Pair Correlation Conjecture for the zeros of the Riemann zeta-function

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Abstract: This note investigates the statistical distribution of the nontrivial zeros of the Riemann zeta-function, focusing on the lower proportions of zeros on the critical line and simple zeros. We review the classical framework established by Montgomery, which utilized the method to establish that 66.6% simplicity under the Riemann Hypothesis (RH), and the refinements by Gallagher and Mueller showing that the Pair Correlation Conjecture (PCC) and RH together imply 100% simplicity. The primary contribution of this work is the introduction of the Horizontal Multiplicity Hypothesis (HMH), an extension of a part of a conjecture by Mueller. We establish that HMH is a flexible condition that guarantees 100% simple and 100% critical zeros of the Riemann zeta-function. Remarkably, we show that these results can be achieved without the assumption of RH. We further clarify the hierarchy of conjectures by proving that PCC implies HMH even in the absence of RH. These findings suggest that the simplicity and criticality of zeros are robust statistical properties that may be decoupled from their precise vertical distribution.

Contents

1. Introduction

The **Riemann zeta-function**, $\zeta(s)$, is defined for $\mathfrak{R}(s) > 1$ by the absolute convergent series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

It extends to a meromorphic function on the whole complex plane $\mathbb{C}$, with its only singularity being a simple pole at $s = 1$.

The zeros of the Riemann zeta-function are classified into two types. The **trivial zeros** are located at the negative even integers, $s \in \{-2, -4, -6, ...\}$. The **nontrivial zeros** lie within the **critical strip**, defined by $0 < \Re(s) < 1$. The **Riemann Hypothesis (RH)** asserts that all nontrivial zeros lie on the **critical line**, $\Re(s) = 1/2$. A nontrivial zero ρ of the Riemann zeta-function is said to be **simple** if it has multiplicity one; that is, if $\zeta'(\rho) \neq 0$. The **Simple Zero Conjecture (SZC)** asserts that all nontrivial zeros of the Riemann zeta-function are simple.

While the ultimate goal is to determine if all zeros lie on the critical line (**RH**) and if all such zeros are simple (**SZC**), these remains two of the most profound open problems in mathematics. In the absence of a complete proof, we focus our attention on the statistical distribution of these zeros. Specifically, we investigate the *lower proportions* $\underline{\mathrm{d}}_{\mathrm{c}}$ and $\underline{\mathrm{d}}_{\mathrm{s}}$ to quantify how closely the known distribution aligns with these conjectures.

Throughout the remainder of this note, we only consider nontrivial zeros and simply call them zeros. For convenience, we denote the set of (nontrivial) zeros of $\zeta(s)$ up to height $T > 0$ by

$$\mathcal{Z}(T) := \{\rho = \beta + i\gamma \in \mathbb{C} \mid \zeta(\rho) = 0, 0 < \beta < 1, 0 < \gamma \leq T\}.$$

Throughout this note, we follow the standard convention of denoting these zeros by $\rho = \beta + i\gamma$, where $\beta, \gamma \in \mathbb{R}$ represent the real and imaginary parts, respectively. The total number of zeros (counted with multiplicity) up to height $T > 0$ is given by the counting function

$$\mathrm{N}(T) := \sum_{\rho \in \mathcal{Z}(T)} \mathfrak{m}(\rho),$$

where $\mathfrak{m}(\rho)$ denotes the analytic multiplicity of $\zeta(s)$ at the point ρ. We further denote by $\mathrm{N}_{\mathrm{c}}(T)$ and $\mathrm{N}_{\mathrm{s}}(T)$ the number of zeros (counted with multiplicity) and the number of simple zeros, respectively, that lie on the critical line $\Re(s) = 1/2$ up to height T. Formally, these are defined as

$$\mathrm{N}_{\mathrm{c}}(T) := \sum_{\substack{\rho \in \mathcal{Z}(T) \\ \beta = \frac{1}{2}}} \mathfrak{m}(\rho) \quad \text{and} \quad \mathrm{N}_{\mathrm{s}}(T) := \sum_{\substack{\rho \in \mathcal{Z}(T) \\ \mathfrak{m}(\rho) = 1}} \mathfrak{m}(\rho) = \sum_{\substack{\rho \in \mathcal{Z}(T) \\ \zeta'(\rho) \neq 0}} 1.$$

The **lower proportions** of these zeros are then defined by the limit infimums

$$\underline{\mathrm{d}}_{\mathrm{c}} = \liminf_{T \to \infty} \frac{\mathrm{N}_{\mathrm{c}}(T)}{\mathrm{N}(T)} \quad \text{and} \quad \underline{\mathrm{d}}_{\mathrm{s}} = \liminf_{T \to \infty} \frac{\mathrm{N}_{\mathrm{s}}(T)}{\mathrm{N}(T)}.$$

With these definitions in place, we may now introduce the statistical formulations of our primary conjectures. The **asymptotic Riemann Hypothesis** asserts that *almost all* zeros of the Riemann zeta-function lie on the critical line. Specifically, it implies that the lower proportion of zeros on the critical line is 1:

$$\underline{\mathrm{d}}_{\mathrm{c}} = 1.$$

Similarly, the **asymptotic Simple Zero Conjecture** asserts that *almost all* zeros of the Riemann zeta-function are simple. This corresponds to the statement that the lower proportion of simple zeros is 1:

$$\underline{\mathrm{d}}_{\mathrm{s}} = 1.$$

It is important to note that while **RH** and **SZC** imply these statistical versions, the converse is not necessarily true; a set of zeros of measure zero could still lie off the critical line or be multiple without violating these “100%” proportions. While establishing the full asymptotic versions of **RH** and **SZC** remains a formidable challenge, many prominent mathematicians have sought to uncover the deep relationships between the horizontal and vertical distributions of zeros. These efforts have yielded increasingly stronger quantitative lower bounds for $\underline{\mathrm{d}}_{\mathrm{c}}$ and $\underline{\mathrm{d}}_{\mathrm{s}}$, as summarized in the following tables.

Summary of Known Results

Year	**Authors**	**Condition**	$\underline{\mathrm{d}}_{\mathrm{c}} \geq$	$\underline{\mathrm{d}}_{\mathrm{s}} \geq$
1942	Selberg [1]	-	ε	-
1973	Montgomery [2]	**RH**	1	0.66666
1973	Montgomery [2]	**RH** & **MC**	1	1
1974	Levinson [3]	-	0.3474	-
1975	Montgomery and Taylor	**RH**	1	0.67250
1978	Gallagher and Mueller [4]	**RH** & **PCC**	1	1
1979	Heath-Brown [5]	-	-	0.3474
1983	Conrey [6]	-	0.3658	0.3485
1983	Anderson [7]	-	-	0.3532
1989	Conrey [8]	-	0.4077	0.401
1993	Cheer and Goldston [9]	**RH**	1	0.67275
1998	Conrey et al. [10]	**RH** & **GLH**	1	0.70370
2011	Bui, Conrey, and Young [11]	-	0.4105	0.4058
2013	Heath-Brown and Bui [12]	**RH** ~~& **GLH**~~	1	0.70370
2020	Pratt, Robles, Zaharescu, and Zeindler [13]	-	0.4172	0.4075
2020	Chirre, Goncalves, and de Laat [14]	**RH**	1	0.67920
2025+	**Goldston, Lee, Schettler, and Suriajaya**	~~**RH** &~~ PCC	**1**	**1**

In the table above, we assume the following conjectures:

- **MC** refers to **Montgomery's Conjecture**, which will be reviewed in detail in Section 3;
- **PCC** refers to the **Pair Correlation Conjecture** (also due to Montgomery), which concerns the distribution of the spacings between the zeros of the Riemann zeta-function (see Section 3 for details).

We remark that the proportion of simple zeros in the table above is actually the proportion of zeros which are both simple and critical. This difference is particularly important in the results of Levinson, Conrey, and others which did not get $\underline{\mathrm{d}}_{\mathrm{c}} = 1$. However, to present our results and in the rest of this note, we never need to distinguish zeros which are only simple and zeros which are both simple and critical, thus we chose to stick to the simpler notation $\underline{\mathrm{d}}_{\mathrm{c}}$ and $\underline{\mathrm{d}}_{\mathrm{s}}$. We explain these arguments below. Let $\underline{\mathrm{d}}_{\mathrm{cs}}$ denote the lower proportion of such zeros. In general, we have the following relationship:

$$\underline{\mathrm{d}}_{\mathrm{s}} \geq \underline{\mathrm{d}}_{\mathrm{cs}} \geq \underline{\mathrm{d}}_{\mathrm{c}} + \underline{\mathrm{d}}_{\mathrm{s}} - 1,$$

where the first inequality follows from the fact that any zero that is both critical and simple is, by definition, a simple zero. The second inequality follows from the superadditivity of the limit infimum applied to the inclusion–exclusion principle. Thus, numerical bounds provided for $\underline{\mathrm{d}}_{\mathrm{cs}}$ are technically stronger than bounds on $\underline{\mathrm{d}}_{\mathrm{s}}$ alone, as they require the zeros to be restricted to the critical line ($\beta = 1/2$). However, in cases where we assume **RH** (as in Montgomery or Gallagher–Mueller) or properties derived from **PCC** (as in our result), we have $\underline{\mathrm{d}}_{\mathrm{c}} = 1$. Under this condition, the inequalities above collapse:

$$\underline{\mathrm{d}}_{\mathrm{s}} = \underline{\mathrm{d}}_{\mathrm{cs}}.$$

Consequently, in these “100%” proportion cases, the distinction between simple zeros and simple critical zeros vanishes, and the values reported in the table effectively represent both quantities.

Let d_c and denote the **proportion of critical zeros**, defined formally as:

$$\mathrm{d}_\mathrm{c} = \lim_{T\to\infty} \frac{\mathrm{N}_\mathrm{c}(T)}{\mathrm{N}(T)}.$$

Strictly speaking, this expression is often considered ill-defined in a general context because the existence of the limit is not guaranteed, unlike the lower proportion $\underline{\mathrm{d}}_\mathrm{c}$, which was defined as the limit infimum. For any $T > 0$, we observe the trivial inequality:

$$1 \geq \frac{\mathrm{N}_\mathrm{c}(T)}{\mathrm{N}(T)} \geq \inf_{0<t\leq T} \frac{\mathrm{N}_\mathrm{c}(t)}{\mathrm{N}(t)}.$$

It is a standard observation that the limit (exists and) equals to 1 if and only if $\underline{\mathrm{d}}_\mathrm{c} = 1$. Since it is significantly more convenient to handle the limit directly, we shall adopt the lim notation throughout our discussion in Section 4.

We also remark that Conrey, Ghosh, and Gonek [10] utilized a different approach from the method we focus on this note. By assuming both **RH** and the **Generalized Lindelöf Hypothesis (GLH)**, they obtained the lower bound $\underline{\mathrm{d}}_\mathrm{s} \geq 0.70370$. The **GLH** asserts that for any $\varepsilon > 0$, the bound

$$L\left(\frac{1}{2} + it, \chi\right) \ll (k(|t| + 2))^\varepsilon$$

holds uniformly in both t and the conductor k of the Dirichlet character χ. Notably, the dependence on **GLH** was later removed by Heath-Brown and Bui [12], who established the same numerical lower bound unconditionally on the Lindelöf Hypothesis, requiring only the assumption of **RH**.

We conclude this section by outlining the structure of the remainder of this note. In Section 2, we review Montgomery’s method and examine its application under the assumption of the Riemann Hypothesis (**RH**). In Section 3, we discuss the **Montgomery Conjecture** (**MC**) and its implications, while briefly reviewing the results of Gallagher and Mueller regarding the **Pair Correlation Conjecture** (**PCC**). Finally, in Section 4, we provide a detailed discussion of our main theorems and the idea of the proof, which improves upon the classical results established by Gallagher and Mueller.

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2. Montgomery’s Method

In this section, we will review the result obtained by Montgomery [2].

Theorem 1 (Montgomery’s method with RH)

*Assume the Riemann Hypothesis (**RH**). Then, at least two thirds of the zeros of the Riemann zeta-function are simple, namely, we have*

$$\underline{\mathrm{d}}_\mathrm{s} = \liminf_{T\to\infty} \frac{\mathrm{N}_\mathrm{s}(T)}{\mathrm{N}(T)} \geq \frac{2}{3} = 0.666\ldots.$$

To establish these bounds, Montgomery utilized a kernel pair $\left(\mathrm{k}_\lambda, \hat{\mathrm{k}}_\lambda\right)$ consisting of a function and its Fourier transform. For a parameter $\lambda > 0$, we define the **Montgomery kernel** as the squared sinc function:

$$\mathrm{k}_\lambda(x) := \operatorname{sinc}^2\left(\frac{\lambda x}{2}\right) = \begin{cases} \left(\frac{\sin(\lambda x/2)}{\lambda x/2}\right)^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

The corresponding Fourier transform,

$$\hat{\mathrm{k}}_\lambda(\xi) = \int_{-\infty}^{\infty} \mathrm{k}_\lambda(x) e^{-2\pi i x \xi} dx = \frac{1}{\lambda} \max\left(1 - \frac{|\xi|}{\lambda}, 0\right),$$

which is given by the triangular function.

Now we define the **Montgomery *F*-function** by

$$F_T(\alpha) := \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{\rho,\rho' \in \mathcal{Z}(T)} \mathfrak{m}(\rho)\,\mathfrak{m}(\rho') T^{i\alpha(\gamma-\gamma')}\, w(\gamma-\gamma')$$

for each $\alpha \in \mathbb{R}$ and $T > 0$, where $w(y) := 4/\left(4 + y^2\right)$ for each $y \in \mathbb{R}$. Montgomery observed that the G-function can be represented via the F-function as follows:

$$G_T(\lambda) := \sum_{\rho,\rho' \in \mathcal{Z}(T)} \mathfrak{m}(\rho)\,\mathfrak{m}(\rho')\,\mathrm{k}_\lambda\left(\frac{\gamma-\gamma'}{2\pi}\log T\right) w(\gamma-\gamma') = \left(\frac{T}{2\pi}\log T\right) \int_{\mathbb{R}} F_T(\alpha)\,\hat{\mathrm{k}}_\lambda(\alpha)\,\mathrm{d}\alpha.$$

Moreover, he observed that G-function contains the information of the simple zeros through the following inequality:

$$\mathrm{N}_{\mathrm{s}}(T) \geq 2\,\mathrm{N}(T) - G_T(\lambda) = 2\,\mathrm{N}(T) - \sum_{\rho,\rho' \in \mathcal{Z}(T)} \mathfrak{m}(\rho)\,\mathfrak{m}(\rho')\,\mathrm{k}_\lambda\left(\frac{\gamma-\gamma'}{2\pi}\log T\right) w(\gamma-\gamma').$$

Meanwhile, F-function provides a formula that is analytically more tractable compared to G-function, as stated in the following theorem.

Theorem 2 (Montgomery's theorem)

*Assume the Riemann Hypothesis (**RH**). Then the Montgomery F-function is non-negative, real-valued, and even. Moreover, as $T \to \infty$, we have the asymptotic formula*

$$F_T(\alpha) = (1 + o(1))T^{-2\alpha}\log T + \alpha + o(1)$$

uniformly for $0 \leq \alpha \leq 1$.

We can now derive Montgomery's result (Theorem 1) on the proportion of simple zeros by formulating the problem as an optimization of the G-functional. By applying the asymptotic formula from Theorem 2, we observe that as $T \to \infty$, the lower density $\underline{\mathrm{d}}_{\mathrm{s}}$ is determined by the behavior of the kernel's integral. Indeed, substituting the asymptotic for $F_T(\alpha)$ into the integral representation of G_T, we have, for any $0 \leq \lambda \leq 1$,

$$\begin{aligned}\int_{\mathbb{R}} F_T(\alpha)\,\hat{\mathrm{k}}_\lambda(\alpha)\,\mathrm{d}\alpha &= \int_{\mathbb{R}} [(1 + o(1))T^{-2\alpha}\log T + \alpha + o(1)]\,\hat{\mathrm{k}}_\lambda(\alpha)\,\mathrm{d}\alpha \\ &\sim 1 + 2\int_0^\lambda \alpha\,\hat{\mathrm{k}}_\lambda(\alpha)\,\mathrm{d}\alpha = 1 + 2\int_0^\lambda \alpha\left(\frac{1}{\lambda}\left(1 - \frac{\alpha}{\lambda}\right)\right)\mathrm{d}\alpha = 1 + \frac{\lambda}{3},\end{aligned}$$

where the first term arises from the delta-like spike of $T^{-2\alpha}\log T$ at $\alpha = 0$. Consequently, as $T \to \infty$, the proportion of simple zeros satisfies

$$\frac{\mathrm{N_s}(T)}{\mathrm{N}(T)} \geq 2 - \frac{G_T(\lambda)}{\mathrm{N}(T)} + o(1) \sim 2 - \left(1 + \frac{\lambda}{3}\right) = 1 - \frac{\lambda}{3}.$$

Setting the parameter $\lambda = 1$ (the limit of the range for which Theorem 2 holds) yields

$$\underline{\mathrm{d}}_{\mathrm{s}} = \liminf_{T\to\infty} \frac{\mathrm{N_s}(T)}{\mathrm{N}(T)} \geq 1 - \frac{1}{3} = \frac{2}{3},$$

which concludes the proof of Theorem 1.

We remark that Montgomery and Taylor [15] were able to improve the estimation of the proportion of simple zeros by refining Montgomery's pair-correlation method. Instead of the standard triangular kernel, they employed an optimized *Montgomery–Taylor kernel*, which minimizes the contribution of the off-diagonal terms in the G-functional.

Theorem 3 (Montgomery method with Montgomery-Taylor kernel)

*Assume the Riemann Hypothesis (**RH**). Then, as T → ∞, at least **67.25%** of the zeros of the Riemann zeta-function are simple. Namely, we have*

$$\underline{\mathrm{d}}_{\mathrm{s}} \geq 0.6725.$$

3. Montgomery's Conjecture and Pair Correlation Conjecture

As observed in the previous result, the lower bound for the proportion of simple zeros increases as we take larger values of $\lambda > 0$. However, current techniques (such as Theorem 2) are limited to the range $0 \leq \alpha \leq 1$. To bypass this restriction, Montgomery conjectured the behavior of the F-function for all $\alpha \geq 1$.

Conjecture 1 (Montgomery Conjecture)

For any fixed $M > 1$, *the Montgomery* F*-function satisfies*

$$F_T(\alpha) = 1 + o(1)$$

uniformly for $1 \leq \alpha \leq M$ *as* $T \to \infty$.

Assuming both the Riemann Hypothesis (**RH**) and Montgomery's Conjecture (**MC**), we can extend the range of the parameter λ indefinitely. As $\lambda \to \infty$, the "extra" multiplicity term $G_T(\lambda)$ becomes negligible relative to the total number of zeros.

Theorem 4 (Montgomery's method with RH and MC)

*Assume the Riemann Hypothesis (**RH**) and the Pair Correlation Conjecture (**PCC**). Then, as T → ∞, the proportion of simple zeros on the critical line satisfies*

$$\underline{\mathrm{d}}_{\mathrm{s}} = \liminf_{T\to\infty} \frac{\mathrm{N_s}(T)}{\mathrm{N}(T)} = 1.$$

This result indicates that, under the strongest distributional assumptions, almost all zeros of the Riemann zeta-function are simple and lie on the critical line.

The proof of Theorem 4 follows the same framework as that of Theorem 1, but allows for an unrestricted optimization over the parameter λ. Indeed, for any $\lambda > 1$, we decompose the integral of the smoothed F-function into its "**RH**" and "**MC**" ranges:

$$\int_{\mathbb{R}} F_T(\alpha)\,\hat{\mathrm{k}}_\lambda(\alpha)\,\mathrm{d}\alpha = \left(\int_{-1}^{1} + 2\int_{1}^{\infty}\right) F_T(\alpha)\,\hat{\mathrm{k}}_\lambda(\alpha)\,\mathrm{d}\alpha = \left(\int_{-1}^{1} + 2\int_{1}^{\lambda}\right) F_T(\alpha)\,\hat{\mathrm{k}}_\lambda(\alpha)\,\mathrm{d}\alpha,$$

where we utilize the fact that $\hat{\mathrm{k}}_\lambda(\alpha) = 0$ for $|\alpha| \geq \lambda$. From the proof of Theorem 1, the first integral, which corresponds to the range where **RH** is sufficient, contributes

$$\int_{-1}^{1} F_T(\alpha)\,\hat{\mathrm{k}}_\lambda(\alpha)\,\mathrm{d}\alpha \sim 1 + 2\int_{0}^{1} \alpha\,\hat{\mathrm{k}}_\lambda(\alpha)\,\mathrm{d}\alpha.$$

Meanwhile, by assuming **MC**, the "off-diagonal" range contributes

$$2\int_{1}^{\lambda} F_T(\alpha)\,\hat{\mathrm{k}}_\lambda(\alpha)\,\mathrm{d}\alpha \sim 2\int_{1}^{\lambda} (1 + o(1))\,\hat{\mathrm{k}}_\lambda(\alpha)\,\mathrm{d}\alpha.$$

Combining the above, the total asymptotic behavior of the integral becomes

$$\begin{aligned}\int_{\mathbb{R}} F_T(\alpha)\,\hat{\mathrm{k}}_\lambda(\alpha)\,\mathrm{d}\alpha &\sim 1 + 2\int_{0}^{1} \alpha\,\hat{\mathrm{k}}_\lambda(\alpha)\,\mathrm{d}\alpha + 2\int_{1}^{\lambda} \hat{\mathrm{k}}_\lambda(\alpha)\,\mathrm{d}\alpha \\ &= 1 + 2\int_{0}^{\lambda} \hat{\mathrm{k}}_\lambda(\alpha)\,\mathrm{d}\alpha - 2\int_{0}^{1} (1-\alpha)\,\hat{\mathrm{k}}_\lambda(\alpha)\,\mathrm{d}\alpha.\end{aligned}$$

For the triangular kernel $\hat{\mathrm{k}}_\lambda(\alpha) = \lambda^{-1}(1 - |\alpha|/\lambda)$, a direct calculation shows that as $T \to \infty$, the ratio of the G-function to the total number of zeros satisfies $G_T(\lambda)/\,\mathrm{N}(T) \sim 1 + O(\lambda^{-1})$. Substituting this into our density inequality as $T \to \infty$, we obtain:

$$\frac{\mathrm{N_s}(T)}{\mathrm{N}(T)} \geq 2 - \frac{G_T(\lambda)}{\mathrm{N}(T)} + o(1) \sim 2 - \left(1 + O\Big(\frac{1}{\lambda}\Big)\right) = 1 + O\Big(\frac{1}{\lambda}\Big).$$

By taking the limit $\lambda \to \infty$, we conclude that $\underline{\mathrm{d}}_{\mathrm{s}} = 1$, as required, so Theorem 4 is proved.

While Theorem 4 establishes the 100% simplicity of zeros under the strong assumption of **MC**, it is natural to ask whether this assumption can be weakened. In [4], Gallagher and Mueller demonstrated that the full strength of **MC** —which specifies the exact asymptotic for $F_T(\alpha)$— is not strictly necessary to preserve the 100% simplicity result. Instead, it can be replaced by a more general distributional property known as the **Pair Correlation Conjecture (PCC)**. To establish **PCC**, we introduce a counting function that captures the number of pairs of zeros whose normalized gaps fall within a specific range. For $T > 0$ and $\lambda > 0$, we define

$$\mathcal{N}_T(\lambda) := \sum_{\substack{\rho,\rho' \in \mathcal{Z}(T) \\ 0 < \gamma - \gamma' \leq \frac{2\pi\lambda}{\log T}}} \mathfrak{m}(\rho)\,\mathfrak{m}(\rho').$$

Here, the condition $\gamma - \gamma' \leq (2\pi\lambda)/\log T$ ensures that we are measuring the gaps in units of the *average spacing* of the zeros near height T, which is approximately $2\pi/\log T$.

As $T \to \infty$, **PCC** predicts that this counting function, properly normalized by the total number of zeros $\mathrm{N}(T)$, converges to the integral of the *GUE kernel.*

Conjecture 2 (Pair Correlation Conjecture)

For any closed interval $[a,b] \subset \mathbb{R}_{>0}$*, we have*

$$\lim_{T\to\infty} \sup_{\lambda\in[a,b]} \left| \frac{2\pi\mathcal{N}_T(\lambda)}{T\log T} - \int_0^{\lambda} \left(1 - \left(\frac{\sin(\pi\alpha)}{\pi\alpha}\right)^2\right) \mathrm{d}\alpha \right| = 0.$$

Gallagher and Mueller (1978) extended Montgomery's work by showing that the full strength of **MC** is not strictly necessary for the 100% result. Instead, they demonstrated that **PCC**, which describes the local distribution of gaps between zeros, is sufficient.

Theorem 5 (Gallagher–Mueller theorem)

*Assume the Riemann Hypothesis (**RH**) and the Pair Correlation Conjecture (**PCC**). Then, as* $T \to \infty$*, almost all zeros of the Riemann zeta-function are simple and lie on the critical line. Namely, we have*

$$\underline{\mathrm{d}}_{\mathrm{s}} = \liminf_{T\to\infty} \frac{\mathrm{N}_{\mathrm{s}}(T)}{\mathrm{N}(T)} = 1.$$

The significance of Theorem 5 is that it shifts the focus from the F-function to the *GUE hypothesis*, which is the idea that the zeros of the zeta-function behave statistically like the eigenvalues of *Gaussian Unitary Ensemble.* Under this statistical distribution, the "clumping" of zeros required for multiple zeros is effectively impossible.

While **MC** and **PCC** are often discussed together, they represent different perspectives on the distribution of zeros. As Montgomery originally noted, the analytical behavior of the F-function for $\alpha \geq 1$ is actually a stronger condition that implies the statistical distribution of the gaps. This implication was first mentioned by Montgomery in [2] without proof. A detailed treatment of this connection can be found in the lecture notes of D. A. Goldston; see Section 6 in [16] for more details.

Theorem 6 (Implication of the Pair Correlation Conjecture)

*Assume that the Riemann Hypothesis (**RH**). Then, the Montgomery Conjecture (**MC**) implies the Pair Correlation Conjecture (**PCC**).*

4. Main Results

We now introduce the main result of this work, which follows the trajectory established by Gallagher and Mueller [4]. A natural question arises: is it possible to further weaken the underlying assumptions while still preserving the 100% simplicity of the zeros?

As we shall demonstrate, the full strength of the Pair Correlation Conjecture (**PCC**) can be relaxed. We introduce a more flexible condition called as the **Horizontal Multiplicity Hypothesis (HMH)**, which partially generalizes a conjecture proposed by Mueller (1983) when RH was assumed. For the definition of *horizontal multiplicity*, we refer the reader to Section 7 of [17]. For more details, see [18].

Conjecture 3 (Horizontal Multiplicity Hypothesis)

We have

$$\lim_{T\to\infty} \frac{1}{\mathrm{N}(T)} \sum_{\substack{\rho,\rho'\in\mathcal{Z}(T)\\ \gamma=\gamma'}} \mathfrak{m}(\rho)\,\mathfrak{m}(\rho') = 1.$$

In [18], we surprisingly found that the assumption of the Riemann Hypothesis (**RH**) can also be dispensed with. Our main result shows that under **HMH**, one obtains not only 100% simplicity but also 100% critical zeros of the Riemann zeta-function.

Theorem 7 (Main result I)

*Assume the Horizontal Multiplicity Hypothesis (**HMH**). Then, as $T \to \infty$, we have*

$$\underline{\mathrm{d}}_{\mathrm{s}} = \liminf_{T\to\infty} \frac{\mathrm{N}_{\mathrm{s}}(T)}{\mathrm{N}(T)} = 1 \quad \text{and} \quad \underline{\mathrm{d}}_{\mathrm{c}} = \liminf_{T\to\infty} \frac{\mathrm{N}_{\mathrm{c}}(T)}{\mathrm{N}(T)} = 1.$$

This result is remarkable because it decouples the distribution of the zeros from the vertical location of the zeros (the critical line), showing that statistical "simplicity" is a more robust property than **RH** itself.

While Theorem 7 establishes 100% simplicity and criticality under **HMH**, it is natural to investigate the dependency of this new hypothesis on classical conjectures. Specifically, one might ask whether **HMH** is truly a "weaker" assumption and whether it implicitly requires the Riemann Hypothesis. Our second main result, detailed in [18], clarifies this relationship by showing that **HMH** is a consequence of **PCC**, even when the latter is assumed without **RH**.

Theorem 8 (Main result II)

*The Pair Correlation Conjecture (**PCC**) implies the Horizontal Multiplicity Hypothesis (**HMH**). Notably, this implication holds without assuming the Riemann Hypothesis (**RH**).*

This confirms that **HMH** is the "essential" property driving the simplicity of zeros, and that **RH** is a sufficient but not necessary condition for the zeros to be statistically simple.

The proof of the 100% criticality and simplicity results under **HMH** relies on a careful decomposition of the total multiplicity sum. We define the sum of products of multiplicities for all pairs of zeros sharing the same height:

$$P(T) := \sum_{\substack{\rho,\rho' \in \mathcal{Z}(T) \\ \gamma=\gamma'}} \mathfrak{m}(\rho)\,\mathfrak{m}(\rho').$$

We split $P(T)$ into a "diagonal" part $P_1(T)$ (where the zeros are identical) and an "off-diagonal" part $P_2(T)$ (where the zeros are distinct but share a height):

$$P(T) = \sum_{\substack{\rho,\rho' \in \mathcal{Z}(T) \\ \gamma=\gamma',\rho=\rho'}} \mathfrak{m}(\rho)\,\mathfrak{m}(\rho') + \sum_{\substack{\rho,\rho' \in \mathcal{Z}(T) \\ \gamma=\gamma',\rho\neq\rho'}} \mathfrak{m}(\rho)\,\mathfrak{m}(\rho') = \underbrace{\sum_{\rho\in \mathcal{Z}(T)} \mathfrak{m}(\rho)^2}_{P_1(T)} + \underbrace{\sum_{\substack{\rho,\rho' \in \mathcal{Z}(T) \\ \gamma=\gamma',\rho\neq\rho'}} \mathfrak{m}(\rho)\,\mathfrak{m}(\rho')}_{P_2(T)}.$$

By **HMH**, we assume the total sum is dominated by the linear count of zeros:

$$\lim_{T\to\infty} \frac{P(T)}{\mathrm{N}(T)} = \lim_{T\to\infty} \frac{1}{\mathrm{N}(T)} \sum_{\substack{\rho,\rho' \in \mathcal{Z}(T) \\ \gamma=\gamma'}} \mathfrak{m}(\rho)\,\mathfrak{m}(\rho') = 1.$$

It also follows from **HMH** that

$$\lim_{T\to\infty} \frac{P_1(T)}{\mathrm{N}(T)} = \lim_{T\to\infty} \frac{1}{\mathrm{N}(T)} \sum_{\rho\in \mathcal{Z}(T)} \mathfrak{m}(\rho)^2 = 1.$$

We partition $P_1(T)$ and $P_2(T)$ based on the location of the zeros relative to the critical line ($\beta = 1/2$). For $P_1(T)$, we separate zeros on and off the line:

$$P_1(T) = \underbrace{\sum_{\beta=\frac{1}{2}} \mathfrak{m}(\rho)^2}_{P_{11}(T)} + \underbrace{\sum_{\beta\neq\frac{1}{2}} \mathfrak{m}(\rho)^2}_{P_{12}(T)}.$$

For the off-diagonal terms $P_2(T)$, the functional equation of the Riemann zeta-function implies that if $\rho = \beta + i\gamma$ is a zero, then $1 - \rho = (1 - \beta) + i\gamma$ is also a zero with identical multiplicity. Thus, the sum $P_{21}(T)$ over pairs symmetric about the critical line ($\beta + \beta' = 1$) is exactly equal to $P_{12}(T)$, namely,

$$P_{21}(T) := \sum_{\substack{\rho,\rho'\in \mathcal{Z}(T)\\ \gamma=\gamma',\rho\neq\rho'\\ \beta+\beta'=1}} \mathfrak{m}(\rho)\,\mathfrak{m}(\rho') = \sum_{\substack{0<\gamma\leq T\\ \beta\neq\frac{1}{2}}} \mathfrak{m}(\beta + i\gamma)\,\mathfrak{m}(1 - \beta + i\gamma) = P_{12}(T).$$

Since all terms are non-negative, and $P_1 + P_2 = P$, we utilize a squeeze argument as $T \to \infty$:

$$0 \leq \frac{P_{12}(T)}{\mathrm{N}(T)} = \frac{P_{21}(T)}{\mathrm{N}(T)} \leq \frac{P_2(T)}{\mathrm{N}(T)} = 1 - \frac{P_1(T)}{\mathrm{N}(T)} \to 0.$$

This forces $\lim_{T\to\infty} P_{12}(T)/\,\mathrm{N}(T) = 0$ so the sum over the critical line must carry the full density:

$$0 \leq \lim_{T\to\infty} \frac{1}{\mathrm{N}(T)} \sum_{\substack{\rho\in \mathcal{Z}(T)\\ \beta\neq\frac{1}{2}}} \mathfrak{m}(\rho) \leq \lim_{T\to\infty} \frac{1}{\mathrm{N}(T)} \sum_{\substack{\rho\in \mathcal{Z}(T)\\ \beta\neq\frac{1}{2}}} \mathfrak{m}(\rho)^2 = 0$$

Consequently, we obtain that

$$\mathrm{d}_{\mathrm{c}} = \lim_{T\to\infty} \frac{1}{\mathrm{N}(T)} \sum_{\substack{\rho\in \mathcal{Z}(T)\\ \beta=\frac{1}{2}}} \mathfrak{m}(\rho) = 1 - \lim_{T\to\infty} \frac{1}{\mathrm{N}(T)} \sum_{\substack{\rho\in \mathcal{Z}(T)\\ \beta\neq\frac{1}{2}}} \mathfrak{m}(\rho) = 1.$$

To conclude simplicity, we invoke the following inequality (the "multiplicity-simplicity" bridge):

$$\mathrm{N}_s(T) \geq 2\,\mathrm{N}(T) - \sum_{\rho\in \mathcal{Z}(T)} \mathfrak{m}(\rho)^2.$$

Normalizing by $\mathrm{N}(T)$ and taking the limit:

$$1 \geq \lim_{T\to\infty} \frac{\mathrm{N}_{\mathrm{s}}(T)}{\mathrm{N}(T)} \geq 2 - \lim_{T\to\infty} \frac{P_1(T)}{\mathrm{N}(T)} = 1 \quad \text{so} \quad \underline{\mathrm{d}}_{\mathrm{s}} = \lim_{T\to\infty} \frac{\mathrm{N}_{\mathrm{s}}(T)}{\mathrm{N}(T)} = 1.$$

This completes the sketch of the proof for our main theorem. We refer readers interested in the formal derivation of the identities and inequalities presented here, as well as the proof of Theorem 8, to our primary work [18].

We conclude with a final remark on the philosophical implications of our main theorem. The narrative of this note follows a clear progression: Montgomery's original conjecture (**MC**), under the assumption of **RH**, implies **PCC**, which in turn implies **HMH**. Together, these force 100% simplicity and criticality. However, one natural question arises: what if **MC** is wrong? If the F-function behaves "pathologically" on the interval $(1, \infty)$, the zeros might not follow the *GUE distribution* at all. This "Alternative Hypothesis" is a profound question that challenges the standard random matrix theory view of the Riemann zeta-function. Surprisingly, as a sequel to our work in [18], we have demonstrated that even in this alternative regime, the 100% simplicity and criticality results remain intact. This suggests that the "simplicity" of the zeros is an even more fundamental property than their specific statistical distribution. For a detailed treatment of this "alternative" case, we refer the interested reader to the sequel [19].

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