Differentiation formulas for deformed Bessel functions and generalized Hardy's identity

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1 Introduction

Firstly, as subject of this paper, let us consider the lattice point problems, which have been studied for a long time. In particular, the geometric subject to look at this time is the closed curves *p-circle of radius* r (>0) $\{x \in \mathbb{R}^2 |\ |x_1|^p + |x_2|^p = r^p\}$ generalizing the circle for positive real number p, as shown in Figure 1. If p is 2, it is the circle, and for p greater than 1 it expands outward and for p less than 1 it depresses inward. Particularly, the case $p = 2/3$ is well known as *the astroid*.

As an introduction, we will discuss the specifics of the problems and previous studies. $N_p(r)$ is the number of lattice points in the p-circle of radius r at the center of the origin. In addition, since a lattice point and area of the unit square at the center of that point are in one-to-one correspondence, we can consider the error term $P_p(r) := N_p(r) -$(area of the p-circle of radius 1)r^2 through the approximation of Figure 1. Then, the subject of problems is order of growth of P_p as the radius is infinitely large. Specifically, the problem is to find a value such that the orders of the Landau symbols $\mathcal{O}$ and Ω match with respect to P_p. Note that, for the functions f and g, $f(t) = \mathcal{O}(g(t))$ and $f(t) = \Omega(g(t))$ respectively mean $\limsup_{t\to\infty} |f(t)/g(t)| < +\infty$ and $\limsup_{t\to\infty} |f(t)/g(t)| > 0$.

In the cases $p > 2$, there is a well-known method like following. Let us consider the following representation of P_p ([8], (3.57)), which is decomposed by the second main term Ψ and the remainder term Δ by E. Krätzel.

$$P_p(r) = \Psi(r; p) + \Delta(r; p).$$

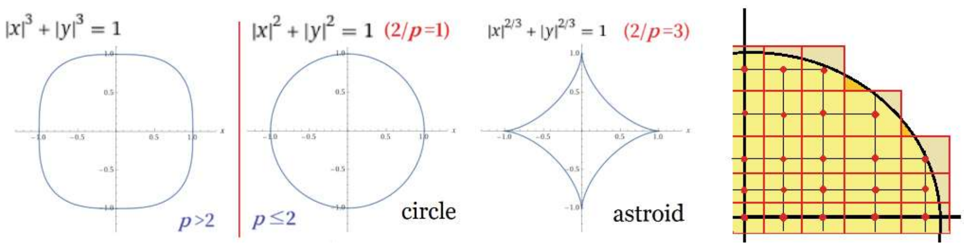

Figure 1: Examples of the p-circle and the approximation by unit squares.

Then, the second main term Ψ is defined as a series consisting of generalized Bessel functions by Krätzel([8], (3.55), Definition 3.3). But note that these are different from deformed Bessel functions stated subsequently in this paper. This representation of P_p and $\mathcal{O}$ - Ω estimates of Ψ, obtained from the asymptotic expansion of Ψ, show that the following important theorem holds.

Theorem 1.1 (*[8], Theorem 3.17 A*). *Let $p > 2$. If $\alpha_p < 1 - \frac{1}{p}$ such that $\Delta(r;p) = \mathcal{O}(r^{\alpha_p})$ exists, then $P_p(r) = \mathcal{O}(r^{1-\frac{1}{p}}), \Omega(r^{1-\frac{1}{p}})$ holds.*

This theorem shows that improving the evaluation for the remainder term Δ is directly related to solving the problem of the cases $p > 2$. As the latest result with this established method, from the G. Kuba's result [9] which $\mathcal{O}$-order of Δ is approximately 46/73 in 1993, the cases with p greater than 73/27 at least have been solved.

On the other hand, in the case of the circle, that is $p = 2$, we emphasize that the following identity which forms the basis of this paper's theme plays an important role in solving this problem.

$$P_2(r) = r\sum_{k=1}^{\infty} \frac{R(k)}{\sqrt{k}} J_1(2\pi\sqrt{k}r) \quad \text{with } R(k) := \#\{n \in \mathbb{Z}^2 \mid |n|^2 = k\}. \tag{1}$$

This identity is a series representation of the error term P_2 composed of the Bessel function of order one J_1 and the number-theoretic function R, derived by G.H. Hardy in 1915 ([1]). Given that *Hardy's identity* (1) converges conditionally and that the growth rate of the error term is greater than half-order, Hardy [2] conjectured the following in 1917.

Conjecture 1.2. *For any small positive real number ε, $P_2(r) = \mathcal{O}(r^{\frac{1}{2}+\varepsilon})$ holds.*

This means that the infimum of P_2 evaluation is half order. Since then, this conjecture has been verified by many mathematicians, but it has not been solved yet. However, as a partial proof of it, many mathematicians have attempted to find an infimum of ε such that this evaluation formula holds. As the latest result by M.N. Huxley [3], it has been found that the conjecture evaluation formula holds for $\varepsilon > 27/208$.

Based on this background, it is natural to look at the remaining cases p, that is, $0 < p < 2$ also. In conclusion, there is no previous study of the classical lattice point problem in these cases. Even if we apply the Krätzel's method for $p > 2$ to these cases, singularities arise in the functions we mainly used, which makes the approach to the problem difficult.

However, these cases are also considered interesting research subject. For example, for a family of *p-ellipse* $\{x \in \mathbb{R}^2 \mid |sx_1|^p + |x_2/s|^p = r^p\}$ $(s > 0)$, the general form of the p-circle for the cases of positive real numbers p, there are problems to find s such that the number of lattice point is the largest. Except for the case $p = 1$, this problem has been solved by R.S. Laugesen et al [11, 12].

Now, as for our research, we focus on the cases p such that $2/p$ is a natural number, and approach the lattice point problem for the figures corresponding to these p, involving the astroid. we call these figures *astroid-type p-circle*.

Thus, with the above, firstly, another method is required. Inspired by the paper of S. Kuratsubo and E. Nakai [10], which gave a harmonic-analytic claim equivalent to Conjecture 1.2, we consider methods like the approach to the circle problem via the Bessel functions and the Fourier transforms. Next, counterparts to the Hardy's identity (1), the main result of this paper, is also necessary as indicators of evaluation improvement.

2 Definition and main result

Firstly, let us recall that definition of the p-circle is a figure represented by p-norm (that is, $|x|_p := (|x_1|^p + |x_2|^p)^{1/p}$ for $x \in \mathbb{R}^2$). Next, we define the deformed Bessel functions $J_\omega^{[p]}$ on $\mathbb{R}^2$ as follows based on *p-radial* (that is, a generalization of *spherical symmetry*). Note that Γ is the gamma function and $J_\omega^{[2]}(x) = J_\omega(|x|)$ holds for $x \in \mathbb{R}^2$.

$$J_\omega^{[p]}(x) := \begin{cases} \frac{1}{\Gamma(\frac{1}{p})^2}(\frac{2}{p})^2 \int_0^1 \cos(x_1\tau^{\frac{1}{p}})\cos(x_2(1-\tau)^{\frac{1}{p}})\tau^{\frac{1}{p}-1}(1-\tau)^{\frac{1}{p}-1}d\tau & (\omega = 0), \\ \frac{|x|_p^\omega}{p^{\omega-1}\Gamma(\omega)} \int_0^1 J_0^{[p]}(\tau x)\tau(1-\tau^p)^{\omega-1}d\tau & (\omega > 0), \end{cases}$$

Then, for p such that $2/p$ is a natural number, $J_\omega^{[p]}$ can be expressed by the following series that converges uniformly on compacts (K. 2024. [5], Proposition 2.6).

$$J_\omega^{[p]}(x) = \frac{(\frac{|x|_p}{p})^\omega(\frac{2}{p})^2}{\Gamma(\frac{1}{p})^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\frac{2}{p}(k+1)+\omega)} \sum_{m\in\mathbb{N}_0^2 \ |m|'=k} \frac{\Gamma(\frac{2m+1}{p})}{(2m)!} x^{2m} \qquad \text{for } \omega \geq 0.$$

Note that, for $n \in \mathbb{N}_0^2$, $x \in \mathbb{R}^2$ and $\xi \in \mathbb{R}_{>0}^2$, we denote $|n|' := n_1 + n_2$, $n! := n_1! \cdot n_2!$, $x^n := x_1^{n_1} \cdot x_2^{n_2}$ and $\Gamma(\xi) := \Gamma(\xi_1) \cdot \Gamma(\xi_2)$ according to multi-index notation.

Furthermore, we define the function $\mathcal{J}_{\omega,\varphi}^{[p]}$ on nonnegative real numbers as

$$\mathcal{J}_{\omega,\varphi}^{[p]}(r) := \frac{(\frac{2}{p})^2}{\Gamma(\frac{1}{p})^2} \sum_{k=0}^{\infty} \frac{p^{2k}(-1)^k}{\Gamma(\frac{2}{p}(k+1)+\omega)} \left(\frac{r}{p}\right)^{2k+\omega} \sum_{m\in\mathbb{N}_0^2 \ |m|'=k} \frac{\Gamma(\frac{2m+1}{p})}{(2m)!} |\cos^{m_1}\varphi \sin^{m_2}\varphi|^{\frac{4}{p}} \quad \text{for } \omega \geq 0$$

(K. 2025. [7], (1.3)), which is defined by fixing *a distorted angle* $\varphi(\in [0, 2\pi))$ via

$$x = (\mathrm{sgn}(\cos\varphi)r|\cos\varphi|^{\frac{2}{p}}, \mathrm{sgn}(\sin\varphi)r|\sin\varphi|^{\frac{2}{p}}) \qquad \text{for } r \geq 0.$$

Then, as our main result, a generalization of the Hardy's identity(1) for the astroid-type p-circle can be expressed by using the function $\mathcal{J}_{\omega,\varphi}^{[p]}$ as follows.

Theorem 2.1 (*Generalized Hardy's identity; K. 2025. [7], Theorem 1.2*)**.**
Let p satisfy $\frac{2}{p} \in \mathbb{N}$ and a finite set $\mathcal{A}_s^{[p]}$ consist of distorted angles φ corresponding to lattice points on p-circle of radius $s^{1/p}$ (≥ 1). Specifically, $\mathcal{A}_s^{[p]}$ is denoted as follows, and $\#\mathcal{A}_s^{[p]} \leq 4[s^{\frac{1}{p}}]$ holds.

$$\mathcal{A}_s^{[p]} := \{\varphi \in [0, 2\pi) \mid (\mathrm{sgn}(\cos\varphi)s^{\frac{1}{p}}|\cos\varphi|^{\frac{2}{p}}, \mathrm{sgn}(\sin\varphi)s^{\frac{1}{p}}|\sin\varphi|^{\frac{2}{p}}) \in \mathbb{Z}^2\}.$$

Then, the following holds for the counting measure μ.

$$P_p(r) = \frac{p\Gamma(\frac{1}{p})^2}{2\pi} r \int_1^\infty \frac{1}{s^{\frac{1}{p}}} \left(\sum\nolimits_{\varphi\in\mathcal{A}_s^{[p]}} \mathcal{J}_{1,\varphi}^{[p]}(2\pi s^{\frac{1}{p}} r)\right) d\mu(s).$$

This shows that asymptotic behavior of $\mathcal{J}_{1,\varphi}^{[p]}$ provides an indicator of the infimum order of P_p.

3 Outline of the proof of Theorem 2.1

For the proof of Theorem 2.1, it is necessary to prepare some properties of the functions $J_\omega^{[p]}$ and $\mathcal{J}_{\omega,\varphi}^{[p]}$. Firstly, a function F on $\mathbb{R}^2$ is said to be *p-radial*, if there exists a function ϕ on non-negative real numbers such that $F(x) = \phi(|x|_p)$ holds for any $x \in \mathbb{R}^2$, then the following holds as lemma.

Lemma 3.1 (*K. 2024. [5], (2.2)*). *Let $p > 0$, then the Fourier transform of an integrable and p-radial function F on $\mathbb{R}^2$ is expressed as*

$$\hat{F}(\xi) = p\Gamma\Big(\frac{1}{p}\Big)^2 \int_0^\infty J_0^{[p]}(2\pi r\xi)\phi(r)\ r dr \qquad \text{for } \xi \in \mathbb{R}^2.$$

Secondly, for p satisfying $(2/p) \in \mathbb{N}$, we derive a differential formula for $\mathcal{J}_{\omega,\varphi}^{[p]}$. As preparation for this, it is necessary to introduce *the Erdély-Kober type fractional integral of order* α ([4], (2.6.1))

$$(I_{0+;p,\eta}^{\alpha})f(r) := \frac{p}{\Gamma(\alpha)}\int_0^1 \frac{\tau^{p(\eta+1)-1}f(\tau r)}{(1-\tau^p)^{1-\alpha}}d\tau \quad \text{for } \alpha, \eta \in \mathbb{C} \text{ such that } \mathrm{Re}(\alpha) > 0. \tag{2}$$

Let $\alpha = \gamma$ and $\eta = (1 - 1/p)\omega + (2/p) - 1$, then, by using the notation for fractional integrals of $\mathcal{J}_{\omega.\varphi}^{[p]}$, the following integral formula (K. 2025. [7], (2.7)).

$$\mathcal{J}_{\omega+\gamma,\varphi}^{[p]}(r) = \frac{r^\gamma}{p^{\gamma-1}\Gamma(\gamma)}\int_0^1 \mathcal{J}_{\omega,\varphi}^{[p]}(\tau r)\tau^{(p-1)\omega+1}(1-\tau^p)^{\gamma-1}d\tau \quad \text{for } \omega \geq 0,\ \gamma > 0,\ r > 0$$

can be rewritten as

$$(I_{0+;p,(1-\frac{1}{p})\omega+\frac{2}{p}-1}^{\gamma})\mathcal{J}_{\omega,\varphi}^{[p]}(r) = \Big(\frac{p}{r}\Big)^\gamma \mathcal{J}_{\omega+\gamma,\varphi}^{[p]}(r). \tag{3}$$

That is, by applying the fractional integral (2), γ is added to the degree ω, and further, the function is multiplied by $(p/r)^\gamma$, which appears to be a highly interesting property.

Next, we introduce *the Erdély-Kober type fractional derivative of order* α ([4], (2.6.29))

$$(D_{0+;p,\eta}^{\alpha})f(r) := r^{-p\eta}\Big(\frac{1}{pr^{p-1}}\frac{d}{dr}\Big)^n r^{p(n+\eta)}(I_{0+;p,\eta+\alpha}^{n-\alpha})f(r),$$
$$\text{for } \alpha \in \mathbb{C}\setminus\{0\} \text{ satisfying } \mathrm{Re}(\alpha) \geq 0,\ n := [\mathrm{Re}(\alpha)] + 1,\ \eta \in \mathbb{C}.$$

In particular, for $0 < \gamma < 1$, note that the recursive property $(D_{0+;p,\eta}^{\alpha} I_{0+;p,\eta}^{\alpha})f = f$ holds ([4], (2.6.43)), then the following differential formula for $\mathcal{J}_{\omega,\varphi}^{[p]}$ is derived by fractionally differentiating both sides of (3) as follows.

$$\begin{aligned}\mathcal{J}_{\omega,\varphi}^{[p]}(r) &= \frac{r^{(1-p)\omega+p-2}}{pr^{p-1}}\frac{d}{dr}r^{(p-1)\omega+2}\Big(I_{0+;p,(1-\frac{1}{p})\omega+\frac{2}{p}-1+\gamma}^{1-\gamma}\Big[\Big(\frac{p}{r}\Big)^\gamma \mathcal{J}_{\omega+\gamma,\varphi}^{[p]}(r)\Big]\Big)\\ &= \cdots = \frac{1}{r^{1+(p-1)\omega}}\frac{d}{dr}r^{1+(p-1)\omega}\mathcal{J}_{(\omega+\gamma)+(1-\gamma)}^{[p]}(r).\end{aligned}$$

Proposition 3.2 (*Differential formula for $\mathcal{J}_{\omega,\varphi}^{[p]}$; K. 2025. [7], Proposition 2.3*). *Let p satisfy $(2/p) \in \mathbb{N}$, then the following holds.*

$$\frac{d}{dr}r^{1+(p-1)\omega}\mathcal{J}_{\omega+1,\varphi}^{[p]}(r) = r^{1+(p-1)\omega}\mathcal{J}_{\omega,\varphi}^{[p]}(r) \quad \text{for } \omega \geq 0,\ 0 \leq \varphi < 2\pi.$$

Hereafter, based on the above, we outline the proof of Theorem 2.1.

Let p satisfy $(2/p) \in \mathbb{N}$. By *the Poisson summation formula* (that is, an equality for periodization of integrable functions), and Lemma 3.1, for a function F which is p-radial and is integrable on $\mathbb{R}^2$, the following holds.

$$\sum_{n\in\mathbb{Z}^2} F(n) = \sum_{n\in\mathbb{Z}^2} \hat{F}(n) = \sum_{n\in\mathbb{Z}^2} \Big(p\Gamma(1/p)^2 \int_0^\infty J_0^{[p]}(2\pi tn)\phi(t)t\ dt \Big). \tag{4}$$

Lemma 3.3 (*Poisson summmation formula: [13], Theorem 2.4*). *For a function F integrable on $\mathbb{R}^d$ $(d \in \mathbb{N})$, the series $f(x) := \sum_{m\in\mathbb{Z}^d} F(x+m)$ converges in the L^1-norm of $\mathbb{T}^d (:= (-\frac{1}{2}, \frac{1}{2}]^d)$ and is integrable on $\mathbb{T}^d$, and $\hat{F}(m) = \hat{f}(m)$, that is, the following holds.*

$$f(x) = \sum_{m\in\mathbb{Z}^d} \hat{F}(m) e^{2\pi i x\cdot m} \qquad \text{for } x \in \mathbb{T}^d.$$

Note that the first equality of (4) follows from the Poisson summation formula based on integrability, and the second equality of it follows from the Lemma 3.1 based on the p-radial assumption.

Particularly, if we set F to be the indicator function on the p-circle of radius r (that is, $F(x) := 1\ (|x|_p < r),\ 0\ (|x|_p \geq r)$), then the left-hand side becomes the total number of lattice points N_p. On the right-hand side, by decomposing the series based on whether n is zero or not, we can find the area of the p-circle as the first term.

$$\begin{aligned} N_p(r) &= p\Gamma(1/p)^2 \Big(\int_0^r \frac{(\frac{2}{p})^2}{\Gamma(\frac{2}{p})} t\ dt + \sum\nolimits_{n\in\mathbb{Z}^2\setminus\{0\}} \Big(\int_0^r J_0^{[p]}(2\pi tn)t\ dt \Big) \Big) \\ &= \frac{2}{p} \frac{\Gamma(\frac{1}{p})^2}{\Gamma(\frac{2}{p})} r^2 + p\Gamma(1/p)^2 \sum\nolimits_{n\in\mathbb{Z}^2\setminus\{0\}} \Big(\int_0^r J_0^{[p]}(2\pi tn)t\ dt \Big) \end{aligned}$$

Thus, from the definition of P_p, the series representation for the error term

$$P_p(r) = p\Gamma(1/p)^2 \sum\nolimits_{n\in\mathbb{Z}^2\setminus\{0\}} \Big(\int_0^r J_0^{[p]}(2\pi tn)t\ dt \Big)$$

is obtained, consisting of $J_0^{[p]}$. Furthermore, by using the set $\mathcal{A}_s^{[p]}$ consisting of distorted angles φ (see the definition in Theorem 2.1) and the counting measure μ, we rewrite $J_0^{[p]}$ as $\mathcal{J}_{0,\varphi}^{[p]}$. Then, by applying the integral form of the previously obtained differential formula (Proposition 3.2; $\int_0^r \tau \mathcal{J}_{0,\varphi}^{[p]}(\tau)d\tau = r\mathcal{J}_{1,\varphi}^{[p]}(r)$), the degree of $\mathcal{J}_{\omega,\varphi}^{[p]}$ increases from 0 to 1. After rearranging the expressions, the desired generalized Hardy's identity is obtained, completing the proof.

$$\begin{aligned} P_p(r) &= p\Gamma(1/p)^2 \int_1^\infty \sum\nolimits_{\varphi\in\mathcal{A}_s^{[p]}} \Big(2\pi r s^{\frac{1}{p}} \mathcal{J}_{1,\varphi}^{[p]}(2\pi s^{\frac{1}{p}} r) \Big) (2\pi s^{\frac{1}{p}})^{-2} d\mu(s) \\ &= \frac{p\Gamma(\frac{1}{p})^2}{2\pi} r \int_1^\infty \frac{1}{s^{\frac{1}{p}}} \Big(\sum\nolimits_{\varphi\in\mathcal{A}_s^{[p]}} \mathcal{J}_{1,\varphi}^{[p]}(2\pi s^{\frac{1}{p}} r) \Big) d\mu(s). \end{aligned}$$

Remark 3.4. For $p = 2$, particularly, from $s \in \mathbb{N}$ and φ-invariance of $\mathcal{J}_{\omega,\varphi}^{[p]}$, the Hardy's identity (1) is obtained. Based on this result, it can be said that the asymptotic behavior of $\mathcal{J}_{1,\varphi}^{[p]}$ which is uniform with respect to the distorted angle φ is essential.

4 Concluding remarks

Firstly, we emphasize the relationship of the functions $\mathcal{J}^{[p]}_{\omega,\varphi}$ and the Erdély-Kober's operators. As the Bessel functions J_ω is one of the solutions of differential equation

$$r^2\frac{d^2}{dr^2}u(r)+r\frac{d}{dr}u(r)+(r^2-\omega^2)u(r)=0 \qquad \text{with } u(r)=J_\omega(r), \tag{5}$$

it is natural to consider whether equation corresponding to $\mathcal{J}^{[p]}_{\omega,\varphi}$ exists. By using the differential formulas of J_ω

$$\frac{d}{dr}r^{\omega+1}J_{\omega+1}(r)=r^{\omega+1}J_\omega(r), \quad \frac{d}{dr}\Big(\frac{J_\omega(r)}{r^\omega}\Big)=-\frac{J_{\omega+1}(r)}{r^\omega},$$

the equation (5) can be obtained. Note that Proposition 3.2 is a ganeralization of the former (the formula of decreasing order). On the other hand, the latter (that of increasing order) seems to be difficult to express in the form of first-order derivative due to the effect of the distorted angle φ.

However, since the connection with *the fractional derivative operator* was clarified by this study, we found a possibility to derive the formula expressed in terms of fractional derivatives. Although not a refined form, we have already obtained the following fractional differential equation whose solution is $\mathcal{J}^{[p]}_{\omega,\varphi}$ for p satisfying $(2/p)\in\mathbb{N}$. Note that E is the identity operator.

$$pr^p\Big(D^1_{0+;p,\eta(p,\omega)}\Big)u(r)+r\frac{d}{dr}\Big(I^1_{0+;p,\eta(p,\omega)+1}-E\Big)u(r)+(p-1)(\omega-2)\Big(I^1_{0+;p,\eta(p,\omega)+1}-E\Big)u(r)=0$$
$$\text{with } u(r)=\mathcal{J}^{[p]}_{\omega,\varphi}(r),\ \eta(p,\omega)=(1-1/p)\omega-2+2/p.$$

Finally, we discuss the importance of uniformly asymptotic evaluations for $J^{[p]}_\omega$. The asymptotic evaluation formula for order zero has already been obtained as the following theorem, but not yet for order positive real numbers.

Theorem 4.1 (*[6], Theorem 1.5; Uniformly asymptotic estimates on* $\mathbb{R}^2$)**.** *For the cases such that $2/p$ are the natural numbers other than 2, the following holds uniformly with respect to $\varphi\in[0,2\pi)$.*

$$J^{[p]}_0(x)=\begin{cases}\mathcal{O}(|x|_p^{-\frac{1}{2}}) & (p=2),\\ \mathcal{O}(|x|_p^{-\frac{p}{2}}) & (\frac{2}{p}\in\mathbb{N}\setminus\{1,2\}),\end{cases} \qquad \text{as } |x|_p\to\infty.$$

For example, if the asymptotic expansion can be obtained by a complex-analytic approach through the analytic connection of the function $\mathcal{J}^{[p]}_{1,\varphi}$, the infimum of *the astroid-type p-circle* error evaluation order can be inferred from the generalized Hardy's identity (Theorem 2.1).

Now, as the key to investigate the asymptotic behavior of $\mathcal{J}^{[p]}_{\omega,\varphi}$, we focus on *the Poisson's integral for the Bessel functions* ([14], p47; 3·3, (1))

$$J_\omega(z)=\frac{2}{\sqrt{\pi}\Gamma(\omega+\frac{1}{2})}\Big(\frac{z}{2}\Big)^\omega\int_0^{\frac{\pi}{2}}\cos(z\cos\theta)\sin^{2\omega}\theta d\theta \quad \text{for } \omega>-\frac{1}{2}. \tag{6}$$

If we define the complex function $\mathcal{J}^{[p]}_{\omega,\varphi}$ and the deformed cosine for φ and p such that $2/p$ is an odd

$$\mathcal{J}^{[p]}_{\omega,\varphi}(z) := \left(\frac{2}{p}\right)^{2+\omega} \frac{\pi}{\Gamma(\frac{1}{p})^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\frac{2}{p}(k+1)+\omega)} \left(\frac{z}{2}\right)^{2k+\omega} \Phi^{[p]}_{k,\varphi},$$

$$\cos^{[p]}_{\varphi}(z) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!!} \left(\frac{\sqrt{\pi}\ \Phi^{[p]}_{k,\varphi}}{\Gamma\big(\frac{1}{p}(2k+1)\big)2^k} \right) z^{2k},$$

$$\text{with } \Phi^{[p]}_{k,\varphi} := \frac{k!\ 2^{2k}}{\pi} \sum_{m_1+m_2=k}^{m\in\mathbb{N}_0^2} \frac{\Gamma(\frac{2}{p}m_1+\frac{1}{p})\Gamma(\frac{2}{p}m_2+\frac{1}{p})}{(2m_1)!\ (2m_2)!} (\cos^{m_1}\varphi \sin^{m_2}\varphi)^{\frac{4}{p}},$$

then the following generalization of the representation (6) is obtained. Note that the former is holomorphic on $\mathbb{C} \setminus \{t \in \mathbb{R} \mid t \le 0\}$, and the latter is an entire function ($\cos^{[2]}_{\varphi}(z) = \cos z$).

$$\mathcal{J}^{[p]}_{\omega,\varphi}(z) = \frac{\sqrt{\pi}(\frac{2}{p})^{2+\omega}2}{\Gamma(\frac{1}{p})^2\Gamma(\omega+\frac{1}{p})} \left(\frac{z}{2}\right)^{\omega} \int_0^{\frac{\pi}{2}} \cos^{[p]}_{\varphi}(z\cos^{\frac{2}{p}}\theta) \sin^{2\omega}\theta (\cos\theta\sin\theta)^{\frac{2}{p}-1} d\theta.$$

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References

[1] G.H. Hardy, On the expression of a number as the sum of two squares, Q. J. Math. 46 (1915) 263-283.

[2] G.H. Hardy, E. Landau, The average order of the arithmetical functions $P(x)$ and $\Delta(x)$, Proc. Lond. Math. Soc. 15 (1917) 192-213.

[3] M.N. Huxley, Exponential sums and lattice points. III, Proc. Lond. Math. Soc. (3) 87(3) (2003) 591-609.

[4] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier. North-Holland mathematics studies. 204 (2006).

[5] M. Kitajima, Series expansions by generalized Bessel functions for certain functions related to the lattice point problems for the p-circle, arXiv:2408.02613v1 (2024).

[6] M. Kitajima, Asymptotic evaluations of generalized Bessel function of order zero related to the p-circle lattice point problem, Res. number theory 11. 71 (2025).

[7] M. Kitajima, Generalized Hardy's identity for the astroid-type p-circle lattice point problem, arXiv:2506.03331v2 (2025).

[8] E. Krätzel, Lattice Points, Kluwer Academic Publication, 1988.

[9] G. Kuba, On sums of two k-th powers of numbers in residue classes II, Abh. Math. Sem. Univ. Hamburg 63 (1993) 87-95.

[10] S. Kuratsubo & E. Nakai, Multiple Fourier series and lattice point problems, J. Func. Anal. 282 (2022) 1-62.

[11] R.S. Laugesen, S. Ariturk, Optimal stretching for lattice points under convex curves, Port. Math. 74 (2017) 91-114.

[12] R.S. Laugesen, S. Liu, Optimal stretching for lattice points and eigenvalues, Ark. Mat. 56 (2018) 111-145.

[13] E.M. Stein & G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton Univ. Press, 1971.

[14] G.N. Watson, A treatise on the theory of Bessel functions, 2nd ed., Cambridge Univ. Press, 1995.

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