A RELATION BETWEEN THE RIEMANN ZETA FUNCTION AND THE FRACTIONAL PARTS OF A GEOMETRIC PROGRESSION

日本大学・理工学部　齋藤耕太
KOTA SAITO
COLLEGE OF SCIENCE & TECHNOLOGY
NIHON UNIVERSITY

Abstract. Kanado and the author found a relation between the Riemann zeta function and the fractional parts of a geometric progression. Furthermore, we obtained a criterion for the normality of algebraic numbers via the Riemann zeta function. In this paper, we aim to give a simpler proof of the relation in easier cases.

1. Introduction

Let $x > 0$ and $b \in \mathbb{Z}_{\geq 2}$. We expand x as

$$x = \sum_{i=-m}^{\infty} c_i b^{-i}, \quad (c_i \in \{0, 1, \ldots, b-1\}),$$

which is called the base-b expansion of x. We say that x is *simply normal* in base b if for every $a \in \{0, 1, \ldots, b-1\}$, we have

$$\lim_{l\to\infty} \frac{1}{l} \sum_{\substack{i\leq l\\ c_i=a}} 1 = \frac{1}{b}.$$

Furthermore, we say that x is *normal* in base b if x is simply normal to base b^k for every $k \in \mathbb{Z}_{>0}$. Borel introduced the notions of simple normality and normality in [Bor09]. He [Bor50] conjectured that all algebraic irrational numbers are normal in any base at the same time. However, we do not get any examples of normal algebraic numbers. We can not determine whether a given non-artificial number is normal or not. We refer [Bug12] to the reader who wants to study normal numbers.

In our previous research [KS25], we discovered a relation between the simple normality of $2^{p/q}$ and the Riemann zeta function. Before asserting the results, $\zeta(s)$ denotes the Riemann zeta function, and $\{x\}$ denotes the fractional part of x.

Theorem 1.1 ([KS25, Theorem 2.1]). *For all co-prime integers p and q with $1 \leq p < q$ and for all $b \in \mathbb{Z}_{\geq 2}$ which is not a q-th power of an integer, we have*

$$\sum_{0\leq d\leq l} \{b^{d+p/q}\} = \frac{l}{2} - \frac{1}{2\pi i} \sum_{1\leq |n|\leq b^l} \zeta\left(\frac{2\pi i n}{\log b}\right) \frac{e^{2\pi i n p/q}}{n} + o_{p,q,b}(l) \quad (\text{as } l \to \infty).$$

Corollary 1.2 ([KS25, Theorem 1.1]). *Let p and q be co-prime positive integers. Then, $2^{p/q}$ is simply normal in base 2 if and only if*

$$\lim_{l\to\infty} \frac{1}{l} \sum_{1\leq |n|\leq 2^l} \zeta\left(\frac{2\pi i n}{\log 2}\right) \frac{e^{2\pi i n p/q}}{n} = 0.$$

We [KS24] improved our results from the simple normality of $2^{p/q}$ in base 2 to the normality of general algebraic numbers in any base. The Benoulli polynomials play a key role in this generalisation. We define the *Bernoulli polynomials* $B_0(x), B_1(x), \ldots \in \mathbb{Q}[x]$ by the coefficients of the Maclaurin expansion of the following generating function:

$$\frac{te^{xt}}{e^t-1} = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} t^k.$$

For example, $B_0(x) = 1$, $B_1(x) = x - 1/2$, $B_2(x) = x^2 - x + 1/6$. It is well-known that

$$B_{k+1}(\{x\}) = -\lim_{N\to\infty} \frac{(k+1)!}{(2\pi i)^{k+1}} \sum_{1\le|n|\le N} \frac{e^{2\pi i\theta n}}{n^{k+1}} \tag{1.1}$$

for every $k \in \mathbb{Z}_{\ge 0}$. We [KS24] showed the following relation between the weighted discrete mean value of $\zeta(s)$ and the mean value of the Bernoulli polynomials over the fractional parts of a geometric progression.

Theorem 1.3 ([KS24, Theorem 2.2])**.** *Let $k \in \mathbb{Z}_{>0}$, let $d > 0$ and θ be real numbers. Then for every integer $N \ge 2$, we have*

$$\begin{aligned}&\frac{(k+1)!}{(2\pi i)^{k+1}} \sum_{1\le|n|\le N} \zeta(-k+2\pi idn)\frac{e^{2\pi i\theta n}}{n^{k+1}}\\ &= -d^k \sum_{0<h<d\log N} B_{k+1}(\{e^{(h+\theta)/d}\}) + O_{k,d,\theta}(\log\log N).\end{aligned}$$

We also obtained a similar result for $k = 0$ (see [KS24, Theorem 2.1]), but the big-O term should be replaced by

$$O_{d,\theta}\left(\sum_{0<h<d\log N} \min\left(1, \frac{1}{\|e^{(h+\theta)/d}\|Ne^{-h/d}}\right) + (\log N)^{2/3}(\log\log N)^{5/9} \right).$$

In [KS24], we simultaneously treated both cases $k \ge 1$ and $k = 0$. The case $k = 0$ is much more complicated than $k \ge 1$ because the Fourier expansion of $B_1(x)$ is not absolutely convergent (see (1.1) with $k = 0$). Thus, in this article, we aim to focus only on the easier case $k \ge 1$ and give a simpler proof.

Combining Theorem 1.3 and the result for $k = 0$, we found a criterion for the normality of algebraic numbers via the Riemann zeta function.

Corollary 1.4 ([KS24, Corollary 1.2])**.** *Let α be a positive algebraic irrational number. Let b be an integer greater than or equal to 2. Then α is normal in base b if and only if for every integer $k \ge 0$, we have*

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{1\le|n|\le N} \zeta\left(-k + \frac{2\pi in}{\log b}\right) \frac{e^{2\pi in\log_b \alpha}}{n^{k+1}} = 0.$$

There are many analogues of a relation between special values of the Riemann zeta function and the Bernoulli numbers. Theorem 1.3 can also be categorised as this analogue.

It is natural to extend our result to other zeta functions. Let us propose the following related question for the readers.

Question 1.5. *Can we prove an analogue of Theorem 1.3 for other zeta functions such as the Dirichlet L-function, Hurwitz zeta function, Lerch zeta function, automorphic L-function, Dedekind zeta function, p-adic zeta function, Multiple zeta function, and so on?*

Notation 1.6. For every set $A \subseteq \mathbb{R}$ and $a \in \mathbb{R}$, we define $A_{\geq a} = \{x \in A \colon x \geq a\}$. We also define $A_{>a}$ in a similar manner. Let $e(x) = e^{2\pi i x}$, and let $\log_b x = (\log x)(\log b)$.

We say that $f(x) = g(x) + o(h(x))$ (as $x \to \infty$) if for all $\epsilon > 0$ there exists $x_0 > 0$ such that $|f(x) - g(x)| \leq h(x)\epsilon$ for all $x \geq x_0$. If x_0 depends on some parameters $\epsilon, a_1, \ldots, a_n$, then we write $f(x) = g(x) + o_{a_1,\ldots,a_n}(h(x))$. We also say that $f(x) = g(x) + O(h(x))$ for all $x \geq x_0$ if there exists $C > 0$ such that $|f(x) - g(x)| \leq Ch(x)$ for all $x \geq x_0$. If C depends on some parameters $a_1, \ldots, a_n$, then we write $f(x) = g(x) + O_{a_1,\ldots,a_n}(h(x))$ for all $x \geq x_0$.

We note that $f(X) \ll g(X)$ and $f(X) \ll_{a_1,\ldots,a_n} g(X)$ as $f(X) = O(g(X))$ and $f(X) = O_{a_1,\ldots,a_n}(g(X))$ respectively, where $g(X)$ is non-negative. In addition, we write $f(X) \asymp g(X)$ if $f(X) \ll g(X) \ll f(X)$.

2. Auxiliary Lemmas

Lemma 2.1 ([Tit86, (2.1.8) and (4.12.3)]). *For every $s \in \mathbb{C} \setminus \{1\}$, we have $\zeta(s) = \chi(s)\zeta(1-s)$, where $\chi(s) = 2^{s-1}\pi^s \sec(\pi s/2)/\Gamma(s)$. Further, for any fixed $\sigma \in \mathbb{R}$ and for sufficiently large $t > 1$, we have*

$$\chi(\sigma + it) = (2\pi/t)^{\sigma+it-1/2}\, e^{i(t+\pi/4)} \left(1 + O\left(\frac{1}{t}\right)\right).$$

Lemma 2.2. *For all $a, b \in \mathbb{Z}$ and finite intervals $I \subset \mathbb{Z}$ with length $L \geq 2$, we have*

$$\int_a^b \left|\sum_{h\in I} e(-hx)\right| dx \ll (b-a)\log L.$$

Proof. By the periodicity of $e(\cdot)$, we have

$$\begin{aligned}
\frac{1}{b-a}\int_a^b \left|\sum_{h\in I} e(-hx)\right| dx &= \int_0^1 \left|\sum_{h\in I} e(-hx)\right| dx \ll \int_0^1 \min(L, \|x\|^{-1})dx \\
&= \left(\int_0^{L^{-1}} + \int_{L^{-1}}^{1-L^{-1}} + \int_{1-L^{-1}}^1\right) \min(L, \|x\|^{-1})dx \\
&= L\left(\int_0^{L^{-1}} + \int_{1-L^{-1}}^1\right) dx + 2\int_{L^{-1}}^{1/2} x^{-1}dx \\
&\ll \log L.
\end{aligned}$$

□

Lemma 2.3 ([Tit86, Lemma 4.2]). *Let $f(x)$ and $g(x)$ be real functions defined on $[a, b]$. Assume that $f(x)$ and $g(x)$ are twice differentiable on $[a, b]$, and $g(x)/f'(x)$ is monotonic throughout the interval $[a, b]$. Suppose that there exists $M > 0$ such that for every $x \in [a, b]$, we have $|f'(x)/g(x)| \geq M$. Then*

$$\left|\int_a^b g(x)e(f(x))dx\right| \ll \frac{1}{M}.$$

Lemma 2.4 ([Tit86, Lemma 4.4]). *Let f be a twice continuously differentiable real function defined on $[a, b]$. Suppose that there exists $\lambda > 0$ such that for every $x \in [a, b]$, we have*

$$|f''(x)| \geq \lambda_2^{-1/2}.$$

Then,

$$\int_a^b e(f(x))dx \ll \lambda_2^{-1/2},$$

where the implicit constant is absolute.

Lemma 2.5 ([Tit86, Lemma 4.5]). *Let $f(x)$ be a real function, twice differentiable, and let $f''(x) \geq r > 0$ or $f''(x) \leq r < 0$, throughout the interval $[a, b]$. Suppose that $g(x)/f'(x)$ is monotonic and $|g(x)| \leq M$. Then*

$$\int_a^b g(x)e(f(x)) \ll Mr^{-1/2}.$$

Lemma 2.6 ([Tit86, Lemma 4.10]). *Let $f(x)$ be a real function with a continuous and steadily decreasing derivative $f'(x)$ in (a, b), and let $f'(b) = \alpha$, $f'(a) = \beta$. Let $g(x)$ be a real positive decreasing function, with a continuous derivative $g'(x)$, and let $|g'(x)|$ be steadily decreasing. Then*

$$\sum_{a<n\leq b} g(n)e(f(n)) = \sum_{\alpha-\eta<h<\beta+\eta} \int_a^b g(x)e(f(x) - hx)dx + O(g(a)\log(\beta - \alpha + 2)) + O(|g'(a)|),$$

where η is any positive constant less than 1.

Lemma 2.7 (Stationary phase integral [Hux94, Theorem 2]). *Let $f(u)$ be a real function, four times continuously differentiable and let $g(u)$ be a real function, three times continuously differentiable on the interval $[a, b]$. Let T_1, T_2, L_1, L_2, B be positive parameters with $L_1 \geq b - a$. Suppose that for all $x \in [a, b]$ we have*

$$f''(u) \leq -T_1/(B^2L_1^2)$$

and

$$|f^{(r)}(u)| \leq B^rT_1/L_1^r, \quad |g^{(s)}(u)| \leq B^sT_2/L_2^s$$

for $r = 2, 3, 4$ and $s = 0, 1, 2$. Assume that $f'(u)$ changes sign from positive to negative at $u = c$ with $a < c < b$. If T_1 is sufficiently large in terms of B, then we have

$$\begin{aligned}\int_a^b g(u)e(f(u))du = &\frac{g(c)e(f(c) - 1/8)}{\sqrt{|f''(c)|}} + \frac{g(b)e(f(b))}{2\pi if'(b)} - \frac{g(a)e(f(a))}{2\pi if'(a)} \\ &+ O\left(\frac{B^4L_1^4T_2}{T_1^2}((c-a)^{-3} + (b-c)^{-3})\right) \\ &+ O\left(\frac{B^{13}L_1T_2}{T_1^{3/2}}\left(1 + \frac{L_1}{B^4L_2}\right)^2\right).\end{aligned}$$

3. Proof of Theorem 1.3

Let σ, d, and θ be positive real numbers. We set

$$S_N = \sum_{1\leq n\leq N} \zeta(-\sigma + 2\pi idn)\frac{e(\theta n)}{n^{1+\sigma}}$$

for every positive integer N. We consider σ, d, and θ as constants, and so we omit the dependency of these parameters.

Step 1: Applying the functional equation. By applying Lemma 2.1, for every $t > 1$, we obtain

$$\begin{aligned}\zeta(-\sigma + 2\pi idt) &= \chi(-\sigma + 2\pi idt)\zeta(1 + \sigma - 2\pi idt) \\ &= (dt)^{\sigma - 2\pi idt + 1/2}\, e^{i(2\pi dt + \pi/4)}\zeta(1 + \sigma - 2\pi idt)\left(1 + O\left(1/t\right)\right).\end{aligned}$$

By $\sigma > 0$, the error term is

$$\ll \left| t^{\sigma + 2\pi idt + 1/2} e^{i(2\pi dt + \pi/4)}\zeta(1 + \sigma - 2\pi idt)\right| (1/t) \ll_\sigma t^{-1/2+\sigma}.$$

This implies that

$$\begin{aligned}S_N &= \sum_{1\le n\le N} (dn)^{\sigma - 2\pi idn + 1/2}\, e^{i(2\pi dn + \pi/4)}\zeta(1 + \sigma - 2\pi idn)\frac{e(\theta n)}{n^{1+\sigma}} + O(1) \\ &= D \sum_{1\le n\le N} \zeta(1 + \sigma - 2\pi idn)\frac{e(\log(\frac{e}{dn}) + \theta n)}{n^{1/2}} + O(1),\end{aligned}$$

where $D = d^{\sigma+1/2}e(1/8)$. By the definition of ζ, we obtain

$$\zeta(1 + \sigma - 2\pi idn) = \sum_{m=1}^{\infty} \frac{1}{m^{1+\sigma-2\pi idn}} = \sum_{m=1}^{\infty} \frac{e(dn\log m)}{m^{1+\sigma}},$$

and hence

$$\begin{aligned}S_N &= D \sum_{1\le n\le N} \frac{e(dn\log(\frac{e}{dn}) + \theta n)}{n^{1/2}} \sum_{m=1}^{\infty} \frac{e(dn\log m)}{m^{1+\sigma}} + O(1) \\ &= D\sum_{m=1}^{\infty} \frac{1}{m^{1+\sigma}} \sum_{1\le n\le N} \frac{e(F_m(n))}{n^{1/2}} + O(1),\end{aligned}$$

where $F_m(x) = dx\log(\frac{me}{dx}) + \theta x$.

Step 2: Transforming the exponential sum to an exponential integral. A simple calculation leads to

$$F_m'(x) = d\log\left(\frac{m}{dx}\right) + \theta.$$

By applying Lemma 2.6 with $a \coloneqq 1$, $b \coloneqq N$, $g(x) \coloneqq x^{-1/2}$, and $f(x) \coloneqq F_m(x)$, we have

$$\begin{aligned}\sum_{1\le n\le N} \frac{e(F_m(n))}{n^{1/2}} &= \sum_{\alpha-\eta<h<\beta+\eta} \int_1^N \frac{e(F_m(x) - hx)}{x^{1/2}}dx \\ &\quad + O(\log(\beta - \alpha + 2)) + O(1),\end{aligned}$$

where η is any positive constant < 1, and we set

$$\begin{aligned}\alpha = \alpha(m) &= F_m'(N) = d\log\left(\frac{m}{dN}\right) + \theta, \\ \beta = \beta(m) &= F_m'(1) = d\log\left(\frac{m}{d}\right) + \theta.\end{aligned} \tag{3.1}$$

Since

$$\beta - \alpha = d\left(\log\left(\frac{m}{d}\right) - \log\left(\frac{m}{dN}\right)\right) = d\log N,$$

we obtain

$$\sum_{1\le n\le N}\frac{e(F_m(n))}{n^{1/2}}=\sum_{\alpha-\eta<h<\beta+\eta}\int_1^N\frac{e(F_m(x)-hx)}{x^{1/2}}dx+O(\log\log N).$$

Therefore,

$$S_N=D\sum_{m=1}^{\infty}\frac{1}{m^{1+\sigma}}\sum_{\alpha-\eta<h<\beta+\eta}\int_1^N\frac{e(F_m(x)-hx)}{x^{1/2}}dx+O(\log\log N). \tag{3.2}$$

Step 3: Truncating the range of the integral. Let X be a parameter of a positive integer satisfying $2\le X\le N^{1/2}$. We will choose X as a sufficiently large positive constant. In this step, we show that

$$S_N=D\sum_{m=1}^{\infty}\frac{1}{m^{1+\sigma}}\sum_{\alpha-\eta<h<\beta+\eta}\int_X^N\frac{e(F_m(x)-hx)}{x^{1/2}}dx+O(X^{1/2}\log\log N). \tag{3.3}$$

To prove (3.3), we see that

$$\begin{aligned}&\sum_{\alpha-\eta<h<\beta+\eta}\int_1^X\frac{e(F_m(x)-hx)}{x^{1/2}}dx\\ &\ll\int_1^X\frac{1}{x^{1/2}}\left|\sum_{\alpha-\eta<h<\beta+\eta}e(-hx)\right|dx\\ &\ll\sum_{1\le r\le\lceil\frac{\log X}{\log 2}\rceil}2^{-r/2}\int_{2^{r-1}}^{\min(2^r,X)}\left|\sum_{\alpha-\eta<h<\beta+\eta}e(-hx)\right|dx.\end{aligned}$$

Therefore, since the length of the interval of h is $\asymp\log N$, Lemma 2.2 leads to

$$\begin{aligned}&\sum_{1\le r\le\lceil\frac{\log X}{\log 2}\rceil}2^{-r/2}\int_{2^{r-1}}^{\min(2^r,X)}\left|\sum_{\alpha-\eta<h<\beta+\eta}e(-hx)\right|dx\\ &\ll\sum_{1\le r\le\lceil\frac{\log X}{\log 2}\rceil}2^{r/2}\log\log N\ll X^{1/2}\log\log N.\end{aligned}$$

Step 4: Restricting the range of h. Let

$$\tilde\beta=\tilde\beta(m)=d\log\left(m/(dX)\right)+\theta. \tag{3.4}$$

In this step, let us show that

$$\begin{aligned}&\sum_{\alpha-\eta<h<\alpha+2d}\int_X^N\frac{e(F_m(x)-hx)}{x^{1/2}}dx\ll 1\\ &\sum_{\tilde\beta-2d<h<\beta+\eta}\int_X^N\frac{e(F_m(x)-hx)}{x^{1/2}}dx\ll X^{1/2}\log X.\end{aligned} \tag{3.5}$$

To show these estimates, let

$$f(x)=f_{m,h}(x)=F_m(x)-hx=dx\log\left(\frac{me}{dx}\right)+(\theta-h)x,$$

and it follows that

$$f'_{m,h}(x) = d\log\left(\frac{m}{dx}\right) + \theta - h, \tag{3.6}$$

$$f''_{m,h}(x) = -d/x. \tag{3.7}$$

Further, we set

$$\xi = \xi_{m,h} = \frac{m}{d}e^{(h-\theta)/d}. \tag{3.8}$$

By a simple calculation, we have $f'_{m,h}(x) = 0 \iff x = \xi$. Furthermore, let

$$G(x) = x^{1/2}f'_{m,h}(x) = x^{1/2}\left(d\log\left(\frac{m}{dx}\right) + \theta - h\right).$$

Then we have

$$G'(x) = \frac{1}{2x^{1/2}}f'_{m,h}(x) + x^{1/2}f''_{m,h}(x) = \frac{1}{2x^{1/2}}\left(d\log\left(\frac{m}{de^2x}\right) + \theta - h\right).$$

Therefore, we obtain the following equivalences:

$$G(x) > 0 \iff x < \xi, \qquad G'(u) > 0 \iff x < e^{-2}\xi,$$

which implies that $G(x)$ is

(1) positive and increasing on $(0, e^{-2}\xi)$;
(2) positive and decreasing on $(e^{-2}\xi, \xi)$;
(3) negative and decreasing on (ξ, ∞).

In the case $\alpha - \eta < h < \alpha + 2d$, we have

$$d\log\left(\frac{m}{dN}\right) + \theta - \eta < h < d\log\left(\frac{m}{dN}\right) + \theta + 2d \iff Ne^{-2} < \xi < Ne^{\eta/d}.$$

We decompose the integral as follows:

$$\int_X^N \frac{e(f(x))}{x^{1/2}}dx = \int_X^{e^{-4}N} \frac{e(f(x))}{x^{1/2}}dx + \int_{e^{-4}N}^N \frac{e(f(x))}{x^{1/2}}dx =: T_1 + T_2.$$

For T_1, the inequality $e^{-4}N < e^{-2}\xi$ implies that $G(x)$ is positive and increasing on $[X, e^{-4}N]$, and hence

$$|G(x)| = G(x) \geq G(X).$$

Lemma 2.3 leads to

$$T_1 \ll \frac{1}{G(X)} = \frac{1}{X^{1/2}(\tilde{\beta} - h)} \ll 1$$

for $\alpha - \eta < h < \alpha + 2d$.

For T_2, by partial integration,

$$\begin{aligned} T_2 &= \left[x^{-1/2}\int_{e^{-4}N}^x e(f(y))dy\right]_{x=e^{-4}N}^N + \frac{1}{2}\int_{e^{-4}N}^N x^{-3/2}\left(\int_{e^{-4}N}^x e(f(y))dy\right)dx \\ &\ll N^{-1/2}\left|\int_{e^{-4}N}^N e(f(y))dy\right| + \int_{e^{-4}N}^N x^{-3/2}\left|\int_{e^{-4}N}^x e(f(y))dy\right|dx. \end{aligned}$$

By Lemma 2.4 and (3.7), we obtain $T_2 \ll 1$. Therefore, we get

$$\sum_{\alpha-\eta<h<\alpha+2d} \int_X^N \frac{e(F_m(x) - hx)}{x^{1/2}}dx \ll 1.$$

In the case $\tilde{\beta} - 2d < h < \beta + \eta$, we have

$$d \log\left(\frac{m}{dX}\right) + \theta - 2d < h < d \log\left(\frac{m}{d}\right) + \theta + \eta \iff e^{-\eta} < \xi < Xe^2.$$

We decompose the integral as follows:

$$\int_X^N \frac{e(f(x))}{x^{1/2}} dx = \int_X^{e^3X} \frac{e(f(x))}{x^{1/2}} dx + \int_{e^3X}^N \frac{e(f(x))}{x^{1/2}} dx =: \tilde{T}_1 + \tilde{T}_2.$$

For $\tilde{T}_1$, by using the trivial estimate, we have $T_1 \ll X^{1/2}$.

For $\tilde{T}_2$, the inequality $\xi < e^2X$ implies that $G(x)$ is negative and decreasing on $[e^3X, N]$, and hence

$$|G(x)| = -G(x) \geq -G(e^3X).$$

Lemma 2.3 leads to

$$\tilde{T}_2 \ll \frac{1}{-G(e^3X)} = \frac{1}{e^{3/2}X^{1/2}(h - (\tilde{\beta} - 3d))} \ll \frac{1}{X^{1/2}}$$

for $\tilde{\beta} - 2d < h < \beta + \eta$. Therefore, we get

$$\sum_{\tilde{\beta}-2d<h<\beta+\eta} \int_X^N \frac{e(F_m(x) - hx)}{x^{1/2}} dx \ll X^{1/2} \log X$$

since $\beta - \tilde{\beta} \asymp \log X$.

Step 5. Decomposing the integral. Combining (3.3) and (3.5), we have

$$S_N = D \sum_{m=1}^{\infty} \frac{1}{m^{1+\sigma}} \sum_{\alpha+2d<h<\tilde{\beta}-2d} \int_X^N \frac{e(F_m(x) - hx)}{x^{1/2}} dx + O\Big(X^{1/2} \log\log N + X^{1/2} \log X\Big).$$

We observe that

$$\alpha + 2d < h < \tilde{\beta} - 2d \iff Xe^2 < \xi < Ne^{-2}.$$

Let δ be a positive real parameter in $(1/\log\log N, 1/2)$, which will be chosen as a sufficiently small positive constant. We now decompose $[X, N]$ into 4 intervals as follows:

$$\begin{aligned}[X, N] &= [X, e^{-2}\xi] \cup [e^{-2}\xi, (1-\delta)\xi] \cup [(1-\delta)\xi, (1+\delta)\xi] \cup [(1+\delta)\xi, N] \\ &=: I_1 \cup I_2 \cup I_3 \cup I_4.\end{aligned}$$

In this step, let us show that

$$\sum_{\alpha+2d<h<\tilde{\beta}-2d} \int_{I_1 \cup I_2 \cup I_4} \frac{e(F_m(x) - hx)}{x^{1/2}} dx \ll X^{-1/2} \log\log N. \tag{3.9}$$

On I_1, the function $G(x)$ is positive and monotonically increasing on I_1. For each $h \in (\alpha + 2d, \tilde{\beta} - 2d) \cap \mathbb{Z}$ and $x \in I_1$, we have

$$|G(x)| = G(x) \geq G(X) = X^{1/2}\left(d \log\left(\frac{m}{dX}\right) + \theta - h\right) = X^{1/2}(\tilde{\beta} - h).$$

Therefore, Lemma 2.3 leads to

$$\sum_{\alpha+2d<h<\tilde{\beta}-2d} \int_{I_1} \frac{e(F_m(x)-hx)}{x^{1/2}}dx \ll \sum_{\alpha+2d<h<\tilde{\beta}-2d} \frac{X^{-1/2}}{\tilde{\beta}-h} \ll X^{-1/2}\log\log N.$$

On I_2, the function $G(x)$ is positive and monotonically decreasing on I_2. For each $h \in (\alpha+2d, \tilde{\beta}-2d) \cap \mathbb{Z}$ and $x \in I_2$, we have

$$\begin{aligned}|G(x)| = G(x) \geq G((1-\delta)\xi) &\gg \xi^{1/2}\left(d\log\left(\frac{m}{d(1-\delta)\xi}\right)+\theta-h\right)\\ &\gg \delta\xi^{1/2} \gg \delta m^{1/2}e^{(\theta-h)/(2d)}.\end{aligned}$$

Therefore, Lemma 2.3 leads to

$$\begin{aligned}\sum_{\alpha+2d<h<\tilde{\beta}-2d} \int_{I_2} \frac{e(F_m(x)-hx)}{x^{1/2}}dx &\ll \frac{1}{\delta m^{1/2}} \sum_{\alpha+2d<h<\tilde{\beta}-2d} e^{(h-\theta)/(2d)}\\ \ll \frac{1}{\delta m^{1/2}}e^{(\tilde{\beta}-\theta)/(2d)} &\ll \frac{1}{\delta X^{1/2}} \leq X^{-1/2}\log\log N,\end{aligned}$$

where we apply $\delta > 1/\log\log N$ to the last inequality.

On I_4, the function $G(x)$ is negative and monotonically decreasing. For each $h \in (\alpha+2d, \tilde{\beta}-2d) \cap \mathbb{Z}$ and $x \in I_4$, we have

$$\begin{aligned}|G(x)| = -G(x) \geq -G((1+\delta)\xi) &\gg \xi^{1/2}\left(d\log\left(\frac{d(1+\delta)\xi}{m}\right)+h-\theta\right)\\ &\gg \delta\xi^{1/2} \gg \delta m^{1/2}e^{(\theta-h)/(2d)}.\end{aligned}$$

Therefore, similarly with I_2, Lemma 2.3 leads to

$$\sum_{\alpha+2d<h<\tilde{\beta}-2d} \int_{I_4} \frac{e(F_m(x)-hx)}{x^{1/2}}dx \ll X^{-1/2}\log\log N,$$

which completes (3.9).

Step 6: Applying the stationary phase integral. By combining (3.3) and (3.9), we have

$$\begin{aligned}S_N = D\sum_{m=1}^{\infty}\frac{1}{m^{1+\sigma}} &\sum_{\alpha+2d<h<\tilde{\beta}-2d} \int_{(1-\delta)\xi}^{(1+\delta)\xi} \frac{e(F_m(x)-hx)}{x^{1/2}}dx\\ &+O\Big(X^{1/2}\log\log N + X^{1/2}\log X\Big).\end{aligned} \tag{3.10}$$

The goal of this step is to show

$$\begin{aligned}S_N = D\sum_{m=1}^{\infty}\frac{1}{m^{1+\sigma}} &\sum_{\alpha+2d<h<\tilde{\beta}-2d} e(-1/8)d^{-1/2}e(d\xi)\\ &+O\Big(\delta^{-3}X^{-3/2} + X^{-1/2}\delta^{-1} + X^{1/2}\log\log N + X^{1/2}\log X\Big).\end{aligned} \tag{3.11}$$

For applying Lemma 2.7, let $g(x) = x^{-1/2}$, and on $I_3 = [(1-\delta)\xi, (1+\delta)\xi]$, the following properties hold:

- $f_{m,h}^{(2)}(x) = -d/x \asymp \xi^{-1} = \xi/\xi^2$;

- $|f_{m,h}^{(r)}(x)| \ll_r \xi^{-r+1} = \xi/\xi^r$ for $r = 2, 3, 4$;
- $|g^{(s)}(x)| \ll \xi^{-1/2-s} = \xi^{-1/2}/\xi^s$ for $s = 0, 1, 2$;
- $f'(x)$ changes sign from positive to negative at $x = \xi$ by (3.6);
- ξ is sufficiently large in terms of d, σ, and θ because $\xi \geq Xe^2$.

Therefore, by Lemma 2.7 with

$$f(x) \coloneqq f_{m,h}(x), \quad g(x) \coloneqq x^{-1/2}, \quad a \coloneqq (1-\delta)\xi, \quad b \coloneqq (1+\delta)\xi$$
$$T_1 \coloneqq \xi, \quad T_2 \coloneqq \xi^{-1/2}, \quad L_1 = L_2 \coloneqq \xi, \quad c \coloneqq \xi,$$

we have

$$\int_{(1-\delta)\xi}^{(1+\delta)\xi} \frac{e(F_m(x) - hx)}{x^{1/2}} dx = \frac{\xi^{-1/2} e(f(\xi) - 1/8)}{\sqrt{|f''(\xi)|}} + \frac{b^{-1/2} e(f(b))}{2\pi i f'(b)} - \frac{a^{-1/2} e(f(a))}{2\pi i f'(a)}$$
$$+ O\left(\frac{\xi^4 \xi^{-1/2}}{\xi^2} \delta^{-3} \xi^{-3}\right) + O\left(\frac{\xi \cdot \xi^{-1/2}}{\xi^{3/2}}\right).$$

In addition, we obtain

$$\frac{b^{-1/2} e(f(b))}{2\pi i f'(b)} - \frac{a^{-1/2} e(f(a))}{2\pi i f'(a)} \ll \xi^{-1/2}\left(\frac{1}{|f'((1+\delta)\xi)|} + \frac{1}{|f'((1-\delta)\xi)|}\right) \ll \xi^{-1/2}\delta^{-1},$$

and the other errors are

$$\frac{\xi^4 \xi^{-1/2}}{\xi^2} \delta^{-3} \xi^{-3} + \frac{\xi \cdot \xi^{-1/2}}{\xi^{3/2}} \ll \delta^{-3} \xi^{-3/2} + \xi^{-1},$$

and hence $f(\xi) = d\xi$ implies that

$$\int_{(1-\delta)\xi}^{(1+\delta)\xi} \frac{e(F_m(x) - hx)}{x^{1/2}} dx = e(-1/8) d^{-1/2} e(d\xi) + O\Big(\delta^{-3}\xi^{-3/2} + \xi^{-1/2}\delta^{-1}\Big).$$

For any fixed $a > 0$, we obtain

$$\sum_{\alpha + 2d < h < \tilde{\beta} - 2d} \xi^{-a} = \sum_{\alpha + 2d < h < \tilde{\beta} - 2d} \left(\frac{d}{m} e^{(h-\theta)/d}\right)^a \ll \frac{1}{m^a} \frac{m^a}{X^a} = X^{-a}.$$

Therefore, the proof of (3.11) is complete.

Step 7: Switching the sums and completion of the proof. By (3.11), we obtain

$$S_N = d^\sigma \sum_{m=1}^{\infty} \frac{1}{m^{1+\sigma}} \sum_{\alpha + 2d < h < \tilde{\beta} - 2d} e\left(m e^{(\theta - h)/d}\right)$$
$$+ O\Big(\delta^{-3} X^{-3/2} + X^{-1/2}\delta^{-1} + X^{1/2} \log\log N + X^{1/2} \log X\Big).$$

We decompose the interval $[1, \infty)$, which is the range of m, into

$$[1, dXe^{2-\theta/d}] \cup (dXe^{2-\theta/d}, dNe^{-2-\theta/d}) \cup [dNe^{-2-\theta/d}, \infty).$$

If $1 \leq m \leq dXe^{2-\theta/d}$, then $\tilde{\beta} - 2d \leq 0$. In this case, we obtain

$$\sum_{1 \leq m \leq dXe^{2-\theta/d}} \frac{1}{m^{1+\delta}} \sum_{\tilde{\beta} - 2d \leq h < 0} e(m e^{(\theta - h)/d}) \ll \sum_{1 \leq m \leq dXe^{2-\theta/d}} \frac{\log m + \log X}{m^{1+\delta}}$$
$$\ll \log X.$$

If $dXe^{2-\theta/d} < m < dNe^{-2-\theta/d}$, then $\tilde{\beta} - 2d > 0$. In this case, we similarly obtain

$$\sum_{dXe^{2-\theta/d}<m<dNe^{-2-\theta/d}} \frac{1}{m^{1+\delta}} \sum_{0<h\le\tilde{\beta}-2d} e(me^{(\theta-h)/d}) \ll \sum_{dXe^{2-\theta/d}<m<dNe^{-2-\theta/d}} \frac{\log m + \log X}{m^{1+\delta}} \ll \log X.$$

If $m \ge dNe^{-2-\theta/d}$, then $\alpha + 2d \ge 0$. In this case, we obtain

$$\sum_{m\ge dNe^{-2-\theta/d}} \frac{1}{m^{1+\sigma}} \sum_{\alpha+2d<h<\tilde{\beta}-2d} e(me^{(\theta-h)/d}) \ll \sum_{m\ge dNe^{-2-\theta/d}} \frac{\log N}{m^{\sigma}} \ll 1.$$

Therefore, setting $M = dNe^{-2-\theta/d}$, we have

$$S_N = d^{\sigma} \sum_{1\le m\le M} \frac{1}{m^{1+\sigma}} \sum_{\alpha+2d<h<0} e\left(me^{(\theta-h)/d}\right) + O\bigg(\delta^{-3}X^{-3/2} + X^{-1/2}\delta^{-1} + X^{1/2}\log\log N + X^{1/2}\log X\bigg).$$

Recalling that $\alpha = \alpha(m) = d\log(\frac{m}{dN}) + \theta$, we get

$$\begin{aligned}
&\sum_{1\le m\le M} \frac{1}{m^{1+\sigma}} \sum_{\alpha+2d<h<0} e\left(me^{(\theta-h)/d}\right)\\
&= \sum_{\alpha(1)+2d<h<0} \sum_{1\le m\le Me^{h/d}} \frac{e\left(me^{(\theta-h)/d}\right)}{m^{1+\sigma}} + O(1)\\
&= \sum_{-d\log N<h<0} \sum_{1\le m\le Me^{h/d}} \frac{e\left(me^{(\theta-h)/d}\right)}{m^{1+\sigma}} + O(1).
\end{aligned}$$

Since

$$\sum_{-d\log N<h<0} \sum_{m>Me^{h/d}} \frac{e\left(me^{(\theta-h)/d}\right)}{m^{1+\sigma}} \ll \sum_{-d\log N<h<0} (Me^{(h/d)})^{-\sigma} \ll 1,$$

by exchanging h with $-h$, we have

$$S_N = d^{\sigma} \sum_{0<h<d\log N} \sum_{m=1}^{\infty} \frac{e\left(me^{(\theta+h)/d}\right)}{m^{1+\sigma}} + O\bigg(\delta^{-3}X^{-3/2} + X^{-1/2}\delta^{-1} + X^{1/2}\log\log N + X^{1/2}\log X\bigg).$$

By choosing δ as a sufficiently small fixed positive real number and X as a sufficiently large fixed positive integer, we obtain

$$S_N = d^{\sigma} \sum_{0<h<d\log N} \sum_{m=1}^{\infty} \frac{e\left(me^{(\theta+h)/d}\right)}{m^{1+\sigma}} + O(\log\log N)$$

By substituting $\sigma = k$, we obtain

$$\sum_{1\le n\le N} \zeta(-k + 2\pi idn)\frac{e(\theta n)}{n^{1+k}} = d^{k} \sum_{0<h<d\log N} \sum_{m=1}^{\infty} \frac{e\left(me^{(\theta+h)/d}\right)}{m^{1+k}} + O(\log\log N).$$

The Schwarz reflection principle yields that

$$\sum_{1\le|n|\le N} \zeta(-k+2\pi idn)\frac{e(\theta n)}{n^{1+k}} = d^k \sum_{0<h<d\log N}\sum_{|m|\neq 0}\frac{e\left(me^{(\theta+h)/d}\right)}{m^{1+k}} + O(\log\log N).$$

The Fourier expansion of $B_{k+1}(x)$ (see (1.1)) completes the proof.

□

Acknowledgement

The author would like to thank Professor Takashi Taniguchi and Professor Yohei Tachiya for the opportunity to give a talk at RIMS Workshop 2025 "Analytic Number Theory and Related Topics". This work was supported by JSPS KAKENHI Grant Number JP25K17223.

References

[Bor09] M. Émile Borel, *Les probabilités dénombrables et leurs applications arithmétiques*, Rend. Circ. Matem. Palermo **27** (1909), 247–271.

[Bor50] ———, *Sur les chiffres décimaux de $\sqrt{2}$ et divers problèmes de probabilités en chaîne*, C. R. Acad. Sci. Paris **230** (1950), 591–593. MR 34544

[Bug12] Y. Bugeaud, *Distribution modulo one and diophantine approximation*, Cambridge Tracts in Mathematics, Cambridge University Press, 2012.

[Hux94] M. N. Huxley, *On stationary phase integrals*, Glasgow Math. J. **36** (1994), no. 3, 355–362. MR 1295511

[KS24] Y. Kanado and K. Saito, *Normality of algebraic numbers and the riemann zeta function*, 2024, Preprint at `https://arxiv.org/abs/2412.02337`.

[KS25] ———, *A criterion for the simple normality of fractional powers of two via the Riemann zeta function*, Canad. J. Math. **First View** (2025), 1–24.

[Tit86] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, second ed., The Clarendon Press, Oxford University Press, New York, 1986, Edited and with a preface by D. R. Heath-Brown.

Kota Saito, Department of Mathematics, College of Science & Technology, Nihon University, Kanda, Chiyoda-ku, Tokyo, 101-8308, Japan

Email address: `saito.kota@nihon-u.ac.jp`