KP CONJECTURE AND BERNOULLIZATION MAP

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This article is a summary of the author's talk, which is based on joint work [YY] with S. Yasuda, presented at the conference "Analytic Number Theory and Related Topics 2025" (October 14-15, 2025) at RIMS, Kyoto. We omit the proofs of the main results (see [YY] and [Y] for details). Instead, we include many examples of holomorphic extensions of the Bernoullization.

Contents

We write $\mathbb{N} := \mathbb{Z}_{\geq 0}$. Let $(x)_n := x(x+1)\cdots(x+n-1)$ denote the Pochhammer symbol for $n \in \mathbb{N}$. We write $\mathbf{n}! := n_0!\cdots n_m!$, and $|\mathbf{n}| := n_0+\cdots+n_m$ for $\mathbf{n} = (n_0, \ldots, n_m) \in \mathbb{N}^{m+1}$.

1. Introduction.

We define the Bernoulli numbers $B_n\,(n \geq 0)$ as $\sum_{n\geq 0} B_n \frac{t^n}{n!} = \frac{te^t}{e^t-1}$, i.e., we use the convention of $B_1 = +\frac{1}{2}$. We also define the Bernoulli polynomials $B_n(x)\,(n \geq 0)$ as $\sum_{n\geq 0} B_n(x)\frac{t^n}{n!} = \frac{te^{xt}}{e^t-1}$. Let $\mathbb{Q}[B]$ denote the polynomial ring of a formal variable B with coefficient $\mathbb{Q}$. We define the **Bernoullization map** to be the $\mathbb{Q}$-linear homomorphism

$$\mathrm{Ber} : \mathbb{Q}[B] \to \mathbb{Q}$$

determined by $\mathrm{Ber}(B^n) := B_n$ (More generally, in Section 6, we will also consider any R-linear homomorphism $\varphi : R[B] \to R$ for a commutative ring R containing $\mathbb{Q}$). We also extend Ber to $\mathrm{Ber} : \mathbb{Q}[B][[t]] \to \mathbb{Q}[[t]]$ by the $\mathbb{Q}[[t]]$-linear manner. Then, we have

$$\mathrm{Ber}(e^{Bt}) = \frac{te^t}{e^t-1}.$$

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Note that $\mathrm{Ber}((B-1+a)^n)=B_n(a)$, since $\mathrm{Ber}(e^{(B-1+a)t})=e^{(-1+a)t}\mathrm{Ber}(e^{Bt})=\frac{te^{at}}{e^t-1}$.

Classically, this map is studied in the context of the umbral calculus, where one informally writes it as "$B^n=B_n$" with the convention of $B_1=-\frac{1}{2}$, and expresses, for example, the recursive identity $\sum_{i=0}^{n-1}\binom{n}{i}B_i=0$ of Bernoulli numbers for $n>1$ as $(B+1)^n=B^n$, the relation $B_n(x)=\sum_{i=0}^{n}\binom{n}{i}B_ix^{n-i}$ with the Bernoulli polynomial as $B_n(x)=(B+x)^n$, and the sum formula $\sum_{i=1}^{n}i^k=\frac{1}{k+1}\sum_{i=0}^{k}\binom{k+1}{i}B_i\cdot(n+1)^{k+1-i}$ as $\sum_{i=1}^{n}i^k=\int_0^{n+1}(B+x)^kdx$ under the convention of $B_1=-\frac{1}{2}$.

For $n\in\mathbb{N}$ and $\mathbf{n}=(n_0,\dots,n_m)\in\mathbb{N}^{m+1}$, we write

$$P_n(B):=\frac{(B)_n}{n!},\qquad P_{\mathbf{n}}(B):=P_{n_0}(B)\cdots P_{n_m}(B).$$

In their studies of linear independence results on special values of the p-adic Hurwitz zeta function, Kawashima-Poëls proposed the following conjecture:

Conjecture 1.1. (Kawashima-Poëls, [KP, Conjecture 8.13]) *For any* $\mathbf{n}\in\mathbb{N}^{m+1}$ *with* $m\geq 1$,

$$\Theta_{\mathbf{n}}:=\det\left(\mathrm{Ber}\left(\frac{d^i}{dB^i}(P_{\mathbf{n}}P_j)\right)\right)_{0\leq i,j\leq m-1}=\frac{m!\mathbf{n}!}{(m+|\mathbf{n}|)!}$$

holds.

Note that there are $m+1$ parameters (i.e., $n_0,\dots,n_m$) in the $m\times m$-matrix case. The reason for the number of parameters comes from Kawashima-Poëls' studies of Padé approximation of the p-adic polygamma functions, and this number of parameters is crucial for the theory (see also [YY, Corollary 2.6, (ii), and Lemma 3.13, (ii)]).

One of the main results of [YY] is the following:

Theorem 1.2. ([YY, Theorem 2.9]) *Conjecture 1.1 is true.*

Remark 1.2.1. Kawashima-Poëls proved $\Theta_{\mathbf{n}}\neq 0$, which implied some $\mathbb{Q}$-linear independence results on special values of p-adic Hurwitz zeta function by constructing Padé approximants of p-adic polygamma functions.

Theorem 1.2 gives us the following corollary which was also conjectured by Kawashima, in a private communication, from the point of view of multivariable beta functions:

Corollary 1.3. (**Distribution relation**, [YY, Corollary 1.3]) *For* $\mathbf{n}\in\mathbb{N}^{m+1}$ *with* $m\geq 1$, *we have* $\Theta_{\mathbf{n}}=\Theta_{\mathbf{n}+(1,0,\dots,0)}+\cdots+\Theta_{\mathbf{n}+(0,\dots,0,1)}$.

Next, in [YY], we extend, *in two ways*, the definition of the Bernoullization map from the polynomials in B to holomorphic functions in B satisfying certain conditions.

- One extension, which is called **series extension** and denoted by Ber^{Σ}, is motivated by the fact that the Bernoulli numbers appear in the special values of Riemann's zeta function at negative integers, as $\zeta(1-n)=-\frac{B_n}{n}$. (Philosophically, this motivation comes from "$\mathrm{Re}(s)<0$".)
- The other extension, which is called **integral extension** and denoted by $\mathrm{Ber}^{\int}$, is motivated by the fact that the Bernoulli numbers appear in the solution of the classical difference problem for polynomials, i.e., for a given polynomial $f(x)$, to find a polynomial $F(x)$ such that $f(x)=F(x+1)-F(x)$, as $x^n=\frac{B_{n+1}(x+1)}{n+1}-\frac{B_{n+1}(x)}{n+1}$. (Philosophically, this motivation comes from "$\mathrm{Re}(s)>-1$".)

- We also show that, in the overlap of the domains of the definition of two extensions of the Bernoullization map, the Bernoullization maps coincide ($\mathrm{Ber}^{\Sigma} = \mathrm{Ber}^{\int}$), which is called the **comparison theorem** (see [YY, Corollary 3.12]).

See Section 3 and Section 4 for the details of the definitions of the extensions.

We give an example of the holomorphic extension of the Bernoullization map:

$$\mathrm{Ber}^{\Sigma}(-\log B) = \mathrm{Ber}^{\int}(-\log B) = \gamma,$$

where $\gamma := \lim_{n\to\infty}(\sum_{i=1}^{n} \frac{1}{i} - \log n)$ is Euler's constant. More generally, $B^i(\log B)^j$ is related to the j-th derivative of Riemann's zeta function at $s = 1 - i$. More precisely, when $i > 0$ (resp. $i = 0$), for the unique $f(B) \in \mathbb{Q}\cdot B^i + \mathbb{Q}\cdot B^i \log B + \cdots + \mathbb{Q}\cdot B^i(\log B)^j$ with $\frac{df}{dB} = B^{i-1}(\log B)^j$ (resp. $f(B) = (\log B)^j$), we have

$$\mathrm{Ber}^{\Sigma}(f(B)) = \mathrm{Ber}^{\int}(f(B)) = \begin{cases} (-1)^{j+1}\zeta^{(j)}(1-i), & \text{if } \ i > 0, \\ (-1)^j \frac{d^j}{ds^j}\big|_{s=1}((s-1)\zeta(s)), & \text{if } \ i = 0. \end{cases} \tag{1}$$

Next, we consider the complex extension of the KP conjecture. Note that $P_n(B)$ can be extended to a holomorphic function

$$P_s(B) := \frac{\Gamma(B+s)}{\Gamma(B)\Gamma(s+1)}$$

in B on $\mathbb{C} \setminus (-s + \mathbb{Z}_{\leq 0})$ for $s \in \mathbb{C} \setminus \mathbb{Z}_{<0}$. We write

$$P_{\mathbf{s}}(B) := P_{s_0}(B) \cdots P_{s_m}(B)$$

for $\mathbf{s} = (s_0, \ldots, s_m) \in (\mathbb{C} \setminus \mathbb{Z}_{<0})^{m+1}$. We also write $\mathbf{s}! := \Gamma(s_0+1)\cdots\Gamma(s_m+1)$, and $|\mathbf{s}| := s_0 + \cdots + s_m$. For $\mathbf{s} \in \mathbb{C} \setminus \mathbb{R}_{\leq -1}$ and $i, j \geq 0$ with $\mathrm{Re}(|\mathbf{s}|) < i - j$, we can also see that $\frac{d^i}{dB^i}(P_{\mathbf{s}}(B)P_j(B))$ is in the overlap of the domain of the definition of two extensions of the Bernoullization map, hence the comparison theorem is applicable to it, and, by the meromorphic continuation (cf. Section 3 and 4), we have $\mathrm{Ber}^{\Sigma}(\frac{d^i}{dB^i}(P_{\mathbf{s}}P_j)) = \mathrm{Ber}^{\int}(\frac{d^i}{dB^i}(P_{\mathbf{s}}P_j))$ as meromorphic functions on $\mathbb{C}^{m+1}$.

The second main result of [YY] is the following:

Theorem 1.4. (**complex KP conjecture**, [YY, Theorem 3.20]) *For $m \geq 1$, we have*

$$\Theta_{\mathbf{s}} := \det\left(\mathrm{Ber}^{\int}\left(\frac{d^i}{dB^i}(P_{\mathbf{s}}P_j)\right)\right)_{0\leq i,j\leq m-1} = \frac{m!\mathbf{s}!}{(m+|\mathbf{s}|)!}$$

as meromorphic functions on $\mathbb{C}^{m+1}$.

The original KP conjecture, which is now a theorem, gives us systematic determinantal relations among Bernoulli numbers, and, by analytic continuations, the complex KP theorem gives us systematic determinantal relations among some special functions. For example, in the case where $m = 1$, by combining the comparison theorem, we obtain the following identity

$$B(s,t) = -\sum_{n\geq 0} \frac{(s)_n(t)_n}{n!^2}\left(\psi(s+n) + \psi(t+n) - 2\psi(n+1)\right)$$

for $(s,t) \in (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})^2$ with $\mathrm{Re}(s+t) < 2$, where $B(s,t)$ is the beta function, and $\psi(s) := \frac{d}{ds}\log\Gamma(s) = \frac{\Gamma'(s)}{\Gamma(s)}$ is the digamma function. After the author's talk in the conference, W. Zudilin kindly pointed out to the author that this equality could be shown by using Gauss' hypergeometric function ${}_2F_1\left[\begin{smallmatrix}a,b\\1\end{smallmatrix}\right]$. The author sincerely thanks him for his observation. On this occasion, we give an elementary proof of it (which is slightly different from the one that Zudilin had in mind).

Proof. We have $\frac{d}{dx}(x)_n = \frac{d}{dx}\frac{\Gamma(x+n)}{\Gamma(x)} = \frac{\Gamma(x+n)}{\Gamma(x)}(\psi(x+n)-\psi(x)) = (x)_n(\psi(x+n)-\psi(x))$, and $\frac{d}{dx}\frac{1}{(x)_n} = -\frac{1}{(x)_n^2}\cdot(x)_n(\psi(x+n)-\psi(x)) = -\frac{1}{(x)_n}(\psi(x+n)-\psi(x))$. Since $\frac{(s)_n(t)_n}{n!^2} = O(\frac{(s+n)^{s+n-1/2}(t+n)^{t+n-1/2}}{((n+1)^{n+1-1/2})^2}) = O(n^{\mathrm{Re}(s+t)-2})$ for $n\to\infty$ by Stirling formula, and since $\mathrm{Re}(s+t)<2$, we can exchange the derivatives in the following infinite sum

$$\left(\frac{d}{ds}+\frac{d}{dt}+2\frac{d}{dc}\right)\Big|_{c\to1}{}_2F_1\begin{bmatrix}s,t\\c\end{bmatrix}(z) = \left(\frac{d}{ds}+\frac{d}{dt}+2\frac{d}{dc}\right)\Big|_{c\to1}\sum_{n\geq0}\frac{(s)_n(t)_n}{(c)_n n!}z^n$$

$$=\sum_{n\geq0}\frac{(s)_n(t)_n}{n!^2}(\psi(s+n)+\psi(t+n)-2\psi(n+1))z^n-(\psi(s)+\psi(t)-2\psi(1)){}_2F_1\begin{bmatrix}s,t\\c\end{bmatrix}(z)$$

for $|z|<1$. By Gauss' formula $\lim_{z\to1^-}{}_2F_1\left[\begin{smallmatrix}a,b\\c\end{smallmatrix}\right](z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$, we have

$$\sum_{n\geq0}\frac{(s)_n(t)_n}{n!^2}(\psi(s+n)+\psi(t+n)-2\psi(n+1))$$

$$=\left(\frac{d}{ds}+\frac{d}{dt}+2\frac{d}{dc}+\psi(s)+\psi(t)-2\psi(1)\right)\Big|_{c\to1}\frac{\Gamma(c)\Gamma(c-s-t)}{\Gamma(c-s)\Gamma(c-t)}$$

$$=\frac{\Gamma(1-s-t)}{\Gamma(1-s)\Gamma(1-t)}(-\psi(1-s-t)+\psi(1-s)-\psi(1-s-t)+\psi(1-t)$$
$$+2\psi(1)+2\psi(1-s-t)-2\psi(1-s)-2\psi(1-t)+\psi(s)+\psi(t)-2\psi(1))$$
$$=\frac{\Gamma(1-s-t)}{\Gamma(1-s)\Gamma(1-t)}(\psi(s)+\psi(t)-\psi(1-s)-\psi(1-t))\quad(*).$$

On the other hand, by taking the logarithmic derivative of the reflection formula $\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin\pi s}$, we have $\psi(1-s)-\psi(s)=\pi\cot\pi s$. Hence,

$$(*) = -\frac{\Gamma(1-s-t)}{\Gamma(1-s)\Gamma(1-t)}\pi(\cot\pi s+\cot\pi t) = -\frac{\pi\Gamma(1-s-t)}{\Gamma(1-s)\Gamma(1-t)}\frac{\sin\pi(s+t)}{\sin\pi s\sin\pi t}$$
$$= -\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} = -B(s,t),$$

as desired. □

Finally, for a function $f(B)$, $a\in\mathbb{C}$, and a Dirichlet character χ of conductor N, we introduce **Hurwitz variant** ${}^af(B)$ and **Dirichlet variant** ${}^\chi f(B)$ as

$${}^af(B) := f(B-1+a),\qquad {}^\chi f(B) := \frac{1}{N}\sum_{1\leq a\leq N}\chi(a)f(N(B-1)+a).$$

Note that

$$\mathrm{Ber}({}^a(B^k)) = B_k(a),\quad \mathrm{Ber}({}^\chi(B^k)) = B_{k,\chi},$$

since $\frac{1}{N}\sum_{1\leq a\leq N}\chi(a)e^{(-N+a)t}\frac{(Nt)e^{Nt}}{e^{Nt}-1} = \sum_{1\leq a\leq N}\frac{\chi(a)te^{at}}{e^{Nt}-1}$, where $B_{n,\chi}$ $(n\geq0)$ is the generalized Bernoulli numbers for χ, i.e., defined by $\sum_{n\geq0}B_{n,\chi}\frac{t^n}{n!} = \sum_{1\leq a\leq N}\frac{\chi(a)te^{at}}{e^{Nt}-1}$.

2. Proof of KP conjecture

In this section, we only explain the strategy of the proof of Theorem 1.2. For the details, see [YY, Section 2]. Theorem 1.2 follows from the following relations between the $\Theta_{\mathbf{n}}$'s with the trivial fact that $\Theta_{(0,0)} = 1$:

Proposition 2.1. (**Boundary relation**, [YY, Proposition 2.7]) $\Theta_{(0,\mathbf{n})} = \frac{m+1}{m+1+|\mathbf{n}|}\Theta_{\mathbf{n}}$.

Proposition 2.2. (**Increment relation**, [YY, Proposition 2.8]) $\Theta_{\mathbf{n}+(1,0,\dots,0)} = \frac{n_0+1}{m+|\mathbf{n}|+1}\Theta_{\mathbf{n}}$.

These two propositions are proved after we show the preliminary relations, i.e., the **Wronskian relation** ([YY, Lemma 2.1]), which works for any $\varphi : R[B] \to R$, and the **Alternating relation** ([YY, Corollary 2.6]), which works only for Ber.

3. Series extension of Bernoullization map

The series extension is motivated by the following heuristic:

$$\mathrm{Ber} : B^k \mapsto B_k = -k\zeta(1-k) \text{ ``}=\text{''} -k\sum_{n\geq 1} n^{k-1} = -\sum_{n\geq 1} \frac{d}{dB}\Big|_{B=n} B^k.$$

Definition 3.1. *For a domain $\Omega \subset \mathbb{C}$ satisfying $\mathbb{Z}_{\geq 1} \subset \Omega$, we define*

$$\mathcal{F}^{\Sigma}(\Omega) := \left\{ \begin{array}{c} \text{meromorphic function } f(B) \text{ on } \Omega \text{ s.t.} \\ f \text{ has no poles on } \mathbb{Z}_{\geq 1}, \text{ and } \sum_{n=1}^{\infty} \frac{df}{dB}(n) \text{ abs. conv.} \end{array} \right\}$$

and **the series extension of Bernoullization map**

$$\mathrm{Ber}^{\Omega,\Sigma} : \mathcal{F}^{\Sigma}(\Omega) \to \mathbb{C}\,;\, f(B) \mapsto -\sum_{n=1}^{\infty} \frac{df}{dB}(n).$$

Remark 3.1.1. For $\mathbf{s} \in (\mathbb{C} \setminus \mathbb{Z}_{<0})^{m+1}$ with $\mathrm{Re}(|\mathbf{s}|) < 0$, we have $P_{\mathbf{s}} \in \mathcal{F}^{\Sigma}(\mathbb{C})$.

We also define the **family version** of $\mathrm{Ber}^{\Omega,\Sigma}$:

Definition 3.2. *For a domain $X \subset \mathbb{C}^N$, we define*

$$\mathcal{M}^{\Omega,\Sigma}(X) := \left\{ \begin{array}{c} \text{meromorphic function } f \text{ on } \Omega \times X \Big| \\ \forall x_0 \in X \setminus \cup_{n\geq 1}(\text{poles of } f|_{B=n}), \exists U \text{ open nbd. of } x_0 \text{ s.t.} \\ \sum_{n=1}^{\infty} \frac{df}{dB}(n)_{|U} \text{ converges uniformly and absolutely} \end{array} \right\}$$

and extend $\mathrm{Ber}^{\Omega,\Sigma}$ *to the $\mathcal{M}(X)$-linear homomorphism*

$$\mathrm{Ber}_X^{\Omega,\Sigma} : \mathcal{M}^{\Omega,\Sigma}(X) \to \mathcal{M}(X) := \{\text{mero. fct. on } X\}.$$

Finally, we extend $\mathrm{Ber}^{\Omega,\Sigma}$ **by the meromorphic continuation**:

Definition 3.3. *Let $\widetilde{X} \subset \mathbb{C}^N$ be a domain containing X. For $f \in \mathcal{M}(\Omega \times \widetilde{X})$, if there exist $g \in \mathcal{M}(\widetilde{X})$ such that* $\mathrm{Ber}_X^{\Omega,\Sigma}(f|_{\Omega\times X}) = g|_X$*, then we define* $\mathrm{Ber}_{\widetilde{X}}^{\Omega,\Sigma}(f) := g$.

Example 3.4. (**Powers**, [YY, Example 3.3, (i)]) For $\Omega = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ and $X = \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\} \subset \widetilde{X} = \mathbb{C}$, $\mathrm{Ber}_{\widetilde{X}}^{\Omega,\Sigma}(B^{-s}) = s\zeta(s+1)$. In particular,

$$\mathrm{Ber}_{\widetilde{X}}^{\Omega,\Sigma}\left(B^{-s}\right)\Big|_{s=-k} = B_k = \mathrm{Ber}(B^k),$$

as intended, for $k \in \mathbb{N}$. If we write $(-)^{[s]} := \frac{(-)^s}{\Gamma(s+1)}$ (divided powers), then **the functional equation** of Riemann's zeta function can be written as

$$\mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}\left((\pi B^2)^{[s/2]}\right) = \mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}\left((\pi B^2)^{[(1-s)/2]}\right).$$

More generally, $a \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, and a Dirichlet character χ of conductor N, we have

$$\mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}\left({}^{a}(B^{-s})\right) = s\zeta(s+1,a), \qquad \mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}\left({}^{\chi}(B^{-s})\right) = sL(s+1,\chi).$$

Since, for $k \in \mathbb{N}$, $\mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}({}^{a}(B^k)) = \mathrm{Ber}({}^{a}(B^k)) = B_k(a)$ and $\mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}({}^{\chi}(B^k)) = \mathrm{Ber}({}^{\chi}(B^k)) = B_{k,\chi}(a)$, these recover the classical formulae

$$\zeta(1-k,a) = -\frac{B_k(a)}{k}, \qquad L(1-k,\chi) = -\frac{B_{k,\chi}}{k}.$$

Note that, in the above argument, we admit the formula $\zeta(1-k) = -\frac{B_k}{k}$, and although the deduction of $L(1-k,\chi) = -\frac{B_{k,\chi}}{k}$ from $\zeta(1-k,a) = -\frac{B_k(a)}{k}$ is just taking a linear combination, the deduction of $\zeta(1-k,a) = -\frac{B_k(a)}{k}$ from $\zeta(1-k) = -\frac{B_k}{k}$ is nontrivial.

Example 3.5. (**Exponentials**, [YY, Example 3.3, (ii)]) For $\Omega = \mathbb{C}$ and $X = \{\mathrm{Re}(t) < 0\} \subset \widetilde{X} = \mathbb{C}$, we have

$$\mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}(e^{Bt}) = \frac{te^t}{e^t-1},$$

since $-\sum_{n\geq 1} \frac{de^{Bt}}{dB}|_{B=n} = \frac{te^t}{e^t-1}$ for $\mathrm{Re}(t) < 0$. For $a \in \mathbb{C}$, a Dirichlet character χ of conductor N, we have

$$\mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}\left({}^{a}(e^{Bt})\right) = \frac{te^{at}}{e^t-1}, \qquad \mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}\left({}^{\chi}(e^{Bt})\right) = \sum_{1\leq a\leq N} \frac{\chi(a)te^{at}}{e^{Nt}-1}.$$

We also have

$$\mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}(\sin(Bt)) = \mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}\left(\frac{e^{iBt}-e^{-iBt}}{2i}\right) = \frac{t}{2},$$

$$\mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}(\cos(Bt)) = \mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}\left(\frac{e^{iBt}+e^{-iBt}}{2}\right) = \frac{t}{2}\cot\frac{t}{2}.$$

Note that $\sin(2\pi nB)\,(n \in \mathbb{Z})$ is an indefinite form "$\infty - \infty$" for the Bernoullization (see Remark 3.6.1).

Example 3.6. (**Logarithms**, [YY, Example 3.3, (iii)]) By differentiating $\mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}(\frac{1}{(B-1+a)^s}) = s\zeta(s+1,a) = -\frac{d}{da}\zeta(s,a)$ with respect to s and specializing $s=0$, we obtain

$$\mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}({}^{a}(\log B)) = \frac{d}{da}\zeta'(0,a) = \frac{d}{da}\log\frac{\Gamma(a)}{\sqrt{2\pi}} = \psi(a)$$

for $\Omega = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ and $X = \{a \in \mathbb{C} \mid |1-a| < 1\} \subset \widetilde{X} = \mathbb{C}$, by *Lerch's formula.*

By differentiating $\mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}(\frac{1}{(B-1+a)^s}) = s\zeta(s+1,a)$ with respect to s and specializing $s=-1$, we obtain

$$\mathrm{Ber}^{\Omega,\Sigma}_{\widetilde{X}}({}^{a}(B\log B)) = -\zeta(0,a) + \zeta'(0,a) = B_1(a) + \log\frac{\Gamma(a)}{\sqrt{2\pi}}$$

for $a \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. It can be written as

$$\mathrm{Ber}_{\widetilde{X}}^{\Omega,\Sigma}({}^{a}(B\log B - B)) = \log \frac{\Gamma(a)}{\sqrt{2\pi}}.$$

For the quadratic character $\chi = \chi_K$ of conductor N associated to an imaginary quadratic field K, by using *class number formula and Chowla-Selberg formula*, we obtain

$$\begin{aligned}
\mathrm{Ber}_{\widetilde{X}}^{\Omega,\Sigma}({}^{\chi}(B\log B - B)) &= \sum_{1\leq a\leq N} \chi(a)\log\Gamma(a/N) + \frac{\log N}{N}\sum_{1\leq a\leq N}\chi(a)a \\
&= -\frac{2h_K}{w_K}\log\frac{2\pi}{N} + \frac{1}{6w_K}\sum_{[\mathfrak{a}]\in \mathrm{Cl}(K)} \log((2\pi)^{24}\Delta(\mathfrak{a})\Delta(\mathfrak{a}^{-1})) - \frac{2h_K}{w_K}\log N \\
&= \frac{1}{6w_K}\sum_{[\mathfrak{a}]\in \mathrm{Cl}(K)} \log((2\pi)^{12}\Delta(\mathfrak{a})\Delta(\mathfrak{a}^{-1})),
\end{aligned}$$

where $\mathrm{Cl}(K)$ is the ideal class group of K, $w_K := \#O_K^{\times}$, h_K is the class number of K, and Δ is Ramanujan's delta function (note that the value $\Delta(\mathfrak{a})\Delta(\mathfrak{a}^{-1})$ does not depend on the choice of the representative $\mathfrak{a}$ of $[\mathfrak{a}] \in \mathrm{Cl}(K)$). Note that $\mathrm{Ber}_{\widetilde{X}}^{\Omega,\Sigma}({}^{a}(B\log B))$ has the "extra" term of $B_1(a)$, however, it disappears in $\mathrm{Ber}_{\widetilde{X}}^{\Omega,\Sigma}({}^{a}(B\log B - B))$. Similarly, the "extra" term of $-\frac{2h_K}{w_K}\log\frac{2\pi}{N}$ in the Chowla-Selberg formula disappears in $\mathrm{Ber}_{\widetilde{X}}^{\Omega,\Sigma}({}^{\chi}(B\log B - B))$. This is related to the fact that $\frac{d}{dB}(B\log B)$ is not equal to $\log B$, but $\frac{d}{dB}(B\log B - B)$ is (see also the condition of $f(B)$ in the formula (1)).

Remark 3.6.1. ([YY, Remark 3.2.1]) (**indefinite form "$\infty - \infty$" for the Bernoullization**) In Definition 3.3, we defined the Bernoullization via the meromorphic continuation. Then, the phenomenon of the indefinite form "$\infty - \infty$" appears for the Bernoullization. For example, the Bernoullization of $e^{2\pi iBt}$ is originally defined on $\mathrm{Im}(t) > 0$, and is extended as $\frac{2\pi i t e^{2\pi i t}}{e^{2\pi i t}-1}$ to the whole plane by the meromorphic continuation, where it has a pole at $t \in \mathbb{Z}$. Hence the Bernoullization of $e^{2\pi iB}$ is not defined. Therefore, $e^{2\pi iB} - e^{-2\pi iB}$ is an indefinite form ("$\infty - \infty$") for the Bernoullization.

As the usual phenomenon of the indefinite forms, if we resolve an indefinite form of the Bernoullization by canceling the poles via meromorphic continuation, then the Bernoullization **depends on how to cancel them**. See the following example:

$$\begin{array}{ccccc}
e^{2\pi iBt} - e^{-2\pi iBt} & \xrightarrow{t=1} & e^{2\pi iB} - e^{-2\pi iB} & \xleftarrow{t=1} & e^{-2\pi it}e^{2\pi iBt} - e^{2\pi it}e^{-2\pi iBt} \\
\downarrow \mathrm{Ber} & & \mathrm{Ber} \downarrow ?? & & \mathrm{Ber} \downarrow \\
\frac{2\pi ite^{2\pi it}}{e^{2\pi it}-1} - \frac{2\pi it}{e^{2\pi it}-1} = 2\pi it & \xrightarrow{t=1} & 2\pi i \neq -2\pi i & \xleftarrow{t=1} & \frac{2\pi it}{e^{2\pi it}-1} - \frac{2\pi ite^{2\pi it}}{e^{2\pi it}-1} = -2\pi it.
\end{array}$$

4. Integral extension of Bernoullization map

Definition 4.1. *For $R > 1$, $0 < \epsilon < 1$, we write*

$$\Omega_{R,\epsilon} := \Big\{ B \in \mathbb{C} \mid \text{either } \mathrm{Re}(B) > R - \epsilon \text{ or } (\mathrm{Re}(B) > 1 - 2\epsilon \text{ and } |\mathrm{Im}(B)| < 2\epsilon) \Big\}.$$

For an unbounded domain $\Omega \subset \mathbb{C}$, we write $\mathcal{F}^{\int}(\Omega)$ for the ring of holomorphic functions on Ω with polynomial growth, i.e., there exists $N \in \mathbb{N}$ such that $|f(B)| < |B|^N$ on Ω for $|B| \gg 0$. Put $\mathcal{F}^{\int}_{R,\epsilon} := \mathcal{F}^{\int}(\Omega_{R,\epsilon})$.

Remark 4.1.1. For $\mathbf{s} \in (\mathbb{C} \setminus \mathbb{R}_{\leq -1})^{m+1}$, $P_{\mathbf{s}}|_{\Omega_{R,\epsilon}} \in \mathcal{F}^{\int}_{R,\epsilon}$ for suitable $R > 1, 0 < \epsilon < 1$.

Definition 4.2. *For $f \in \mathcal{F}^{\int}_{R,\epsilon}$, we define* **the integral extension of Bernoullization map** *by*

$$\mathrm{Ber}^{\int}(f) := 2\pi i \left(\int_{R-\infty i}^{R-\epsilon i} + \int_{R-\epsilon i}^{1-\epsilon-\epsilon i} + \int_{1-\epsilon-\epsilon i}^{1-\epsilon+\epsilon i} + \int_{1-\epsilon+\epsilon i}^{R+\epsilon i} + \int_{R+\epsilon i}^{R+\infty i} \right) \frac{f(B)e^{2\pi i B}}{(e^{2\pi i B}-1)^2} dB.$$

Note that $\frac{f(B)e^{2\pi i B}}{(e^{2\pi i B}-1)^2}$ exponentially decays for both $\mathrm{Im}(B) \to \pm\infty$, hence the integral converges.

Remark 4.2.1. (weakening the growth condition and **Reflection formula**) We may weaken the growth condition by the following condition: $|f(B)| < e^{\alpha|\mathrm{Im}(B)|}$ on Ω for $|\mathrm{Im}(B)| \to \infty$ for some $0 \leq \alpha < 2\pi$. Then, the new $\mathcal{F}^{\int}_{R,\epsilon}$ is not closed under the product any longer, however, we have more applicable function to the Bernoullization, and we can show **Reflection formula** ([YY, Proposition 3.7])

$$\mathrm{Ber}^{\int}(f(B)) = \mathrm{Ber}^{\int}(f(1-B))$$

for entire functions $f(B)$ with the above weakened growth condition, which is a generalization of classical equality $\mathrm{Ber}(B^k) = \mathrm{Ber}((1-B)^k)$ (this formula follows from $\mathrm{Ber}(e^{Bt}) = \frac{te^t}{e^t-1} = e^t\frac{-te^{-t}}{e^{-t}-1} = \mathrm{Ber}(e^{(1-B)t})$). Note that any entire function of polynomial growth is a polynomial, hence we need to weaken the growth condition to generalize to non-polynomial functions.

By using the residue theorem, we can easily prove the following proposition:

Proposition 4.3. (**Bernoullization via solution of difference problem**) *Let $R > 1$, $0 < \epsilon < 1$ and $f \in \mathcal{F}^{\int}_{R,\epsilon}$. Assume that there exists $F \in \mathcal{F}^{\int}_{R,\epsilon}$ such that $f(B) = F(B+1) - F(B)$. Then, we have*

$$\mathrm{Ber}^{\int}(f(B)) = \frac{dF}{dB}\Big|_{B=1}.$$

Example 4.4. (**nonnegative integral Powers**, [YY, Example 3.3, (i)]) Let $k \in \mathbb{N}$. Since $B_{k+1}(x+1) - B_{k+1}(x) = (k+1)x^k$, we have

$$^a(B^k) = \frac{1}{k+1}(^aB_{k+1})(B+1) - \frac{1}{k+1}(^aB_{k+1})(B)$$

for $a \in \mathbb{C}$. We also have $\frac{1}{k+1}(^aB_{k+1})(B) \in \mathcal{F}^{\int}_{R,\epsilon}$ for any $R > 1$, $0 < \epsilon < 1$. Hence, by $\frac{dB_{k+1}(x)}{dx} = (k+1)B_k(x)$, we have

$$\mathrm{Ber}^{\int}(^a(B^k)) = \frac{1}{k+1}\frac{d}{dB}(^aB_{k+1})(B)\Big|_{B=1} = B_k(a).$$

In particular, for $a = 1$, we have $\mathrm{Ber}^{\int}(B^k) = B_k$, as intended.

Example 4.5. (**positive integral Powers**, [YY, Example 3.3, (i)]) Let $k \in \mathbb{Z}_{>0}$ and $|1-a| < 1$. Since $\Gamma(s+1) = s\Gamma(s)$, we have $\psi^{(k-1)}(s+1) - \psi^{(k-1)}(s) = \frac{(-1)^{k-1}(k-1)!}{s^k}$, where $\psi^{(n)}(s) := \frac{d^n}{ds^n}\psi(s)$ is the n-th polygamma function. Hence

$$^a(B^{-k}) = \frac{(-1)^{k-1}}{(k-1)!}{}^a(\psi^{(k-1)})(B+1) - \frac{(-1)^{k-1}}{(k-1)!}{}^a(\psi^{(k-1)})(B).$$

By Stirling's asymptotic formula for gamma function, $\frac{(-1)^{k-1}}{(k-1)!}{}^a(\psi^{(k-1)})(B) \in \mathcal{F}^{\int}_{R,\epsilon}$ for some $R > 1$, $0 < \epsilon < 1$. Hence, we have

$$\mathrm{Ber}^{\int}({}^a(B^{-k})) = \frac{(-1)^{k-1}}{(k-1)!}\frac{d}{dB}{}^a(\psi^{(k-1)})(B)\Big|_{B=1} = \frac{(-1)^{k-1}}{(k-1)!}\psi^{(k)}(a) = k\zeta(k+1,a).$$

Example 4.6. (**Exponentials**, [YY, Example 3.3, (ii)]) Let $\mathrm{Re}(t) < 0$ and $a \in \mathbb{C}$. Since

$${}^a(e^{Bt}) = \frac{{}^a(e^{(B+1)t})}{e^t - 1} - \frac{{}^a(e^{Bt})}{e^t - 1},$$

and $\frac{{}^a(e^{Bt})}{e^t-1} \in \mathcal{F}^{\int}_{R,\epsilon}$ for any $R > 1$, $0 < \epsilon < 1$, we have

$$\mathrm{Ber}^{\int}({}^a(e^{Bt})) = \frac{d}{dB}\frac{{}^a(e^{Bt})}{e^t - 1}\Big|_{B=1} = \frac{te^{at}}{e^t - 1}.$$

Example 4.7. (**Logarithms**, [YY, Example 3.3, (iii)]) Let $a \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$. Since $\Gamma(s+1) = s\Gamma(s)$, we have

$${}^a(\log)(B) = {}^a(\log\Gamma)(B+1) - {}^a(\log\Gamma)(B)$$

By Stirling's asymptotic formula for gamma function, we have ${}^a(\log\Gamma)(B) \in \mathcal{F}^{\int}_{R,\epsilon}$ for some $R > 1$, $0 < \epsilon < 1$. Hence, we have

$$\mathrm{Ber}^{\int}({}^a(\log)(B)) = \frac{d}{dB}{}^a(\log\Gamma)(B)\Big|_{B=1} = \psi(a).$$

Similarly as the series extension, we consider the **family version** of the integral extension, and we extend it **by the meromorphic continuation**.

Definition 4.8. *For a domain $X \subset \mathbb{C}^N$, we define*

$$\mathcal{M}^{\Omega,\int}(X) := \left\{ \begin{array}{c} \text{meromorphic function } f \text{ on } \Omega \times X \Big| \\ \forall x_0 \in X, \exists U \text{ open nbd. of } x_0, R_U > 1, 0 < \epsilon_U < 1 \text{ s.t.} \\ \Omega_{R_U,\epsilon_U} \subset \Omega,\ f|_{\Omega_{R_U,\epsilon_U}\times U} \ : \text{hol. and} \\ \text{of polynomial growth uniformly w.r.t. } U \end{array} \right\}$$

and extend $\mathrm{Ber}^{\Omega,\int}$ *to the $\mathcal{O}(X)$-linear homomorphism*

$$\mathrm{Ber}^{\Omega,\int}_X : \mathcal{M}^{\Omega,\int}(X) \to \mathcal{O}(X) := \{\text{hol. fct. on } X\}.$$

Definition 4.9. *Let $\widetilde{X} \subset \mathbb{C}^N$ be a domain containing X. For $f \in \mathcal{M}(\Omega \times \widetilde{X})$, if there exist $g \in \mathcal{M}(\widetilde{X})$ such that $\mathrm{Ber}^{\Omega,\int}_X(f|_{\Omega\times X}) = g|_X$, then we define $\mathrm{Ber}^{\Omega,\int}_{\widetilde{X}}(f) := g$.*

The comparison theorem is the following:

Theorem 4.10. (**Comparison theorem**, [YY, Corollary 3.12]) *For domains $\Omega \subset \mathbb{C}$, $X \subset \mathbb{C}^N$, and $f \in \mathcal{M}^{\Omega,\Sigma}(X) \cap \mathcal{M}^{\Omega,\int}(X)$, we have $\mathrm{Ber}^{\Omega,\Sigma}_X(f) = \mathrm{Ber}^{\Omega,\int}_X(f)$.*

5. Proof of complex KP conjecture

The idea of the proof of Theorem 1.4 is based on the following idea. We use the induction on the number of the parameters that are not nonnegative integers (the first step of the induction is Theorem 1.2), and we apply classical Carlson's theorem:

Theorem 5.1. (Carlson, 1914) *$f(z)$ is a holomorphic function on $\mathrm{Re}(z) \geq 0$ of order $e^{k|z|}$ with $k < \pi$, and $f(n) = 0$ for $n \in \mathbb{N}$, then $f(z) \equiv 0$.*

The difficulty is in obtaining the required estimate (see [YY, Theorem 3.19]).

6. Maps other than Bernoullization

Let R be a commutative ring containing $\mathbb{Q}$, and

$$\varphi : R[B] \to R$$

be an R-linear homomorphism. Note that φ is an R-*algebra* homomorphism if and only if $\varphi = \mathrm{ev}_a$ for some $a \in R$, where ev_a denotes the evaluation at a, i.e., $\mathrm{ev}_a(B^k) = a^k$. For $\mathbf{n} \in \mathbb{N}^{m+1}$, we define

$$\Theta_{\mathbf{n}}^{\varphi} := \det\left(\varphi\left(\frac{d^i}{dB^i}(P_{\mathbf{n}}P_j)\right)\right)_{0\leq i,j\leq m-1}.$$

The following is a corollary of **Wronskian relation**, which is mentioned in Section 2:

Proposition 6.1. (**KP-determinants for algebra homomorphisms**, [YY, Corollary 2.4]) *For any $m \geq 1$, $\mathbf{n} \in \mathbb{N}^{m+1}$ and $a \in R$, it holds that*

$$\Theta_{\mathbf{n}}^{\mathrm{ev}_a} = (P_{\mathbf{n}}(a))^m.$$

The following identity is remarkable from the point of view that it holds for any φ:

Theorem 6.2. (**Contiguous relation**, [YY, Theorem 4.2], Kawashima for $m = 1, 2$) *For $m \geq 1$ and $\mathbf{n} = (n_0, \ldots, n_m) \in \mathbb{N}^{m+1}$, it holds that*

$$\Theta_{\mathbf{n}}^{\varphi} = \sum_{0\leq r\leq m} \frac{(n_r+1)^m}{\prod_{\substack{0\leq i\leq m,\\ i\neq r}}(n_r - n_i)} \Theta^{\varphi}_{\mathbf{n}+(0,\ldots,0,1\,(r\text{-th}),0,\ldots,0)}$$

for distinct $n_0, \ldots, n_m$.

The proof of Theorem 6.2 is based on the following observation:

$$\begin{aligned}(n+1)&P_{n+1}(B)P_m(B) - (m+1)P_n(B)P_{m+1}(B)\\ &= P_n(B)P_m(B)((B+n)-(B+m)) = (n-m)P_n(B)P_m(B).\end{aligned}$$

We interpret this relation in terms of the *generating function* and *differential operators*. By this interpretation, we show **Vandermonde orthogonality** (see [YY, Lemma 4.1]) for differential operators, and finally Theorem 6.2 is shown by using the Vandermonde orthogonality.

We extend φ to $\varphi : R[B][[t]] \to R[[t]]$ in the $R[[t]]$-linear manner. We write

$$f_{\varphi}(t) := \varphi(e^{Bt}) = \sum_{n\geq 0} \frac{\varphi(B^n)}{n!}t^n.$$

For φ satisfying $\varphi(1) = 1$, we have $f_{\varphi}(t) \in R[[t]]^{\times}$, and we define the dual φ^* of φ to be the R-linear homomorphism $\varphi^* : R[B] \to R$ determined by

$$f_{\varphi^*}(t) = 1/f_{\varphi}(-t).$$

For example, we have $\mathrm{ev}_a^* = \mathrm{ev}_a$ for $a \in R$, and $\mathrm{Ber}^*(B^n) = \frac{1}{n+1}$, since $1/e^{-at} = e^{at}$, and $\frac{e^{-t}-1}{(-t)e^{-t}} = \frac{e^t-1}{t} = \sum_{n\geq 0} \frac{t^n}{(n+1)!}$.

Theorem 6.3. (**Duality relation for some KP-determinants**, [YY, Corollary 4.6]) *Assume that* $\varphi(1) = 1$. *For* $m, m^* \geq 1$, $\mathbf{n} = (n_0, \dots, n_m) \in \mathbb{N}^{m+1}$, *and* $\mathbf{n}^* = (n_0^*, \dots, n_{m^*}^*) \in \mathbb{N}^{m^*+1}$ *with* $0 \leq n_0, n_0^*, \dots, n_m, n_{m^*}^* \leq 1$ *satisfying* $m^* = |\mathbf{n}|$, *and* $m = |\mathbf{n}^*|$. *Then it holds that*

$$\Theta_{\mathbf{n}}^{\varphi} = \Theta_{\mathbf{n}^*}^{\varphi^*}.$$

The condition on the parameters in Theorem 6.3 is restrictive, hence a further generalization should be expected. Theorem 6.3 is deduced from the **skew-Young duality** (see [YY, Theorem 4.5]) which is a duality for skew-Young determinants, essentially due to Macdonald. In the case where the skew-Young diagram is a rectangular, then the skew-Young duality is a duality for Toeplitz determinants (see [YY, Remark 4.5.1]), i.e., the Toeplitz determinant for φ with center k of size n is equal to the Toeplitz determinant for φ^* with center n of size k:

$$\det\left(\varphi\left(\frac{B^{k-i+j}}{(k-i+j)!}\right)\right)_{1\leq i,j\leq n} = \det\left(\varphi^*\left(\frac{B^{n-i+j}}{(n-i+j)!}\right)\right)_{1\leq i,j\leq k},$$

where $\varphi(\frac{B^\ell}{\ell!}) = \varphi^*(\frac{B^\ell}{\ell!}) := 0$ for $\ell < 0$. For $\varphi = \mathrm{Ber}$, it is

$$\det\left(\frac{B_{k-i+j}}{(k-i+j)!}\right)_{0\leq i,j\leq n-1} = \det\left(\frac{1}{(n+1-i+j)!}\right)_{0\leq i,j\leq k-1},$$

where $\frac{B_\ell}{\ell!} = \frac{1}{(\ell+1)!} := 0$ for $\ell < 0$. In the case where $n = 1$ or $k = 1$, this is the classical determinantal identity for Bernoulli numbers.

7. Further work (in progress)

Finally, we would like to mention a little bit an elliptic analogue of KP-theory. For $R := \mathbb{C}[\mathrm{Eis}_2, \mathrm{Eis}_4, \mathrm{Eis}_6] \subset$ (holomorphic functions on upper half plane), **Eisensteinization map** is the R-linear homomorphism

$$\mathrm{Eis} : R[B] \to R; \quad B^k \mapsto \mathrm{Eis}_k,$$

where $\mathrm{Eis}_k(\tau) := B_k - 2k\sum_{n\geq 1}\sigma_{k-1}(n)q^n$ for k is even, 0 for k is odd ($q := e^{2\pi i\tau}$). We have

$$f_{\mathrm{Eis}}(z) = \mathrm{Eis}_2(\tau)\frac{z^2}{2} + z\frac{d}{dz}\log\sigma(z/2\pi i, \tau) = z\frac{d}{dz}\log\theta_{11}(z/2\pi i, \tau),$$

where σ is *Weierstrass' σ-function*, and θ_{11} is one of *Jacobi's theta functions*. Note that $\lim_{\tau\to i\infty}\mathrm{Eis}(B^n)(\tau) = \mathrm{Ber}(B^n)$ for $n \neq 1$. An analogous theory of [KP] for the elliptic situation is a work in progress.

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