On Sun's $x + ny$ conjecture

Songlin Han

ABSTRACT. We report on our work approaching a 2015 conjecture of Zhi-Wei Sun, which asserts that every integer $n > 1$ can be written as $n = x + y$ $(x, y > 0)$ with both $x + ny$ and $x^2 + ny^2$ prime. Using Richert's weighted sieve, we establish partial results: for all sufficiently large n, the decomposition $n = x + y$ can be achieved with $\Omega(x + ny) \le 3$ or $\Omega(x^2 + ny^2) \le 4$. A further application of the DHR sieve yields $\Omega\big((x + ny)(x^2 + ny^2)\big) \le 11$.

1. Background and motivation

In 2015, Zhi-Wei Sun proposed a family of conjectures on representations of integers involving primes [7]. Among them is the following.

Conjecture A ([7, Conjecture 2.21 (i)])**.** *Every integer $n > 1$ has a decomposition $n = x + y$ with $x, y > 0$ such that $x + ny$ and $x^2 + ny^2$ are simultaneously prime.*

Since $n = x + y$ forces $x = n - y$, the two target expressions become polynomials in a single variable y:

$$F_n^1(y) = n + (n-1)y, \qquad F_n^2(y) = (n+1)y^2 - 2ny + n^2.$$

Detecting primes of this form is closely related to Linnik's problem on the least prime in arithmetic progressions. Linnik [4] showed that $\max_{(l,k)=1} p(k, l) \ll k^L$ for an absolute constant L; the current best bound is $L \le 5.18$ due to Xylouris [8]. In the opposite direction, Li–Pratt–Shakan [3] proved

$$\max_{(l,k)=1} p(k, l) \gg \phi(k)\,(\log k)\,(\log\log k)\,\frac{\log\log\log\log k}{\log\log\log k},$$

illustrating that prime-level results in such settings are inherently delicate. This motivates the study of almost-prime relaxations.

This is joint work with Jinbo Yu and the related paper [2] has been subscribed.

2. Main results

Our main theorem provides two separate almost-prime substitutes for Conjecture A.

Theorem 1. *Every sufficiently large integer n can be written as $n = x + y$ with $x, y > 0$ such that*

(i) $\Omega(x + ny) \le 3$, *or*
(ii) $\Omega(x^2 + ny^2) \le 4$.

By applying the multi-dimensional weighted sieve of Diamond–Halberstam–Richert [1], Chapter 11, we further obtain: for every sufficiently large n there exist $x, y > 0$ with $n = x + y$ and

$$\Omega\big((x + ny)(x^2 + ny^2)\big) \le 11.$$

Key words and phrases. linear sieve, weighted sieve, integer partitions, almost prime numbers.

3. Method of proof

For each of the polynomials F_n^i $(i=1,2)$, we sift the sequence $\mathcal{A}_n^i = \{F_n^i(y) : 1 \le y < n\}$ of size $X = n-1$. Let $\rho(\mathcal{A}, p)$ denote the number of roots of F_n^i modulo p, extended multiplicatively to square-free moduli via the Chinese Remainder Theorem. The local density satisfies $0 \le \rho(\mathcal{A}, p) \le (1 - 1/A_1)\, p$ for a constant $A_1 \ge 1$. Neither F_n^1 nor F_n^2 has a fixed prime divisor, so $\rho(\mathcal{A}, p) < p$ for every p and the global density $G(z) = \prod_{p<z}(1 - \rho(\mathcal{A}, p)/p)$ is strictly positive.

Using Nagell's theorem [5] and Mertens' estimate, one verifies the standard sieve axioms:

- $\displaystyle \Bigg| \sum_{z \le p < w} \frac{\rho(\mathcal{A}, p) - 1}{p} \log p \Bigg| \le A_2$ for all $w \ge z \ge 2$,
- $\displaystyle \sum_{d \le X/\log^{3g+2} X} \mu^2(d)\,(3g)^{\omega(d)}\,\eta(\mathcal{A}, d) \le A_3 \frac{X}{\log^2 X}$,
- $\displaystyle \sum_{\substack{a \in \mathcal{A} \\ a \equiv 0 \pmod{p^2}}} 1 \le A_4 \Big(\frac{X \log X}{p^2} + 1 \Big)$,

where $g = \deg F_n^i$ and $\eta(\mathcal{A}, d)$ is the remainder in the equidistribution estimate.

Following Richert [6], we use the weighted sifting function

$$W(\mathcal{A}, u, \lambda) = \sideset{}{'}\sum_{\substack{a \in \mathcal{A} \\ (a, P(X^{1/4}))=1}} \Bigg(1 - \lambda \sum_{\substack{X^{1/4} \le p < X^{1/u} \\ p \mid a}} \Big(1 - u \frac{\log p}{\log X} \Big) \Bigg),$$

where $\sum'$ restricts to a not divisible by p^2 for any $p \in [X^{1/4}, X^{1/u})$. The key property is that, for a suitable pair (u, λ), we have $W(\mathcal{A}, u, \lambda) \le \#\{a \in \mathcal{A} : \Omega(a) \le r\}$.

Richert's theorem [6, Theorem 1] provides the lower bound

$$W(\mathcal{A}, u, \lambda) \ge X\, G(X^{1/4}) \Bigg(\frac{e^\gamma}{2} \log 3 - \lambda \int_u^4 F\big(4(1 - \tfrac{1}{t})\big)\big(1 - \tfrac{u}{t}\big) \frac{dt}{t} - \frac{b}{(\log X)^{1/15}} \Bigg),$$

where F, f are the continuous solutions for some linear sieve differential-delay system and b depends only on u. Evaluating the integral in closed form yields $\frac{e^\gamma}{2} D(u)$ with $D(u) = u \log \frac{4}{u} - (u-1) \log \frac{3}{u-1}$. For the admissibility condition $\max_{a \in \mathcal{A}} \frac{\log a}{\log X} \le \Lambda_r - \delta$, one checks:

- Case F_n^1 $(\deg = 1)$: $\max a < X^2 + X + 1$, so $\frac{\log a}{\log X} < \Lambda_3 \approx 2.7$. Hence $r = 3$ is admissible and $\Omega(F_n^1(y)) \le 3$ is attained.
- Case F_n^2 $(\deg = 2)$: An analogous calculation shows $r = 4$ is admissible, giving $\Omega(F_n^2(y)) \le 4$.

In both cases, the lower bound for W is positive for all sufficiently large n, completing the proof of Theorem 1.

4. Remarks

The almost-prime bounds P_3 and P_4 obtained here are consequences of Richert's general result that an irreducible polynomial of degree g represents infinitely many integers with at most $g+1$ prime factors [6]. The simultaneous constraint $\Omega((x+ny)(x^2+ny^2)) \le 11$ requires the multi-dimensional extension of [1], Chapter 11. Improving these bounds toward the full prime case of Conjecture A would likely require progress on the parity problem in sieve theory.

Acknowledgements. The author thanks Professor Ade Irma Suriajaya for her supervision, and Professors Kohji Matsumoto and Henrik Bachmann for their kind advice.

References

1. H. G. Diamond and H. Halberstam, *A higher-dimensional sieve method*, Cambridge Tracts in Mathematics, vol. 177, Cambridge University Press, Cambridge, 2008, With an appendix ("Procedures for computing sieve functions") by W. F. Galway.
2. Songlin Han and Jinbo Yu, *Almost-primes in sun's $x^2 + ny^2$ conjecture*, 2026.

3. J. Li, K. Pratt, and G. Shakan, *A lower bound for the least prime in an arithmetic progression*, Quarterly Journal of Mathematics **68** (2017), no. 3, 729–758.
4. Y. V. Linnik, *On the least prime in an arithmetic progression. I. the basic theorem*, Recueil Mathématique [Matematicheskii Sbornik] **15** (1944), no. 57, 139–178.
5. T. Nagel, *Généralisation d'un théorème de Tchebycheff*, Journal de Mathématiques Pures et Appliquées **4** (1921), 343–356.
6. H.-E. Richert, *Selberg's sieve with weights*, Mathematika **16** (1969), 1–22.
7. Z.-W. Sun, *Conjectures on representations involving primes*, Combinatorial and Additive Number Theory II: CANT, New York, NY, USA, 2015 and 2016 (M. Nathanson, ed.), Springer Proceedings in Mathematics & Statistics, vol. 220, Springer, New York, 2017, pp. 279–310.
8. T. Xylouris, *On the least prime in an arithmetic progression and estimates for the zeros of Dirichlet L-functions*, Acta Arithmetica **150** (2011), no. 1, 65–91.

Graduate School of Mathematics, Kyushu University, Fukuoka, Japan.
Email address: han.songlin.638@s.kyushu-u.ac.jp