HALF-INTEGER INDICES FOR ONE-DIMENSIONAL GAPLESS QUANTUM WALKS

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§1. Introduction

These notes are based mainly on index theory for *chiral-symmetric* unitary operators. More precisely, we shall focus on an abstract unitary operator U on a Hilbert space $\mathcal{H}$, which satisfies the following algebraic condition;

$$U^* = \varGamma U \varGamma, \tag{1}$$

where $\varGamma$ can be any unitary self-adjoint operator on $\mathcal{H}$. Let us call (1) the ***chiral-symmetry condition***, and $(\varGamma, U)$ a ***chiral pair*** on $\mathcal{H}$. Note that the unitary self-adjoint operator $\varGamma$ allows us to decompose the underlying Hilbert space $\mathcal{H}$ into an orthogonal sum of the form $\mathcal{H} = \ker(\varGamma - 1) \oplus \ker(\varGamma + 1)$ as is well-known, and that the spectrum of U, denoted by $\sigma(U)$, is symmetric about the real axis by (1).

If $R := (U + U^*)/2$ denotes the real part of U, then it follows from (1) that the self-adjoint operator R can be written as a diagonal block-operator matrix of the form $R = R_1 \oplus R_2$ with respect to $\mathcal{H} = \ker(\varGamma - 1) \oplus \ker(\varGamma + 1)$. We obtain

$$\ker(U \mp 1) = \ker(R \mp 1) = \ker(R_1 \mp 1) \oplus \ker(R_2 \mp 1). \tag{2}$$

This motivates us to introduce $\operatorname{ind}_\pm(\varGamma, U) := \dim\ker(R_1 \mp 1) - \dim\ker(R_2 \mp 1)$. Note that this formal index is well-defined, if ± 1 does not belong to the essential spectrum of U, denoted by $\sigma_{\mathrm{ess}}(U) := \{z \in \mathbb{T} \mid U - z \text{ is not Fredholm}\}$. We get $|\operatorname{ind}_\pm(\varGamma, U)| \leq \dim\ker(U \mp 1)$ by (2).

§2. A concrete example

We consider the following block-operator matrices on the Hilbert space $\ell^2(\mathbb{Z}, \mathbb{C}^2) = \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ of square-summable $\mathbb{C}^2$-valued sequences on $\mathbb{Z}$:

$$\varGamma := \begin{pmatrix} 1 & 0 \\ 0 & L^* \end{pmatrix} \begin{pmatrix} p & \sqrt{1-p^2} \\ \sqrt{1-p^2} & -p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & L \end{pmatrix}, \qquad \varGamma' := \begin{pmatrix} a & \sqrt{1-a^2} \\ \sqrt{1-a^2} & -a \end{pmatrix}, \tag{3}$$

where L is the left-shift operator on $\ell^2(\mathbb{Z})$, and where $p = (p(x))_{x\in\mathbb{Z}}$ and $a = (a(x))_{x\in\mathbb{Z}}$ are two arbitrary sequences taking values in $(-1,1)$. Here, p, a are viewed as the multiplication operators on $\ell^2(\mathbb{Z})$ in the obvious way. Since $\varGamma, \varGamma'$ are unitary self-adjoint by construction, the unitary operator $U := \varGamma\varGamma'$ satisfies the chiral-symmetry condition (1). This unitary operator is often referred to as the time-evolution operator of the so-called ***(one-dimensional) split-step quantum walk***.

Theorem A. Let us assume the existence of the following limits for each $\star = -\infty, +\infty$:

$$p(\star) := \lim_{x\to\star} p(x), \qquad a(\star) := \lim_{x\to\star} a(x). \tag{A1}$$

Then $\pm 1 \notin \sigma_{\mathrm{ess}}(U)$ if and only if $p(\star) \mp a(\star) \neq 0$ for each $\star = -\infty, +\infty$. In this case, we have

$$\mathrm{ind}_{\pm}(\varGamma, U) = \frac{\mathrm{sign}\,(p(+\infty) \mp a(+\infty)) - \mathrm{sign}\,(p(-\infty) \mp a(-\infty))}{2}, \tag{A2}$$

where sign denotes the sign function.

The aim of the current section is to prove the index formula (A2).

§2.1. **Preliminaries.**

§2.1.1. A unitary transform through the half-step decomposition. Note first that the indices $\mathrm{ind}_{\pm}(\varGamma, U)$ are unitary invariants. More precisely, it can be easily shown that for any unitary operator ϵ on $\mathcal{H}$ we have that $\mathrm{ind}_{\pm}(\varGamma, U) = \mathrm{ind}_{\pm}(\epsilon^*\varGamma\epsilon, \epsilon^* U\epsilon)$, provided that $\sigma_{\mathrm{ess}}(U) = \sigma_{\mathrm{ess}}(\epsilon^* U \epsilon)$ does not contain ± 1. We shall prove the following technical lemma with this result in mind;

Lemma 1. *There exist two unitary operators ϵ, η on $\ell^2(\mathbb{Z}, \mathbb{C}^2)$, such that the following decompositions hold true with respect to $\ell^2(\mathbb{Z}, \mathbb{C}^2) = \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$:*

$$\epsilon^*\varGamma\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \eta^*\varGamma'\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \epsilon^* U \epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\eta^*\epsilon)^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\eta^*\epsilon). \tag{4}$$

More explicitly, if we let $\zeta_\pm := \sqrt{1 \pm \zeta}$ for each $\zeta = p, a$, then the unitary operators ϵ, η are given respectively by

$$\epsilon := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & L^* \end{pmatrix} \begin{pmatrix} p_+ & -p_- \\ p_- & p_+ \end{pmatrix}, \qquad \eta := \frac{1}{\sqrt{2}} \begin{pmatrix} a_+ & -a_- \\ a_- & a_+ \end{pmatrix}. \tag{5}$$

Proof. Note first that we have the following unitary diagonalisation for each $\zeta = p, a$ and each $x \in \mathbb{Z}$;

$$\begin{pmatrix} \frac{\zeta_+(x)}{\sqrt{2}} & \frac{-\zeta_-(x)}{\sqrt{2}} \\ \frac{\zeta_-(x)}{\sqrt{2}} & \frac{\zeta_+(x)}{\sqrt{2}} \end{pmatrix}^* \begin{pmatrix} \zeta(x) & \sqrt{1-\zeta(x)^2} \\ \sqrt{1-\zeta(x)^2} & -\zeta(x) \end{pmatrix} \begin{pmatrix} \frac{\zeta_+(x)}{\sqrt{2}} & \frac{-\zeta_-(x)}{\sqrt{2}} \\ \frac{\zeta_-(x)}{\sqrt{2}} & \frac{\zeta_+(x)}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This result motivates us to introduce the unitary operators ϵ, η defined by (5). Indeed,

$$\epsilon^* \Gamma \epsilon = \begin{pmatrix} \frac{p_+}{\sqrt{2}} & \frac{-p_-}{\sqrt{2}} \\ \frac{p_-}{\sqrt{2}} & \frac{p_+}{\sqrt{2}} \end{pmatrix}^* \begin{pmatrix} p & \sqrt{1-p^2} \\ \sqrt{1-p^2} & -p \end{pmatrix} \begin{pmatrix} \frac{p_+}{\sqrt{2}} & \frac{-p_-}{\sqrt{2}} \\ \frac{p_-}{\sqrt{2}} & \frac{p_+}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\eta^* \Gamma' \eta = \begin{pmatrix} \frac{a_+}{\sqrt{2}} & \frac{-a_-}{\sqrt{2}} \\ \frac{a_-}{\sqrt{2}} & \frac{a_+}{\sqrt{2}} \end{pmatrix}^* \begin{pmatrix} a & \sqrt{1-a^2} \\ \sqrt{1-a^2} & -a \end{pmatrix} \begin{pmatrix} \frac{a_+}{\sqrt{2}} & \frac{-a_-}{\sqrt{2}} \\ \frac{a_-}{\sqrt{2}} & \frac{a_+}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Finally,

$$\epsilon^* U \epsilon = \epsilon^* \Gamma \Gamma' \epsilon = (\epsilon^* \Gamma \epsilon)(\epsilon^* \eta)(\epsilon^* \Gamma \epsilon)(\eta^* \epsilon) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\eta^* \epsilon)^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (\eta^* \epsilon).$$

□

With the notation introduced in Lemma 1, it is easy to see that the operator $F := \eta^* \epsilon$ is given explicitly by

$$F = \frac{1}{2} \begin{pmatrix} p_+ a_+ + a_- L^* p_- & -p_- a_+ + a_- L^* p_+ \\ -p_+ a_- + a_+ L^* p_- & p_- a_- + a_+ L^* p_+ \end{pmatrix} =: \begin{pmatrix} A_- & * \\ A_+ & * \end{pmatrix}. \tag{6}$$

It follows from [CGWW21, Lemma 3.2] that $U \mp 1$ is Fredholm if and only if $A_\pm$ is Fredholm. In this case, we have

$$\operatorname{ind}_\pm(\Gamma, U) = \operatorname{ind}_\pm(\epsilon^* \Gamma \epsilon, \epsilon^* U \epsilon) = \operatorname{ind} A_\pm, \tag{7}$$

where the first equality follows from the unitary invariance of the indices $\operatorname{ind}_\pm$, and where the second equality follow from [CGWW21, Lemma 3.2]. Therefore, it remains to compute the Fredholm index of $A_\pm = \mp p_+ a_\mp + p_-(\cdot - 1) a_\pm L^*$, and we shall make use of the standard Toeplitz index theorem.

§2.1.2. The Toeplitz index theorem. The Hilbert space $L^2(\mathbb{T})$ of square-summable functions on the unit-circle $\mathbb{T}$ admits the standard complete orthonormal basis $(e_x)_{x \in \mathbb{Z}}$ defined by $\mathbb{T} \ni z \longmapsto z^x \in \mathbb{C}$. The ***Hardy-Hilbert space*** $\mathrm{H}^2(\mathbb{T})$ is the closure of the linear span of $\{e_x \mid x \geq 0\}$. Let $\iota : \mathrm{H}^2(\mathbb{T}) \hookrightarrow L^2(\mathbb{T})$ be the inclusion mapping, and let $f \in C(\mathbb{T})$. Then the ***Toeplitz operator*** T_f with symbol f is defined by

$$T_f := \iota^* M_f \iota, \tag{8}$$

where $M_f : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is the bounded multiplication operator by f. The Toeplitz index theorem states that the Toeplitz operator T_f is Fredholm if and only if the curve $\mathbb{T} \ni z \longmapsto f(z) \in \mathbb{C}$ does *not* pass through the origin, and that the following equality holds true in this case;

$$\operatorname{ind} T_f = -\mathrm{wn}(f), \tag{9}$$

where $\mathrm{wn}(f)$ denotes the winding number of the continus function f.

Lemma 2. *For each* $\star = -\infty, +\infty$, *let*

$$c_\pm(\star) := \sqrt{(1 + p(\star))(1 \mp a(\star))}, \tag{10}$$

$$r_\pm(\star) := \sqrt{(1 - p(\star))(1 \pm a(\star))}, \tag{11}$$

$$\hat{A}_\pm(z, \star) := \mp c_\pm(\star) + r_\pm(\star) z^*, \qquad z \in \mathbb{T}. \tag{12}$$

Then the operator $A_\pm$ *is Fredholm if and only if the continuous function* $\mathbb{T} \ni z \longmapsto \hat{A}_\pm(z, \star) \in \mathbb{C}$ *is nowhere vanishing for each* $\star = -\infty, +\infty$. *In this case,*

$$\operatorname{ind} A_\pm = \mathrm{wn}\left(\hat{A}_\pm(\cdot, +\infty)\right) - \mathrm{wn}\left(\hat{A}_\pm(\cdot, -\infty)\right). \tag{13}$$

Proof. Here, we only give a sketch of proof, since an analogous argument can be found in the existing literature (see, for example, [Mat20] or [Tan21, Theorem A]). The underlying Hilbert space $\ell^2(\mathbb{Z})$ admits a natural orthogonal decomposition $\ell^2(\mathbb{Z}) = \ell^2_{\mathrm{L}}(\mathbb{Z}) \oplus \ell^2_{\mathrm{R}}(\mathbb{Z})$, where

$$\ell^2_{\mathrm{L}}(\mathbb{Z}) := \{\psi \in \ell^2(\mathbb{Z}) \mid \psi(x) = 0 \,\forall x \geq 0\}, \qquad \ell^2_{\mathrm{R}}(\mathbb{Z}) := \{\psi \in \ell^2(\mathbb{Z}) \mid \psi(x) = 0 \,\forall x < 0\}.$$

For each $\sharp = \mathrm{L}, \mathrm{R}$ and each (bounded) operator A on $\ell^2(\mathbb{Z})$, we let $A_\sharp := \iota_\sharp^* A \iota_\sharp$, where $\iota_\sharp$ denotes the inclusion mapping $\iota_\sharp : \ell^2_\sharp(\mathbb{Z}) \hookrightarrow \ell^2(\mathbb{Z})$. Here, each Hilbert space $\ell^2_\sharp(\mathbb{Z})$ can be naturally identified with the Hardy-Hilbert space $\mathrm{H}^2(\mathbb{T})$ via unitary transforms $\mathscr{F}_{\mathrm{L}} : \mathrm{H}^2(\mathbb{T}) \to \ell^2_{\mathrm{L}}(\mathbb{Z})$ and $\mathscr{F}_{\mathrm{R}} : \mathrm{H}^2(\mathbb{T}) \to \ell^2_{\mathrm{R}}(\mathbb{Z})$ defined by $\mathscr{F}_{\mathrm{L}} e_x := \delta_{-x-1}$ and $\mathscr{F}_{\mathrm{R}} e_x := \delta_x$ for each $x \geq 0$, where $(\delta_x)_{x \in \mathbb{Z}}, (e_x)_{x \geq 0}$ are the standard bases for $\ell^2(\mathbb{Z}), \mathrm{H}^2(\mathbb{T})$ respectively.

An operator A on $\ell^2(\mathbb{Z})$ is said to be ***strictly local***, if it is a finite sum of the form $A = \sum_{j=-n}^{n} \alpha_j L^j$, where each $\alpha_j = (\alpha_j(x))_{x \in \mathbb{Z}}$ is a bounded sequence viewed as a multiplication operator. It is easy to show that the difference $A - A_{\mathrm{L}} \oplus A_{\mathrm{R}}$ is finite-rank, if A is strictly local.

A stricly local operator of the form $A = \sum_{j=-n}^{n} \alpha_j L^j$ is said to be ***uniform***, if each sequence $\alpha_j = (\alpha_j(x))_{x \in \mathbb{Z}}$ does not depend on the choice of $x \in \mathbb{Z}$. In this case, we can define the Fourier transform of A by $\hat{A}(z) := \sum_{j=-n}^{n} \alpha_j z^j$ for each $z \in \mathbb{T}$. It follows from a direct computation that for each $\sharp = \mathrm{L}, \mathrm{R}$ the operator $A_\sharp = \iota_\sharp^* A \iota_\sharp$ on $\ell^2_\sharp(\mathbb{Z})$ can be viewed as a Toeplitz operator via

$$\mathscr{F}_\sharp^* A_\sharp \mathscr{F}_\sharp = \begin{cases} T_{\hat{A}}, & \sharp = \mathrm{L}, \\ T_{\hat{A}(-^*)}, & \sharp = \mathrm{R}, \end{cases}$$

where $\hat{A}(-^*)$ denotes the continuous function $z \longmapsto \hat{A}(z^*)$. Recall that the Fredholm index is invariant under unitary transforms and compact perturbations, and that the difference $A - A_\mathrm{L} \oplus A_\mathrm{R}$ is of finite-rank. As such, we have that the uniform stricly local operator A is Fredholm if and only if $A_\mathrm{L}, A_\mathrm{R}$ are Fredholm if and only if the Fourier transform $\mathbb{T} \ni z \longmapsto \hat{A}(z) \in \mathbb{C}$ is nowhere vanishing. In this case, we have $\operatorname{ind} A = \operatorname{ind} A_\mathrm{L} + \operatorname{ind} A_\mathrm{R} = 0$, and $\operatorname{ind} A_\mathrm{R} = \operatorname{wn}(\hat{A}(\cdot))$ by the Toeplitz index theorem.

Finally, let $A = \sum_{j=-n}^{n} \alpha_j L^j$ be a strictly local operator with the property that each α_j has the limit $\alpha_j(\star) := \lim_{x \to \star} \alpha_j(x)$ for each $\star = -\infty, +\infty$. Note that for each $\star = -\infty, +\infty$ we can construct a uniform strictly local operator of the form $A(\star) := \sum_{j=-n}^{n} \alpha_j(\star) L^j$. Since $A_\mathrm{L} \oplus A_\mathrm{R} - A(-\infty)_\mathrm{L} \oplus A(+\infty)_\mathrm{R}$ is compact, it follows that the operator A is Fredholm if and only if the continuous function $\hat{A}(\cdot, \star)$ defined by $\hat{A}(z, \star) := \sum_{j=-n}^{n} \alpha_j(\star) z^j$ is nowhere vanishing for each $\star = -\infty, +\infty$. In this case,

$$\operatorname{ind} A = \operatorname{wn}\left(\hat{A}(\cdot, +\infty)\right) - \operatorname{wn}\left(\hat{A}(\cdot, -\infty)\right), \tag{14}$$

where $\operatorname{ind}\left(T_{\hat{A}_\pm(-^*, \star)}\right) = \operatorname{wn}\left(\hat{A}_\pm(\cdot, \star)\right)$ for each $\star = -\infty, +\infty$. Clearly, the above formula is applicable to the strictly local operators A_-, A_+, and so the claim follows. $\square$

§2.2. **Proof of Theorem A.**

Proof of Theorem A. It remains to compute the winding number of the continuous function $z \longmapsto \hat{A}_\pm(z, \star)$ in (13). The image of $\hat{A}_\pm(z, \star) = \mp c_\pm(\star) + r_\pm(\star) z^*$ is a *circle* with centre $\mp c_\pm(\star)$ and radius $r_\pm(\star)$. If the notation $\gtreqless$ simultaneously denotes the three binary relations $>, =, <$, then the relation $c_\pm(\star) \gtreqless r_\pm(\star)$ is equivalent to

$$\frac{c_\pm(\star)}{r_\pm(\star)} \gtreqless 1 \iff \left(\frac{1 + p(\star)}{1 - p(\star)}\right)\left(\frac{1 \mp a(\star)}{1 \pm a(\star)}\right) \gtreqless 1 \iff p(\star) \pm a(\star) \gtreqless 0,$$

where the last equivalence follows from the fact that the increasing function Λ defined by $(-1, 1) \ni t \longmapsto (1 + t)/(1 - t) \in (0, \infty)$ has the property that $\Lambda(t)\Lambda(t') \gtreqless 1$ if and only if $t + t' \gtreqless 0$ for each $t, t' \in (-1, 1)$, since $\Lambda(t)\Lambda(t') = \Lambda((t + t')/(1 + tt'))$. It is then geometrically obvious that the circle $z \longmapsto \hat{A}_\pm(z, \star)$ does not pass through the origin iff $p(\star) \mp a(\star) \neq 0$.

Moreover,

$$\operatorname{ind}\left(T_{\hat{A}_\pm(-^*,\star)}\right) = \operatorname{wn}\left(\hat{A}_\pm(\cdot,\star)\right) = \frac{\operatorname{sign}(p(\star) \mp a(\star)) - 1}{2}. \tag{15}$$

Theorem A now follows from Lemmas 1 to 2. □

§3. A generalisation of the index formula

A bounded operator A on $\mathcal{H}$ is called ***trace-compatible***, if $A^*A - AA^*$ is of trace-class. For such A, we define

$$\operatorname{ind}_t(A) := \operatorname{Tr}(e^{-tA^*A} - e^{-tAA^*}), \qquad t \in \mathbb{R}.$$

If the limit $w(A) := \lim_{t\to\infty} \operatorname{ind}_t(A)$ exists, then we call $w(A)$ the ***Witten index*** of A. It is well-known that if A is a trace-compatible Fredholm operator, then $w(A) = \operatorname{ind} A$. In this case,

$$w(A) = \operatorname{ind} A = \dim\ker(A) - \dim\ker(A^*) = \dim\ker(A^*A) - \dim\ker(AA^*),$$

where $H_0 := A^*A$ and $H := AA^*$ are often referred to as superhamiltonians (this construction comes from supersymmetric quantum mechanics [BGGSS87]). The purpose of the current section is to show that the index formula (A2) can be genralised by replacing $\operatorname{ind}_\pm(\Gamma, U)$ by $w(A_\pm)$. In order to do so, let us first consider the following simple example;

Example 3. Let us compute the Witten index of the Toeplitz operator T_z. Let $\iota : H^2(\mathbb{T}) \hookrightarrow L^2(\mathbb{T})$ be the inclusion mapping, and let $P := \iota\iota^*$. We have $T_z^*T_z = \iota^* M_{z^*} P M_z \iota$. For each $x \geq 0$, we have

$$T_z^*T_z e_x = \iota^* M_{z^*} P M_z \iota e_x = \iota^* M_{z^*} P e_{x+1} = \iota^* e_x = e_x.$$

Similarly, $T_zT_z^* = \iota^* M_z P M_{z^*} \iota$, and so

$$T_zT_z^* e_x = \iota^* M_z P M_{z^*} \iota e_x = \iota^* M_z P e_{x-1} = \begin{cases} 0, & x = 0, \\ e_x, & x > 0. \end{cases}$$

We get $T_z^*T_z = 1$ and $T_zT_z^* = 1 - \langle e_0, -\rangle\, e_0$. It follows that the operator T_z is a Fredholm trace-compatible operator, and so $w(T_z)$ is well-defined. We get

$$e^{-tT_z^*T_z} - e^{-tT_zT_z^*} = e^{-t} - e^{-t(1-P_0)} = e^{-t}(1 - e^{tP_0}) = e^{-t}(1 - e^t)P_0 = (e^{-t} - 1)P_0.$$

It follows that $\operatorname{ind}_t(T_z) = e^{-t} - 1 \to -1$ as $t \to \infty$. In fact, T_z is Fredholm with $w(T_z) = \operatorname{ind} T_z = -1$, since $\mathbb{T} \ni z \longmapsto z \in \mathbb{C}$ does not pass through the origin. On a side note this result is not surprising, since T_z is nothing but the right-shift operator, since $T_z e_y = e_{y+1}$ for each $y \geq 0$.

Recall that the Freholm index of the Toeplitz operator $T_{\hat{A}_\pm(-^*,\star)}$ is given by (15), and that this operator fails to be Fredholm if $p(\star) \mp a(\star) = 0$. We are now in a position to compute the Witten index of $T_{\hat{A}_\pm(-^*,\star)}$;

Lemma 4. *The following assertions hold true for each* $\star = -\infty, +\infty$:

(i) *The Toeplitz operator* $T_{\hat{A}_\pm(-^*,\star)}$ *is trace-compatible.*

(ii) *The Witten index of* $T_{\hat{A}_\pm(-^*,\star)}$ *is given by*

$$w\left(T_{\hat{A}_\pm(-^*,\star)}\right) = \frac{\operatorname{sign}(p(\star) \mp a(\star)) - 1}{2}, \tag{16}$$

where we define $\operatorname{sign}(0) := 0$.

Note that (16) is equivalent to

$$w\left(T_{\hat{A}_\pm(-^*,\star)}\right) = \begin{cases} -1, & p(\star) \mp a(\star) < 0, \\ -1/2, & p(\star) \mp a(\star) = 0, \\ 0, & p(\star) \mp a(\star) > 0. \end{cases} \tag{17}$$

Proof. (i) We get

$$T_{\hat{A}_\pm(-^*,\star)} = \iota^* M_{\mp c_\pm(\star) + r_\pm(\star) z^*} \iota = \iota^* \left(M_{\mp c_\pm(\star)} + r_\pm(\star) M_z\right) \iota = T_{\mp c_\pm(\star)} + r_\pm(\star) T_z = \mp c_\pm(\star) + r_\pm(\star) T_z,$$

where the last equality follows from $T_{\mp c_\pm(\star)} = \mp c_\pm(\star)$. We have

$$T^*_{\hat{A}_\pm(-^*,\star)} T_{\hat{A}_\pm(-^*,\star)} = \mp c_\pm(\star)^2 + \mp c_\pm(\star) r_\pm(\star) T_z + \mp c_\pm(\star) r_\pm(\star) T_{z^*} + r_\pm(\star)^2 T_{z^*} T_z,$$
$$T_{\hat{A}_\pm(-^*,\star)} T^*_{\hat{A}_\pm(-^*,\star)} = \mp c_\pm(\star)^2 + \mp c_\pm(\star) r_\pm(\star) T_{z^*} + \mp c_\pm(\star) r_\pm(\star) T_z + r_\pm(\star)^2 T_z T_{z^*}.$$

Therefore,

$$T^*_{\hat{A}_\pm(-^*,\star)} T_{\hat{A}_\pm(-^*,\star)} - T_{\hat{A}_\pm(-^*,\star)} T^*_{\hat{A}_\pm(-^*,\star)} = -r_\pm(\star)^2 (T_{z^*} T_z - T_z T_{z^*}) = -r_\pm(\star)^2 \langle e_0, - \rangle e_0,$$

where the last equality follows from $T_{z^*} T_z = 1$ and $T_z T_{z^*} = 1 - \langle e_0, - \rangle e_0$ (see Example 3). It follows that $T_{\hat{A}_\pm(-^*,\star)}$ is trace-compatible.

(ii) To prove the formula (17) it remains to show $w(T_{\hat{A}_\pm(-^*,\star)}) = -1/2$ under the assumption $p(\star) \mp a(\star) = 0$. In this case, we get $c_\pm(\star) = r_\pm(\star) = \sqrt{1 - a(\star)^2} \neq 0$, and so

$$T_{\hat{A}_\pm(-^*,\star)} = \mp c_\pm(\star) + r_\pm(\star) T_z = c_\pm(\star)(T_z \mp 1).$$

Therefore, it remains to compute the Witten index of the operator $T_z \mp 1$. We shall only consider $T_z + 1$ in what follows, since the other case can be proved similarly. We get

$$(T_z + 1)^*(T_z + 1) = T_{z^*}T_z + T_{z^*} + T_z + 1 = (T_z + T_{z^*}) + 2,$$
$$(T_z + 1)(T_z + 1)^* = T_zT_{z^*} + T_z + T_{z^*} + 1 = (T_z + T_{z^*}) - \langle e_0, -\rangle\, e_0 + 2.$$

We have

$$\begin{aligned} \mathrm{ind}\,_t(T_z + 1) &= \mathrm{Tr}\,(e^{-t((T_z+T_{z^*})+2)} - e^{-t((T_z+T_{z^*})-\langle e_0,-\rangle e_0+2)}) \\ &= -e^{-2t}\mathrm{Tr}\,(e^{-t(T_z+T_{z^*}-\langle e_0,-\rangle e_0)} - e^{-t(T_z+T_{z^*})}) \\ &= -e^{-2t}\mathrm{Tr}\,(e^{-tH_1} - e^{-tH_0}), \end{aligned}$$

where the two hamiltonians $H_1 := T_z + T_{z^*} - \langle e_0, -\rangle\, e_0$ and $H_0 := T_z + T_{z^*}$ have a one-rank difference, and so the associated spectral shift function can be explicitly computed;

$$\xi(x) = \begin{cases} 0, & |x| > 2, \\ -\frac{1}{2\pi}\cos^{-1}\left(\frac{x}{2}\right) & |x| \leq 2. \end{cases} \tag{18}$$

This spectral shift function satisfies the following trace-formula;

$$\mathrm{Tr}\,(e^{-tH_1} - e^{-tH_0}) = \int_{\mathbb{R}} \xi(x)(e^{-tx})'\,dx = \frac{1}{2}e^{2t} - \frac{1}{2\pi}\int_{-2}^{2} \frac{e^{-tx}}{\sqrt{4-x^2}}\,dx.$$

Therefore,

$$\mathrm{ind}\,_t(T_z + 1) = -\frac{1}{2} + \frac{1}{2\pi}\int_{-2}^{2} \frac{e^{-(x+2)t}}{\sqrt{4-x^2}}\,dx \to -\frac{1}{2} + 0$$

as $t \to +\infty$ by the dominated convergence theorem. It follows that $w(T_z+1) = -1/2$ as required. On a final (18) can be proved as follows. The spectral shift function is given by the well-known formula $\xi(x) = \pi^{-1}\lim_{y\to+0}\mathrm{Arg}\,\Delta_{H_1/H_0}(x+iy)$, where the perturbation determinant Δ_{H_1/H_0} is of the form $\Delta_{H_1/H_0}(z) = 1 - \left\langle e_0, (H_0 - z)^{-1}e_0\right\rangle$. We can then compute $\left\langle e_0, (H_0 - z)^{-1}e_0\right\rangle$, and take the limit. An analogous argument can be found in the existing litrerature (see, for example, [MSTTW23, §3]). □

We are now in a position to prove the following generalisation of Theorem A;

Theorem B. Let U be the time-evolution operator of the split-step quantum walk, and let us assume the existence of limits of the form (A1). We also impose the following additional assumption;

$$\sum_{x=0}^{\infty} |\zeta(-x) - \zeta(-\infty)| + \sum_{x=1}^{\infty} |\zeta(x) - \zeta(+\infty)| < \infty, \qquad \zeta = p, a. \tag{B1}$$

Then $A_\pm$ is trace-compatible, and the Witten index $w(A_\pm)$ is given by the right hand side of (A2), where we define $\mathrm{sign}\,(0) := 0$.

Proof. For each $\star = -\infty, +\infty$ let

$$\hat{A}_\pm(\star) := \mp c_\pm(\star) + r_\pm(\star) L^*,$$
$$\hat{A}_\pm(z, \star) := \mp c_\pm(\star) + r_\pm(\star) z^*, \qquad z \in \mathbb{T}.$$

It is shown in the proof of Lemma 2 that the two-phase assumption (A1) implies the compactness of $A_\pm - A_\pm(-\infty)_{\mathrm{L}} \oplus A_\pm(+\infty)_{\mathrm{R}}$. Similarly, the additional assumption (B1) implies that the same operator is of trace-class. Since the Witten index is invariant under unitary transforms and trace-class perturbations as with the Fredhom case, we get

$$w(A_\pm) = w(A_\pm(-\infty)_{\mathrm{L}}) + w(A_\pm(+\infty)_{\mathrm{R}}) = w\left(T_{\hat{A}_\pm(-^*,+\infty)}\right) - w\left(T_{\hat{A}_\pm(-^*,-\infty)}\right).$$

The claim now follows from Lemma 4. □

§4. Concluding remarks

Recall that the two indices $\mathrm{ind}_-(\varGamma, U), \mathrm{ind}_+(\varGamma, U)$ we introduced in §1 are associated with the *real part* of the chiral unitary U, but there is in fact yet another index $\mathrm{ind}\,(\varGamma, U)$ which can be assigned to the *imaginary part* of U (see, for example, [Tan21, §3.1]). If neither -1 nor $+1$ belongs to $\sigma_{\mathrm{ess}}(U)$, then we have $\mathrm{ind}\,(\varGamma, U) = \mathrm{ind}_-(\varGamma, U) + \mathrm{ind}_+(\varGamma, U)$. These indices naturally arise in the context of (discrete-time) quantum walks on the one-dimensional integer lattice $\mathbb{Z}$, and there is extensive amount of literature [CGSVWW16, CGGSVWW18, CGSVWW18, Suz19, ST19, Mat20, AFST21, Tan21, CGWW21]. Note that the main theorem of the present article, Theorem B, extends integer-valued $\mathrm{ind}_\pm(\varGamma, U)$ to half-integer-valued $w(A_\pm)$, and this result was entirely motivated by [MSTTW23, Theorem 1.3] with an analogous extension method for $\mathrm{ind}\,(\varGamma, U)$. The main difference comes from the fact that Theorem B deals with strictly local operators of the form $\alpha + \beta L^*$, while one needs to consider an additional term γL in [MSTTW23, Theorem 1.3].

There are some open problems associated with the extension methods described above. For example, the operator $A_{\pm}$ comes from the unitary transform ϵ we constructed in Lemma 1, but it is not known to the author whether or not $w(A_{\pm})$ depends on ϵ. Some geometric perspectives of the index formulas might also be worth considering; indeed, the formula (17) seems to suggest that the Witten index can be understood as the half-integer-valued winding number. What has been described so far is a work in progress (joint work with Y. Matsuzawa, A. Suzuki, and K. Wada).

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