Linear independence measures for Chowla–Selberg periods

Wadim Zudilin*

Department of Mathematics, IMAPP, Radboud University,
PO Box 9010, 6500 GL Nijmegen, Netherlands

October 2025

Abstract

We use simultaneous Padé approximations to ${}_3F_2$ hypergeometric functions to estimate from below linear forms in 1, $\pi\sqrt{d}$, Ω_D/π and π/Ω_D with integral coefficients, for certain choices of positive integer d and negative integer D, where Ω_D is (the square of) a Chowla–Selberg period attached to the imaginary quadratic field $Q(\sqrt{D})$.

This report is not an independent research investigation but rather an addendum to our paper [10] published some twenty years ago. There we used Ramanujan's and Ramanujan-type hypergeometric formulas for $1/\pi$ and explicit simultaneous Padé approximations to the corresponding hypergeometric functions to produce reasonable estimates for the irrationality measures of $\pi\sqrt{d}$, for the preselected list $d \in \{1, 2, 3, 10005\}$. What we have realised now is that we do a bit more in [10], as some other interesting periods besides the multiples of π are approximated as well. The text below gives some details on the extra.

We consider an integer $D < 0$ such that either $D \equiv 1 \pmod 4$, in which case we assume that D is square-free, or $D \equiv 0 \pmod 4$, in which case we assume that $D/4$ is square-free. Denote by $h = h_D$ the class number of the inaginary quadratic field $\mathbb{Q}(\sqrt{D})$. Then we define

$$\Omega_D = \frac{2\pi}{|D|}\Bigg(\prod_{j=1}^{|D|}\Gamma\bigg(\frac{j}{|D|}\bigg)^{\left(\frac{D}{j}\right)}\Bigg)^{1/h_D},$$

where $\left(\frac{j}{k}\right)$ stands for the Kronecker–Legendre symbol. Observe that Ω_D is, up to a positive algebraic multiple, the square of the period constructed in the original paper

*I am grateful for the financial support and hospitality of the Sydney Mathematical Research Institute (SMRI) during July–August 2025. This work was supported in part from the NWO grant OCENW.M.24.112.

[8] of Selberg and Chowla, but for the purpose of this note it will be convenient to refer to Ω_D as to the Chowla–Selberg period.

In [10], we construct Padé type II approximations to 1 and the functions

$$f_i(z) = \left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^i {}_3F_2\left(\begin{matrix} s,\ \frac{1}{2},\ 1-s \\ 1,\ 1 \end{matrix} \,\middle|\, z\right) = \sum_{\nu=0}^{\infty} \frac{(s)_\nu(\frac{1}{2})_\nu(1-s)_\nu}{n!^3} n^i z^n \quad \text{for } i = 0, 1, 2,$$

where $s \in \{\frac{1}{3}, \frac{1}{4}, \frac{1}{6}\}$, and apply them at special values of z exclusively for particular $\mathbb{Q}$-linear combinations of $f_0(z)$ and $f_1(z)$. These linear combinations are known to evaluate to a number $\sqrt{d}/\pi$ for some positive integer d, thus giving us good rational approximations to the latter. At the same time we deliberately ignore the fact that our Padé approximations lead to the $\mathbb{Q}$-linear independence, in a quantitative form, of the *four* numbers $1, f_0(1/Z), f_1(1/Z), f_2(1/Z)$ for integral Z with $|Z|$ sufficiently large. It happens that for the special values $z = 1/Z$ treated in [10] the $\mathbb{Q}$-linear span of these four numbers coincides with the $\mathbb{Q}$-linear span of

$$1,\ \frac{\sqrt{d}\,\Omega_D}{\pi^2},\ \frac{\sqrt{d}}{\pi},\ \frac{\sqrt{d}}{\Omega_D} \tag{1}$$

for a related choice of $D < 0$ and $d > 0$. This leads (after rescaling the latter span by $\pi/\sqrt{d}$) to the following theorem, which summarises the actual outcomes of the construction and analysis in [10].

Theorem 1. *For* $(D, d) \in \{(-148, 1), (-232, 2), (-267, 3), (-163, 10005)\}$*, define*

$$\mu_{D,d} = 57.53011083\ldots,\ 13.93477619\ldots,\ 44.12528464\ldots,\ 10.02136339\ldots,$$

respectively; these are the estimates for the corresponding irrationality measures of $\pi\sqrt{d}$ *computed in* [10, Theorem].[1] *Then for any* $\varepsilon > 0$ *and any collection of integers* m_0, m_1, m_2, m_3 *with* $m = \max\{|m_0|, |m_1|, |m_2|, |m_3|\} \ge m^*(\varepsilon)$ *we have the following estimate:*

$$|m_0 + m_1\pi\sqrt{d} + m_2\,\Omega_D/\pi + m_3\pi/\Omega_D| \ge m^{1-\mu_{D,d}-\varepsilon}.$$

In the appendix we give the explicit connection between the transcendental entries in (1) and the values of hypergeometric functions; related algorithms can be found in [4, 5, 6]. The origin of those expressions rests on the modular parameterisation of the hypergeometric function $f_0(z)$, the closedness of differentiation in the ring of quasimodular forms and the Selberg–Chowla formula [8] evaluating the CM values of modular forms as quotiens of the gamma values. For example, in the 'classical' case $s = \frac{1}{6}$ we have

$${}_3F_2\left(\begin{matrix} \frac{1}{2},\ \frac{1}{6},\ \frac{5}{6} \\ 1,\ 1 \end{matrix} \,\middle|\, \frac{1728}{j(\tau)}\right) = E_4(\tau)^{1/2},$$

[1] The class numbers of the quadratic fields $\mathbb{Q}(\sqrt{-148}), \mathbb{Q}(\sqrt{-232}), \mathbb{Q}(\sqrt{-267})$ are equal to 2, while the class number of $\mathbb{Q}(\sqrt{-163})$ is 1.

where $j(\tau)=1728E_4(\tau)^3/(E_4(\tau)^3-E_6(\tau)^2)$ is the modular invariant and

$$E_2(\tau)=1-24\sum_{m=1}^{\infty}\frac{mq^m}{1-q^m},$$
$$E_4(\tau)=1+240\sum_{m=1}^{\infty}\frac{m^3q^m}{1-q^m} \quad\text{and}\quad E_6(\tau)=1-504\sum_{m=1}^{\infty}\frac{m^5q^m}{1-q^m}, \qquad q=e^{2\pi i\tau},$$

are the Eisenstein series. Then $j\big((1+\sqrt{-163})/2\big)=-1728\cdot 53360^3$ and the Selberg–Chowla formula leads to explicit evaluations of $E_2(\tau), E_4(\tau), E_6(\tau)$ at $\tau=(1+\sqrt{-163})/2$ in terms of π and Ω_D. At the same time, $f_0(1/Z), f_1(1/Z), f_2(1/Z)$ for $Z=-53360^3$ received their expressions in terms of the values of the Eisenstein series. Very similar parameterisations and the same scheme work for $s=\frac13,\frac14$ (and $s=\frac12$ as well, though this case is neither discussed in [10] nor here).

Honestly speaking, the proof of the arithmetic aspects in [10, Section 2] should be slightly adapted to accommodate the presence of $f_2(z)$ in approximations (originally, only linear combinations of $f_0(z)$ and $f_1(z)$ were taken into account). This modifies the required denominators D_n (in the notation of [10]) but does not affect their asymptotics as given in [10, Lemmas 3–5].

More quadruples can be treated for which linear independence estimates can be similarly produced from the general results in [10]; those are linked to the known formulas for $\sqrt{d}/\pi$ carefully listed in [2]. Furthermore, Theorem 1 implies estimates for the nonquadraticity measure of Ω_D/π (by setting $m_1=0$). Finally, notice that very fine bounds for the irrationality measures of the quantities ω_D related to our Chowla–Selberg periods via the formula $|D|\Omega_D/(2\pi)=\omega_D^2$ are obtained in our recent joint paper [3] with Cohen.

Note that our theorem only discusses the estimates; the fact that the numbers Ω_D and π are algebraically independent is already a consequence of the results of Chudnovsky [1, 9], while Nesterenko's theorem in [7] implies that the three numbers Ω_D, π and $\exp(\pi\sqrt{|D|})$ are algebraically independent over the rationals, for any choice of $D<0$ with $D\equiv 0,1 \pmod 4$.

Appendix

For $D=-148$, $d=1$ we take $s=\frac14$, $Z=-882^2$. Then

$$f_0(1/Z)=\frac{42\sqrt{d}\,\Omega_D}{\pi^2}, \quad 1123f_0(1/Z)+21460f_1(1/Z)=\frac{3528\sqrt{d}}{\pi},$$
$$157655f_0(1/Z)+6024969f_1(1/Z)+115132900f_2(1/Z)=\frac{37044\sqrt{d}}{\Omega_D}.$$

For $D=-232$, $d=2$ we take $s=\frac14$, $Z=99^4$. Then

$$f_0(1/Z)=\frac{99\sqrt{d}\,\Omega_D}{\pi^2},\quad 4412f_0(1/Z)+105560f_1(1/Z)=\frac{9801\sqrt{d}}{\pi},$$
$$77862889f_0(1/Z)+3725845296f_1(1/Z)+89143308800f_2(1/Z)=\frac{3881196\sqrt{d}}{\Omega_D}.$$

For $D=-267$, $d=3$ we take $s=\frac13$, $Z=-500^2$. Then

$$f_0(1/Z)=\frac{75\sqrt{d}\,\Omega_D}{\pi^2},\quad 827f_0(1/Z)+14151f_1(1/Z)=\frac{1500\sqrt{d}}{\pi},$$
$$684107f_0(1/Z)+23406555f_1(1/Z)+400501602f_2(1/Z)=\frac{30000\sqrt{d}}{\Omega_D}.$$

For $D=-163$, $d=10005$ we take $s=\frac16$, $Z=-53360^3$. Then

$$f_0(1/Z)=\frac{2\sqrt{d}\,\Omega_D}{\pi^2},\quad 13591409f_0(1/Z)+545140134f_1(1/Z)=\frac{426880\sqrt{d}}{\pi},$$
$$277089597908329f_0(1/Z)+22227667570529352f_1(1/Z)$$
$$+891533297092613868f_2(1/Z)=\frac{136669900800\sqrt{d}}{\Omega_D}.$$

Acknowledgements. An inspiration for this write-up came from recent discussions with several colleagues on two disjoint topics: modular forms, their periods and CM evaluations — on one hand, and simultaneous Padé (or Padé type II) approximations — on the other. I thank Dmitry Badziahin, Florian Breuer, Francis Brown, Heng Huat Chan, Vesselin Dimitrov, Tiago Jardim da Fonseca, Vasily Golyshev and Pengcheng Zhang for related conversations on these themes. I further acknowledge the hospitality of the University of Sydney and of the University of Newcastle during my stays in these institutions, in the unusually cold and wet August 2025.

References

[1] G. V. Chudnovsky, Algebraic independence of constants connected with the exponential and the elliptic functions, *Dokl. Akad. Nauk Ukrain. SSR Ser. A* (1976), no. 8, 698–701.

[2] H. Cohen and J. Guillera, Rational hypergeometric Ramanujan identities for $1/\pi^c$: Survey and generalizations, *Preprint* `arXiv:2101.12592 [math.NT]` (2021), 20 pages.

[3] H. Cohen and W. Zudilin, Continued fractions and irrationality measures for Chowla–Selberg gamma quotients, *Preprint* `arXiv:2510.00215 [math.NT]` (2025), 23 pages.

[4] J. Guillera, Proof of a rational Ramanujan-type series for $1/\pi$. The fastest one in level 3, *Intern. J. Number Theory* **17** (2021), no. 2, 473–477.

[5] J. Guillera, Proof of Chudnovsky' hypergeometric series for $1/\pi$ using Weber modular polynomials, in: *Transcendence in algebra, combinatorics, geometry and number theory*, A. Bostan and K. Raschel (eds.), *Springer Proc. Math. Stat.* **373** (Springer 2021), 341–354.

[6] L. Milla, An efficient determination of the coefficients in the Chudnovskys' series for $1/\pi$, *Ramanujan J.* **57** (2022), 803–809.

[7] Yu. V. Nesterenko, Modular functions and transcendence questions, *Sb. Math.* **187** (1996), no. 9, 1319–1348.

[8] A. Selberg and S. Chowla, On Epstein's zeta-function, *Crelle's J.* **227** (1967), 86–110.

[9] M. Waldschmidt, Les travaux de G.V. Čudnovskiĭ sur les nombres transcendants, in: *Séminaire Bourbaki* (1975/76), 28ème année, Exp. 488; *Lecture Notes in Math.* **567** (Springer-Verlag, Berlin–New York, 1977), 274–292.

[10] W. Zudilin, Ramanujan-type formulae and irrationality measures of certain multiples of π, *Sb. Math.* **196** (2005), no. 7, 983–998; available from the author's website.