let $\boldsymbol{u}_\lambda(\boldsymbol{s}, \boldsymbol{t}) \in ST(\lambda, \mathbb{C})$ be a tableau whose entries are the variables from the set $\{s_{ij}\} \cup \{t_{ij}\}$, with each variable appearing exactly once. Then the following equality holds for any such $\boldsymbol{u}_\lambda(\boldsymbol{s}, \boldsymbol{t})$:*

$$\sum_{\mathrm{Sym}(\boldsymbol{s},\boldsymbol{t})} \zeta^P_\mu(\boldsymbol{s}) \zeta^P_\nu(\boldsymbol{t}) = \sum_{\mathrm{Sym}(\boldsymbol{s},\boldsymbol{t})} \sum_{\lambda \in \mathcal{G}^P(\mu,\nu)} f^{\lambda}_{\mu\nu} \zeta^P_\lambda(\boldsymbol{u}_\lambda(\boldsymbol{s}, \boldsymbol{t}))$$

where the $\sum_{\mathrm{Sym}(\boldsymbol{s},\boldsymbol{t})}$ is the summation over the symmetric group permuting all the variables in $\{s_{ij}\} \cup \{t_{ij}\}$.

Here, $\mathcal{G}^P(\mu, \nu)$ is the set of all strict partitions λ such that $f^{\lambda}_{\mu\nu} > 0$. It is known that $|\lambda| = |\mu| + |\nu|$, and then there exists $\boldsymbol{u}_\nu(\boldsymbol{s}, \boldsymbol{t})$ satisfying the conditions in Main Theorem 1. For the expansion of skew Schur Q-multiple zeta function $\zeta^Q_{\lambda/\mu}(\boldsymbol{v})$, we obtain an analogue of (2) similarly by taking the summation over the symmetric group on all the variables v_{ij} in $SB(\boldsymbol{v})$, which is a part of the tableau $\boldsymbol{v} \in ST(\lambda/\mu)$ of variables defined as follows:

Definition 3.1 (Shifted arm, shifted body)**.** Let λ/μ be a skew strict partition, and $T \in ST(\lambda/\mu)$.

- The *shifted arm* of T is the part of the diagram which consists of the boxes $(1, m + \mu_1 + 1), \ldots, (1, \lambda_1)$ with the entries in these boxes where $m = \min\{\lambda_2, \sum_{i\geq 2}(\lambda_i - \mu_i)\}$. We denote the shifted arm of T by $SA(T)$.
- The *shifted body* of T is the part of the tableau obtained by removing $SA(T)$ from T, and is denoted by $SB(T)$.

Here, $\sum_{i\geq 2}(\lambda_i - \mu_i)$ is the number of boxes in the second row or below in the shifted diagram of λ/μ.

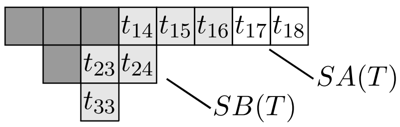


FIGURE 2. Example of the shifted arm and the shifted body : $\lambda/\mu = (8, 3, 1)/(3, 1)$

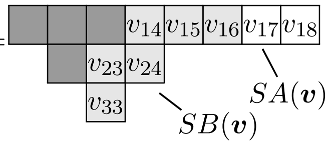
Main Theorem 2. *Let λ/μ be a skew strict partition. Let $\boldsymbol{v} = (v_{ij}) \in ST(\lambda/\mu, \mathbb{C})$ be variables. Assume that the real parts of the variables v_{1j} with $\min\{\lambda_2, \sum_{i\geq 2}(\lambda_i - \mu_i)\} + \mu_1 + 1 \leq j \leq \lambda_1 - 1$ are greater than or equal to 1 and those of other variables are greater than 1. Then the following equality holds:*

$$\sum_{\mathrm{Sym}(SB(\boldsymbol{v}))} \zeta^Q_{\lambda/\mu}(\boldsymbol{v}) = \sum_{\mathrm{Sym}(SB(\boldsymbol{v}))} \sum_{\nu \in \mathcal{G}^Q(\lambda/\mu)} f^{\lambda}_{\mu\nu} \zeta^Q_\nu(\boldsymbol{u}_\nu(\boldsymbol{v}))$$

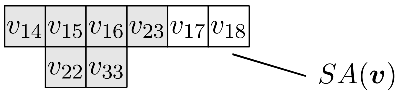
where the $\sum_{\mathrm{Sym}(SB(\boldsymbol{v}))}$ is the summation over the symmetric group permuting the variables $\{v_{ij}\}$ in $SB(\boldsymbol{v})$, and $\boldsymbol{u}_\nu(\boldsymbol{v})$ is a shifted tableau satisfying the following conditions:

- *The entries are the variables from the set $\{v_{ij}\}$, with each variable appearing exactly once.*
- *The shifted tableau contains $SA(\boldsymbol{v})$ in the right-most part of first row.*

Here, $\mathcal{G}^Q(\lambda/\mu)$ is the set of all partitions ν such that $f^{\lambda}_{\mu\nu} > 0$. Since the symmetric group $\mathrm{Sym}(SB(\boldsymbol{v}))$ is a subgroup of $\mathrm{Sym}(\boldsymbol{v})$, Main Theorem 2 can be said to be more refined than Main Theorem 1. We restrict the symmetric group by considering Littlewood-Richardson rule for semi-standard marked shifted tableau, which is a combinatorial rule calculating the coefficients $f^{\lambda}_{\mu\nu}$ by counting shifted tableaux satisfying some conditions. These rule were developed by Sagan [Sag] and Worley [W].

Remark 3.2. For example, for $\lambda/\mu = (8,3,1)/(3,1)$, $\nu = (6,2)$, and $\boldsymbol{v} =$  ,

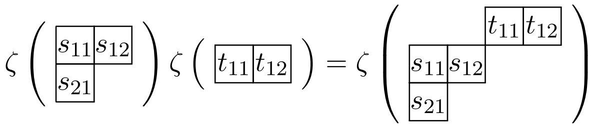
the following tableau $\boldsymbol{u}_\nu(\boldsymbol{v})$ is an example of shifted tableau satisfying the conditions in Main Theorem 2:

$\boldsymbol{u}_\nu(\boldsymbol{v}) =$  .

For $\nu \in \mathcal{G}^Q(\lambda/\mu)$, it is not obvious but ν_1 is greater than or equal to the number of boxes in $SA(\boldsymbol{v})$, and then there exists such a shifted tableau.

4. Future Work

For ordinary Schur multiple zeta functions, we obtained a refined formula not only for the skew type formula, but also the product formula ([H1]). This is because the product of the ordinary Schur multiple zeta functions is expressed as the skew Schur MZF as follows:



However, the product of Schur P-MZFs cannot be expressed as a skew Schur Q-multiple zeta function in the same way as in the case of ordinary Schur MZFs. Our future work is finding another way of restricting the symmetric group of the product formula of Schur P-MZFs.

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References

[H1] H. Hanaki, The Littlewood-Richardson rule for Schur multiple zeta functions, arXiv:2601.03970.
[H2] H. Hanaki, The Littlewood-Richardson rule for Schur P-, Q-multiple zeta functions, preprint.
[NT] M. Nakasuji, W. Takeda, Symmetric multiple zeta functions, *Canadian Journal of Mathematics*, 2025, 1-35.
[Sag] B. E. Sagan, Shifted tableaux, Schur Q-functions, and a conjecture of R. Stanley, *Journal of Combinatorial Theory*, Series A **45** (1987), 62-103.
[Ser] L. Serrano, The shifted plactic monoid, *Mathematische Zeitschrift*, **266** (2010), 363-392.
[W] D. R. Worley, A theory of shifted Young tableaux, Ph. D. thesis, Massachusetts Institute of Technology, 1984.

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