SEMICLASSICAL RESONANCES FOR MATRIX SCHRÖDINGER OPERATORS ABOVE AN ENERGY-LEVEL CROSSING WITH VANISHING INTERACTIONS

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Abstract. This paper is based on a talk given on November 29th, 2023 at the Research Institute for Mathematical Sciences (RIMS) of Kyoto University. In this conference paper, we present a recent development on the study of the semiclassical distribution of resonances for matrix Schrödinger operators in the case where the underlying classical trajectories intersect at a finite order and their interaction is described by a non-elliptic (vanishing) operator. A recent result [2] on the elliptic case reduces the problem to proving a microlocal connection formula at the crossing point, which itself reduces to proving a stationary phase estimate with degenerate phase. In this paper, we adapt this method and study the non-elliptic case when the crossing of classical trajectories is not a turning point. The material of this paper is borrowed from [12].

1. Introduction and framework

In this paper, we study the distribution of semiclassical quantum resonances of the 2×2 Schrödinger operator defined as

$$P := \begin{pmatrix} P_1 & hU \\ hU^* & P_2 \end{pmatrix} \tag{1.1}$$

with

$$P_j := (hD_x)^2 + V_j(x),\ j \in \{1,2\}, \quad \text{and} \quad D_x := -i\partial_x,$$

where $h > 0$ denotes the semiclassical parameter and $U = r_0(x)$ a smooth multiplication operator. Such operator arises in the framework of the Born-Oppenheimer model. This paper as well as [12] are based on the point of view adopted in [2], and we send the reader to this article and the references therein for more background on the subject.

We choose an energy level $E_0 \in \mathbb{R}$ such that V_1 has a simple well and V_2 is "non-trapping", and we assume that those two potentials intersect at a finite order. The situation is sum up in the two following assumptions and on Figure 1.

Assumption 1. *There exist real numbers $a_0 < 0 < a_0'$ and $b_0 > 0$ such that, for all $x \in \mathbb{R}$,*

$$\frac{V_1(x) - E_0}{(x-a_0)(x-a_0')} > 0 \tag{1.2}$$

and

$$\frac{V_2(x) - E_0}{x - b_0} > 0. \tag{1.3}$$

Assumption 2. *The set $\{x \in \mathbb{R};\ V_1(x) = V_2(x) < E_0\}$ of potential crossings is reduced to $\{0\}$ and the function $V_2 - V_1$ vanishes at $x = 0$ at a finite order $n \in \mathbb{N} \setminus \{0\}$:*

$$(V_2 - V_1)^{(j)}(0) = 0 \text{ for all } j \in \{0, \dots, n-1\}, \quad \text{and} \quad (V_2 - V_1)^{(n)}(0) \neq 0. \tag{1.4}$$

As the title of this paper hints, we only consider the case where the energy E_0 is above the crossing of the potentials; this corresponds to cases (I-a) and (I-b) of [12]. One can find an analysis of the case where $V_1(x) = V_2(x) = E_0$ in [12, Section 5], based on [3] and [5].

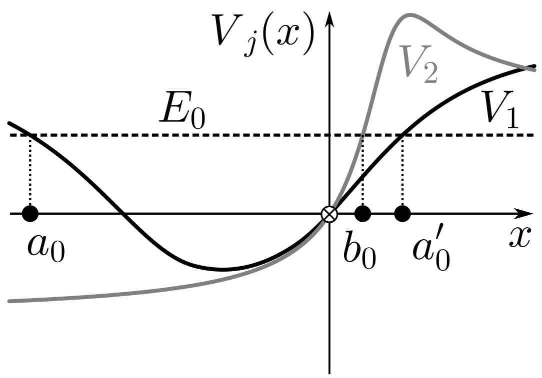

FIGURE 1. Potential crossing at $x = 0$

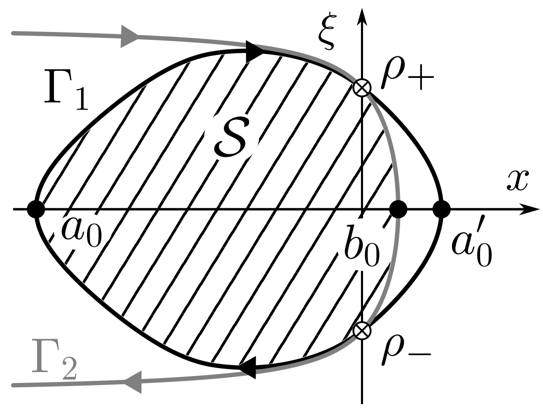

FIGURE 2. Phase space crossings of the associated classical trajectories

We note $p_j(x, \xi) = \xi^2 + V_j(x)$ the semiclassical symbol of P_j. We call the *classical trajectory*, or (bi)characteristic curve associated to p_j the set defined for any $E \in \mathbb{R}$ by

$$\Gamma_j(E) := \{(x, \xi) \in \mathbb{R}^2;\ p_j(x, \xi) = E)\}.$$

The above coincides with the trajectory $\{e^{tH_{p_j - E}}\rho, t \in \mathbb{R}\}$ of the classical Hamiltonian $p_j - E$, where ρ is any point of $\Gamma_j(E)$. This explains the terminology *classical trajectory* as well as the orientation of Γ_j (arrows on Figure 2). Here, $H_p := \partial_\xi p \partial_x - \partial_x p \partial_\xi = \{p, \cdot\}$.

Under the two above assumptions, the set of *crossing points*, referring to the elements of $\mathscr{C} := \Gamma_1 \cap \Gamma_2$, is reduced to $\{\rho_-, \rho_+\}$ (see Figure 2). The *contact order* at a crossing point ρ refers to the geometric contact order m of the characteristic sets at ρ. In this context, $m = n$. We will write $\Gamma(E) := \Gamma_1(E) \cup \Gamma_2(E)$ the union of the two classical trajectories. The *turning points* will refer to the points of $\Gamma(E) \cap \{\xi = 0\}$; they correspond to the points $a_0(E)$, $b_0(E)$ and $a_0'(E)$ defined in Assumption 1. Note that they differ from crossing points here.

In the following, we study the spectral behavior of P in a small complex box around E_0 of the form

$$\mathcal{R}_h := [E_0 - Lh, E_0 + Lh] + i[-Lh, Lh] \quad \text{with} \quad L > 0 \text{ independent of } h. \tag{1.5}$$

In the case where $r_0(x)$ vanishes identically on $\mathbb{R}$, it is known that the spectrum of $P = \operatorname{diag}(P_1, P_2)$ in this box is composed of eigenvalues, coming from P_1, embedded into an essential spectrum, coming from P_2 (Figure 3). It is also known that the eigenvalues of P_1 are approximated by the Bohr-Sommerfeld quantization condition. The set of energies satisfying this condition is

$$\mathfrak{B}_h := \left\{ E \in [E_0 - Lh, E_0 + Lh];\ \cos\left(\frac{\mathcal{A}(E)}{2h}\right) = 0 \right\} \tag{1.6}$$

where $\mathcal{A}(E) := \displaystyle\int_{\Gamma_1(E)} \xi \, dx$ is the classical action along $\Gamma_1(E)$.

However, in the general case, the *golden rule* of Fermi states that we can expect the embedded eigenvalues to shift below the real axis. This is what we call resonances for P (Figure 4). Mathematically speaking one defines, under an appropriate analytic assumption on the potentials, resonances near E_0 to be energies $E \in \mathcal{R}_h$ such that there exist a nontrivial microlocal solution u (resonant state associated to E) to the equation $(P - E)u = 0$, in a sense specified in Section 3. For a more rigorous definition, we refer to the analytic dilation method of [1]. We write $\mathrm{Res}_h(P)$ the set of resonances of P near E_0. Physically speaking, the so-called Born-Oppenheimer approximation in quantum chemistry states that P describes a particle system. Resonances correspond to semi-bound states, and the imaginary part of the resonances is shown to be inversely proportional to the half-life of this particle system. Our goal is to compute the imaginary part of those resonances.

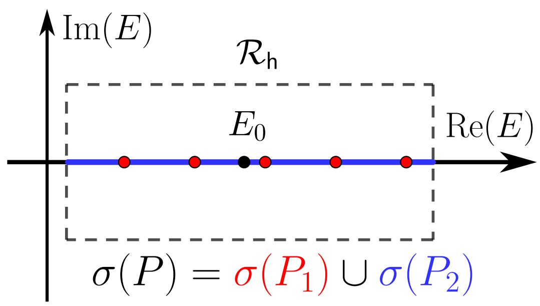

FIGURE 3. Spectrum of $\mathrm{diag}(P_1, P_2)$ around E_0 (σ stands for spectrum in $\mathcal{R}_h$).

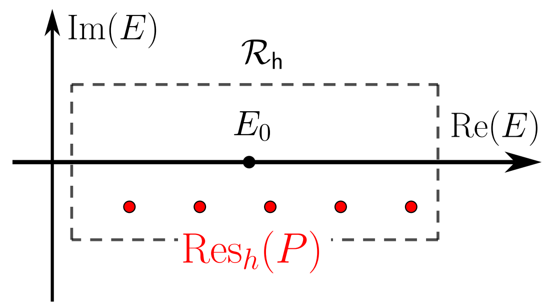

FIGURE 4. Resonances of P around E_0. (Fermi's golden rule)

Considering the above assumptions, the distribution of resonances in this tangential case ($n \geq 1$) is known (see [2], [3]) under some elliptic assumption on $U = r_0(x)$, namely $r(0) \neq 0$. Here however, we allow r_0 to vanish at a finite order. This is sum up in the following assumption.

Assumption 3. *The function r_0 vanishes at $x = 0$ at a finite order $k \in \mathbb{N}$:*

$$r_0^{(j)}(0) = 0 \quad \textit{for all} \quad j \in \{0, \dots, k-1\} \quad \textit{and} \quad r_0^{(k)}(0) \neq 0 \tag{1.7}$$

Moreover, we assume that $k < n$.

In the following, we will "ignore" everything smaller or equal to $\mathcal{O}(h)$. The assumption $k < n$ implies $h = o\left(h^{(k+1)/(m+1)}\right)$. This ensures the quantity $h^{(k+1)/(m+1)}$, appearing as the principal of the stationary phase expansion (A.2) and also as the sub-principal term of the transfer matrix of the microlocal connection formula (3.3) does not get absorbed by the best precision $\mathcal{O}(h)$ we have on the asymptotic expansions (3.7) of solutions.

2. Main result

The main result is as follows. It generalizes [2, Theorem 2.1] as it expresses the width (imaginary part) of the resonances in terms of the contact order m as well as some geometric data of the classical trajectories ($\mathcal{S}(E)$ and $\mathcal{A}(E)$ below), but also in terms of the vanishing order k of r_0.

Theorem 2.1. *Assume that assumptions 1 to 3 hold true. Then for all small $h>0$, there exists a bijective map $z_h : \mathfrak{B}_h \to \mathrm{Res}_h(P)$ such that for any $E \in \mathfrak{B}_h$ one has*

$$|z_h(E) - E| = \mathcal{O}\left(h^{1+2\frac{k+1}{m+1}}\right) \tag{2.1}$$

and

$$\operatorname{Im} z_h(E) = -D(E)h^{1+2\frac{k+1}{m+1}} + \mathcal{O}\left(h^{1+2\frac{k+1}{m+1}+s}\right) \tag{2.2}$$

with $s := \min(1/3, 1/(m+1))$ and

$$D(E) := \frac{2|\omega|^2}{\mathcal{A}'(E_0)} \left|\sin\left(\frac{\operatorname{sgn}(v_m)(k+1)}{2(m+1)}\pi + \frac{\mathcal{S}(E)}{2h}\right)\right|^2. \tag{2.3}$$

Here $\mathcal{S} = \mathcal{S}(E) := 2\left(\int_{a_0'(E)}^0 \sqrt{E - V_1(x)}dx + \int_0^{b_0(E)} \sqrt{E - V_2(x)}dx\right)$ is the area bounded by $\Gamma_1(E)$ and $\Gamma_2(E)$ (shaded area on Figure 2), and ω is defined by

$$\omega := \frac{\mu_{k,m}\left(\frac{\operatorname{sgn}(v_m)(k+1)}{2(m+1)}\pi\right)}{(m+1)k!} \mathbf{\Gamma}\left(\frac{k+1}{m+1}\right) E_0^{\frac{k-m}{2(m+1)}} \left(\frac{2(m+1)!}{|v_m|}\right)^{\frac{k+1}{m+1}} r_0^{(k)}(0) \tag{2.4}$$

where $v_m := (V_2 - V_1)^{(m)}(0)$ and

$$\mu_{k,m}(\theta) := \frac{e^{i\theta} + (-1)^k e^{i(-1)^{m+1}\theta}}{2} \quad \textit{for } \theta \textit{ in } \mathbb{R}. \tag{2.5}$$

In the case where both k and m are odd, $D(E)$ vanishes and the lower order term of $\operatorname{Im} z_h(E)$ shrinks to an order of $h^{1+2\frac{k+2}{m+1}}$, as stated in [12, Theorem 2.2].

3. Microlocal connection formula at a crossing point

Our goal for this section is to convince the reader that the proof of the main result essentially relies on an appropriate (degenerate) stationary phase estimate. The method of proof, developed in [7] and [2], consists in reducing the study to a study of microlocal solutions around crossing points. This is done by establishing a microlocal connection formula (Theorem 3.3). The proof of this formula relies on the construction of two bases of exact solutions near the crossing point and on the study of the linear relationship between these two bases (Lemma 3.6). The asymptotic behavior resulting from this linear relationship is studied using the stationary phase method with both degenerate phase and degenerate amplitude. We refer to Appendix A, which is taken from [12, Appendix A].

To this purpose, we briefly introduce some notations of semiclassical analysis. In this paper, we say that a function $f \in L^2(\mathbb{R}, \mathbb{C}^2)$ depending on h is microlocally infinitely small around a point $(x_0, \xi_0) \in \mathbb{R}^2$, and note

$$f \equiv 0 \quad \text{near} \quad (x_0, \xi_0) \tag{3.1}$$

if there exists $\chi \in \mathcal{C}_b^\infty(\mathbb{R}^2, \mathbb{R})$ independent of h such that $\chi(x_0, \xi_0) \neq 0$ and

$$\left\|\chi^W(x, hD_x)f\right\|_{L^2(\mathbb{R},\mathbb{C}^2)} = \mathcal{O}(h^\infty)$$

where χ^W refers to the Weyl quantization ([16, (4.11)]) of χ. We say that $f \equiv 0$ near Ω, with $\Omega \subset \mathbb{R}^2$, when the previous definition applies to every point of Ω.

We now need to construct microlocal solutions to the equation $(P-E)u \equiv 0$ near a well chosen set. The two following facts will help us to reduce this problem to a microlocal study near the crossing point.

Fact 3.1. *The space of microlocal solutions to the equation $(P-E)w \equiv 0$ near any connected subset of $\Gamma(E) \setminus \mathscr{C}$ is of dimension 1.*

Fact 3.2. *Let w satisfy $(P-E)w \equiv 0$ near all $\mathbb{R}^2$. Then $w \equiv 0$ near $\mathbb{R}^2 \setminus \Gamma$.*

From the second fact, we see that it suffices to study the behavior of microlocal solutions on classical trajectories. From the first fact, we see that the problem consists in describing the space of microlocal solutions around the crossing point.

We now fix $j \in \{1,2\}$ and any point $\rho_j = (x_j, \xi_j) \in \Gamma_j(E_0) \setminus (\mathscr{C} \cup \{\xi = 0\})$ on Γ_j which is neither a crossing point nor a turning point. Then, we construct via a WKB method a non-trivial microlocal solution f_{ρ_j} to the equation $(P-E)f_{\rho_j} \equiv 0$ near ρ_j (see for example [7, Proposition 5.4]) of the form

$$f_{\rho_j}(x,h) = e^{\frac{i}{h}\operatorname{sgn}(\xi_j)\phi_j(x)} \begin{pmatrix} \sigma_{j,1}(x,h) \\ \sigma_{j,2}(x,h) \end{pmatrix}, \tag{3.2}$$

where the phase function ϕ_j is defined starting from the x-coordinate $\rho_x = 0$ of the crossing points $(0, \pm\sqrt{E_0})$ by $\phi_j(x) := \int_0^x \sqrt{E_0 - V_j(y)}\, dy$ and $\sigma_{j,k}(x,h) \sim \sum_{l \geq 0} h^l \sigma_{j,k,l}(x)$ with

$$\sigma_{j,j,0}(x) = (E_0 - V_j(x))^{-1/4}, \quad \sigma_{j,\hat{j},0}(x) = 0 \quad \text{and} \quad \sigma_{j,j,l}(x_j) = 0 \quad \text{for all} \quad l \geq 1.$$

Here, $\hat{j}$ is the complementary of j in $\{1,2\}$, meaning that $\{j, \hat{j}\} = \{1,2\}$.

Secondly, we study the behavior of microlocal solutions near crossing points. We split $\Gamma \setminus (\mathscr{C} \cup \{\xi = 0\})$ in eight connected components according to the crossing points and the turning points (see Figure 5 for the labelling at ρ_+):

$$\gamma_j^\flat := \Gamma_j \cap \{x < 0, \xi > 0\}, \quad \gamma_j^\sharp := \Gamma_j \cap \{x > 0, \xi > 0\}, \tag{3.3}$$

$$\breve{\gamma}_j^\flat := \Gamma_j \cap \{x > 0, \xi < 0\}, \quad \breve{\gamma}_j^\sharp := \Gamma_j \cap \{x < 0, \xi < 0\}. \tag{3.4}$$

Let $\rho_j^\bullet$ be any point of $\gamma_j^\bullet$. Since the space of microlocal solutions away from crossing points is of dimension 1, any microlocal solution $w \in L^2(\mathbb{R}, \mathbb{C}^2)$ of

$$(P-E)w \equiv 0 \quad \text{near} \quad \rho_\pm$$

is a (complex) multiple

$$w \equiv \alpha_j^\bullet f_j^\bullet \quad \text{near} \quad \rho_j^\bullet$$

of the previously introduced microlocal solution $f_j^\bullet := f_{\rho_j^\bullet}$. We symmetrically define $\breve{\alpha}_j^\bullet$ and $\breve{f}_j^\bullet$ on $\breve{\gamma}_j^\bullet$. The following theorem sums this remark up and gives an asymptotic microlocal connection formula near the crossing points.

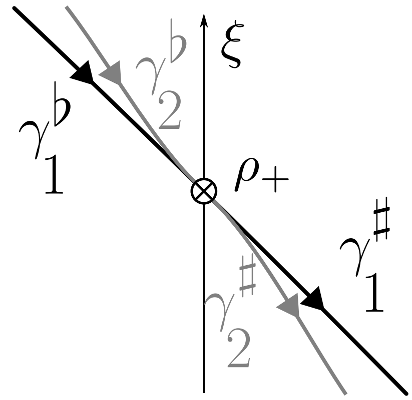

FIGURE 5. Labeling around the upper crossing point ρ_+

Theorem 3.3. *Assume that assumptions 1 to 3 hold true. Then the space of microlocal solutions w of the equation $(P-E)w \equiv 0$ near the crossing point $\rho_\pm$ is of dimension 2 : there exists a 2×2 matrix $T^\pm = (t^\pm_{i,j})_{1\le i,j\le 2}$ depending on E, h and $\rho_j^\bullet$ (respectively E, h and $\breve\rho_j^\bullet$ for T^-) such that, for any microlocal solution $w \in L^2(\mathbb{R},\mathbb{C}^2)$ of $(P-E)w \equiv 0$ near $\rho_\pm$ with $\|w\|_{L^2(\mathbb{R},\mathbb{C}^2)} \le 1$,*

$$\begin{pmatrix}\alpha_1^\sharp\\ \alpha_2^\sharp\end{pmatrix} = T^+(E,h)\begin{pmatrix}\alpha_1^\flat\\ \alpha_2^\flat\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\breve\alpha_1^\sharp\\ \breve\alpha_2^\sharp\end{pmatrix} = T^-(E,h)\begin{pmatrix}\breve\alpha_1^\flat\\ \breve\alpha_2^\flat\end{pmatrix} \tag{3.5}$$

We call ***transfer matrix at $\boldsymbol{\rho}_\pm$*** *the matrix $T^\pm$. Moreover, T^+ is given for all $E\in\mathcal{R}_h$ by*

$$T^+(E,h) = I_2 - ih^{\frac{k+1}{m+1}}\begin{pmatrix}0 & \overline{\omega}\\ \omega & 0\end{pmatrix} + \mathcal{O}\left(h^{\frac{k+2}{m+1}}\right) \tag{3.6}$$

where ω is given by (2.4). A symmetric expansion for T^- at ρ_- is obtained from (3.11).

Remark 3.4. *The first part of the theorem claims the existence of the transfer matrix, which is invertible for h small enough. This proves in particular that any microlocal solution u to $(P-E)u = 0$ near ρ_+ satisfying $\alpha_1^\flat = \alpha_2^\flat = 0$ is microlocally small near ρ_+. This can be seen as an uniqueness property around the crossing point. We skip some computations for this part of the proof (which is essentially the same as in* [2, Section 3.4]*) and focus more on the asymptotical expansion, which is the main contribution of* [12] *compared to previous works.*

Let us now turn to the proof of Theorem 3.3. We study the behavior of microlocal solutions on a small, compact, h-independent interval I around the x-coordinate $x=0$ of the crossing points. In the phase space, this means that we consider $\Gamma_j \cap (I\times\mathbb{R}_\xi)$. We follow the idea of [5, Annex 2], which adapts an idea from D.Yafaev ([15]) and constructs exact solutions for the scalar equations of $(P_j - E)u_j = 0$ based on Picard's successive approximation method (see for example [14, Chapter 5]). We obtain pairs $(u_j, \breve u_j)$ of solutions to $(P_j-E)u_j = 0$ in I which asymptotically behave as

$$u_j(x) = \frac{1+\mathcal{O}(h)}{\sqrt[4]{E-V_j(x)}}e^{\frac{i}{h}\int_0^y \sqrt{E-V_j(y)}dy}, \quad \breve u_j(x) = \frac{1+\mathcal{O}(h)}{\sqrt[4]{E-V_j(x)}}e^{-\frac{i}{h}\int_0^y \sqrt{E-V_j(y)}dy} \tag{3.7}$$

where $j\in\{1,2\}$. This asymptotic behavior when $h\to 0^+$ is uniform on I. Next, we use those scalar solutions to construct vector-valued solutions w of the system $(P-E)w = 0$. Those solutions, constructed with the appropriate integral operators, are close at order $\mathcal{O}(h)$ to the solutions $f_j^\bullet$ and small at order $\mathcal{O}(h)$ elsewhere (see [5, Section 4]). Essentially, this construction relies on the estimates of Proposition 3.5 below.

We fix a small $\delta > 0$ and define for $f \in \mathcal{C}(I)$ and $x \in I$ the integral operators $\widetilde{K}_{j,S}$ by

$$\widetilde{K}_{j,S} f(x) := \frac{i}{2}\left(\breve{u}_j(x)\int_{I_S(x)} u_j(y) r_0(y) f(y) dy - u_j(x)\int_{I_S(x)} \breve{u}_j(y) r_0(y) f(y) dy\right) \tag{3.8}$$

where $I_R(x) := [\delta, x]$ and $I_L(x) := [-\delta, x]$ with $j \in \{1,2\}$ and $S \in \{L, R\}$. The next proposition corresponds to [12, Proposition 3.2] and is proved therein.

Proposition 3.5. *For all $j \in \{1,2\}$ and $S \in \{L, R\}$, $\widetilde{K}_{j,S}$ is a bounded operator of $\mathcal{C}(I)$. Moreover,*

$$\left\| \widetilde{K}_{j,S} \widetilde{K}_{\hat{j},S} \right\|_{\mathcal{B}(\mathcal{C}(I))} = \mathcal{O}\left(h^{\frac{k+1}{m+1}}\right). \tag{3.9}$$

Following the method of [5, Section 4], one can construct two basis $(\breve{w}_1^\sharp, w_1^\flat, \breve{w}_2^\sharp, w_2^\flat)$ and $(\breve{w}_1^\flat, w_1^\sharp, \breve{w}_2^\flat, w_2^\sharp)$ of exact solutions on I to $(P - E)w = 0$ and a matrix $A = (a_{i,j}(E,h))_{1 \le i,j \le 4}$ such that

$$(\breve{w}_1^\sharp, w_1^\flat, \breve{w}_2^\sharp, w_2^\flat) = A(\breve{w}_1^\flat, w_1^\sharp, \breve{w}_2^\flat, w_2^\sharp)$$

and

$$T^+ = \begin{pmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{pmatrix} + \mathcal{O}(h).$$

We skip the reasoning leading up to the existence of such matrix A (and refer to [2]) and focus on the asymptotic expansion of the coefficients of A.

Lemma 3.6. *The matrix A is of the form $A = \begin{pmatrix} I_2 & \widetilde{A}_1 \\ \widetilde{A}_2 & I_2 \end{pmatrix} + \mathcal{O}\left(h^{\frac{k+2}{m+1}}\right)$ where $\widetilde{A}_j = \begin{pmatrix} \breve{F}_j \breve{u}_j & \breve{F}_j u_j \\ F_j \breve{u}_j & F_j u_j \end{pmatrix}$ and $F_j := \alpha_{j,L} - \alpha_{j,R}$, $\breve{F}_j := \breve{\alpha}_{j,L} - \breve{\alpha}_{j,R}$. In particular the coefficients of T^+ are given by $a_{22} = a_{44} = 1$ and*

$$a_{42} = F_1 u_1 = -i\omega h^{\frac{k+1}{m+1}} + \mathcal{O}\left(h^{\frac{k+2}{m+1}}\right), \quad a_{24} = F_2 u_2 = -i\overline{\omega} h^{\frac{k+1}{m+1}} + \mathcal{O}\left(h^{\frac{k+2}{m+1}}\right),$$

where ω is given by (2.4). Here, $\alpha_{j,S}$ and $\beta j, S$ stand for

$$\alpha_{j,S}(f) := \pm\frac{i}{2}\int_{I_S(0)} \breve{u}_{\hat{j}} U_{\hat{j}} J_{j,S} f \quad \text{or} \quad \beta_{j,S}(f) := \pm\frac{i}{2}\int_{I_S(0)} \breve{u}_{\hat{j}} U_j \widetilde{K}_{\hat{j},S} J_{j,S} f \tag{3.10}$$

where $J_{j,S} := \sum_{l \ge 0} \left(\widetilde{K}_{j,S}\widetilde{K}_{\hat{j},S}\right)^l$ (when defined), $j \in \{1,2\}$ and I_S is defined in (3.8). The $\pm$ sign is $+$ when $S = R$ and $-$ when $S = L$. The quantities $\breve{\alpha}_{j,S}(f)$ and $\breve{\beta}_{j,S}(f)$ are symmetrically defined replacing $\breve{u}_j$ by u_j.

Proof. This lemma is essentially due to Lemma A.1. The integrals we have to compute are of the same type as

$$F_1 u_1 = \frac{-i}{2}\left[\int_{-\delta}^{\delta} \breve{u}_2 r_0 u_1 + \sum_{l \ge 1}\left(\int_{-\delta}^{0} \breve{u}_2 r_0 (\widetilde{K}_{1,L}\widetilde{K}_{2,L})^l u_1 + \int_0^{\delta} \breve{u}_2 r_0 (\widetilde{K}_{1,R}\widetilde{K}_{2,R})^l u_1\right)\right.$$

where a_j and $\breve{a}_j$ are the amplitudes of the exact scalar solutions u_j and $\breve{u}_j$ defined in (3.7). Essentially, the right term (sum from $l = 1$) can be dealt with using Proposition 3.5, that is to say with the estimate (A.1). The first term, corresponding to $l = 0$ (and yielding the coefficient ω of the sub-principal term of the transfer matrix) is computed using the asymptotic expansion (A.2).

□

Remark 3.7. *One can show that*

$$T^- = T^-(E,h) = \left(\overline{T^+(\overline{E},h)}\right)^{-1} = \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} + \mathcal{O}(h) \tag{3.11}$$

4. Computation of the width of resonances

In this section, we give a sketch of the computation of the with of the resonances (2.2). We leave aside the proof of existence of z_h, which is essentially the same as the proof of [2, Section 4.3], which is done in the more general case of an arbitrary number of crossings.

Let us consider a resonant state u for a resonance $E \in \mathcal{R}_h$. Using the notations of the previous section and the WKB solutions $f_j^\bullet$ and $\breve{f}_j^\bullet$ defined therein, there exist eight constants $\alpha_j^\bullet$ and $\breve{\alpha}_j^\bullet$ such that

$$u \equiv \alpha_j^\bullet f_j^\bullet, \quad \text{respectively} \quad u \equiv \breve{\alpha}_j^\bullet \breve{f}_j^\bullet \tag{4.1}$$

microlocally near $\rho_j^\bullet$, respectively microlocally near $\breve{\rho}_j^\bullet$, for $j \in \{1,2\}$, $\bullet \in \{\flat, \sharp\}$.

Then, according to [7, Proposition 7.1], we have

$$\mathrm{Im}(E) = -\frac{|\breve{\alpha}_2^\sharp|^2}{\|u\|^2_{L^2(x_0,+\infty)}} h(1 + \mathcal{O}(h)). \tag{4.2}$$

We send the reader to [12] for the computation of

$$\|u\|^2_{L^2(x_0,+\infty)} = 2\mathcal{A}'(E) + \mathcal{O}\left(h^{\frac{1}{3}} + h^{\frac{k+1}{m+1}}\right).$$

Our goal for the rest of this section is then to compute $|\breve{\alpha}_2^\sharp|^2$. The first step is to remark that $\alpha_2^\flat = 0$. Essentially, this comes from the fact that, on $\gamma_2^\flat$, one can reduce the microlocal equation $(P - E)u \equiv 0$ to the scalar case since $P_1 - E$ is elliptic. One can then, as in the scalar case, prove that u is microlocally small near $\gamma_2^\flat$ using an escape function (see [4, Section 8]). We also claim that $\alpha_1^\flat \neq 0$. This is actually due to the uniqueness property (Remark 3.4). One can then scale u up so as to have $\alpha_1^\flat = 1$. The idea is now to use the transfer matrix T^+ at the crossing point ρ_+ to deduce the asymptotic behavior of $\alpha_1^\sharp$ and $\alpha_2^\sharp$ as illustrated below. More precisely, we use

$$\begin{pmatrix} \alpha_1^\sharp \\ \alpha_2^\sharp \end{pmatrix} = T \begin{pmatrix} \alpha_1^\flat \\ \alpha_2^\flat \end{pmatrix} = \begin{pmatrix} \alpha_1^\flat \\ \alpha_2^\flat \end{pmatrix} - ih^{\frac{k+1}{m+1}} \begin{pmatrix} \overline{\omega}\alpha_2^\flat \\ \omega\alpha_1^\flat \end{pmatrix} + \mathcal{O}(h).$$

We represent the situation on Figure 6, where $\simeq$ stands for an equality up to $\mathcal{O}(h)$.

By an argument of propagation of semiclassical singularities, we remark that the microlocal equalities (4.1) are valid microlocally near any point of $\gamma_j^\bullet$ (resp. $\breve{\gamma}_j^\bullet$). This is why represent the situation with grey zones on Figure 6.

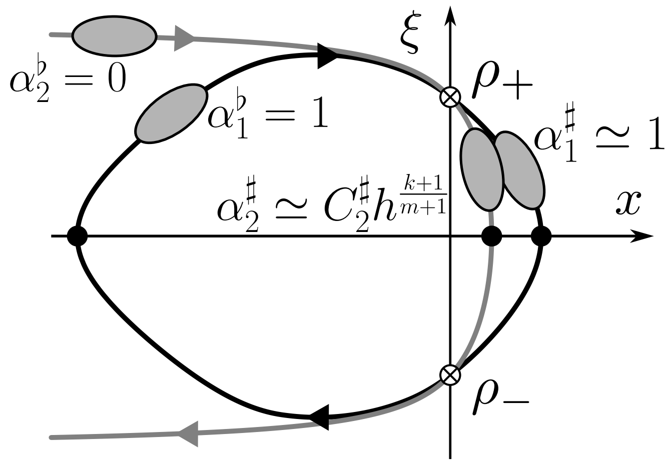

FIGURE 6. Computing $\alpha_1^\sharp$ and $\alpha_2^\sharp$.

Now, we apply a "Maslov correction" (see [7, Lemma 6.1]) to pass through the turning points b and a_0' and we deduce $\breve{\alpha}_1^\flat$ and $\breve{\alpha}_2^\flat$ from $\alpha_1^\sharp$ and $\alpha_2^\sharp$, see Figure 7. This yields

$$\begin{cases} \breve{\alpha}_1^\flat e^{\frac{i}{h}\mathcal{S}_1 - i\frac{\pi}{2}} & = \alpha_1^\sharp \\ \breve{\alpha}_2^\flat e^{\frac{i}{h}\mathcal{S}_2 - i\frac{\pi}{2}} & = \alpha_2^\sharp \end{cases}$$

where $\mathcal{S}_j = \mathcal{S}_j(E)$ is the classical action from ρ_+ to ρ_- on Γ_j.

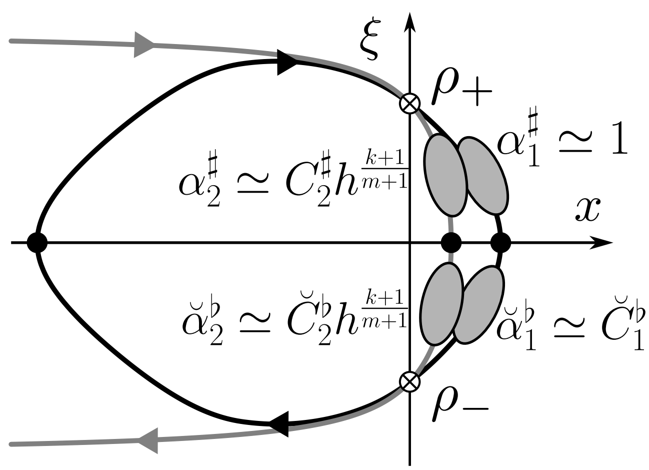

FIGURE 7. Computing $\breve{\alpha}_1^\flat$ and $\breve{\alpha}_2^\flat$ (passing the turning points).

Finally, we repeat the first step of Figure 6 with the transfer matrix T^- at the crossing point ρ_-, that is to say

$$\begin{pmatrix} \breve{\alpha}_1^\sharp \\ \breve{\alpha}_2^\sharp \end{pmatrix} = T^- \begin{pmatrix} \breve{\alpha}_1^\flat \\ \breve{\alpha}_2^\flat \end{pmatrix}.$$

We deduce $\breve{\alpha}_1^\sharp$ and $\breve{\alpha}_2^\sharp$ from $\breve{\alpha}_1^\flat$ and $\breve{\alpha}_2^\flat$, see Figure 8.

We then obtain indeed an imaginary part $\mathrm{Im}(E) \simeq (2\mathcal{A}'(E_0))^2 \left|\breve{C}_j^\sharp\right|^2 h^{1+2\frac{k+1}{m+1}}$, where the constant $\breve{C}_j^\sharp$ is computed using the asymptotic expansion of the transfer matrices $T^\pm$ as above. Remark that

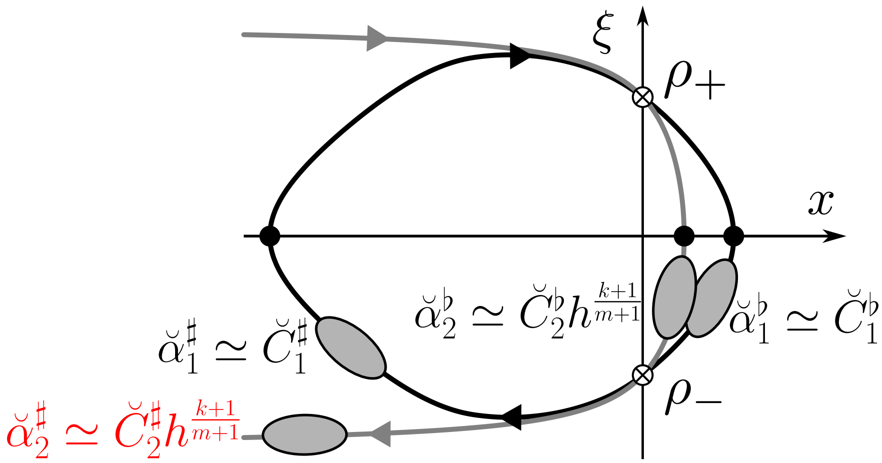

FIGURE 8. Computing $\breve{\alpha}_1^\sharp$ and $\breve{\alpha}_2^\sharp$.

the last step also allows to compute $\breve{\alpha}_1^\sharp$. Using again a Maslov correction at a_0, we can deduce $\alpha_1^\flat = C_1^\flat + \mathcal{O}(h) = 1$. This yields a quantization condition $C_1^\flat = 1$ on the resonance E. One can check that this corresponds to the Bohr-Sommerfeld condition $\cos(\mathcal{A}(E)/(2h)) = 0$ of (1.6).

5. AN OPEN QUESTION

During a similar presentation of [12] at the Symposium on spectral and scattering theory of Kitasato university in January 2024, professor Haruya Mizutani asked a question regarding the contact order m of Theorem 3.3: does the theorem still hold for an infinite contact order $m = +\infty$? In particular, the main problem is to know whether one can extend the uniqueness property (existence of the transfer matrix) to this case. The question is still open to this day.

APPENDIX A. A DEGENERATE STATIONARY PHASE ESTIMATE

Lemma A.1. *Let I be a bounded open interval of $\mathbb{R}$ containing 0 and fix $\alpha \in I$. Let $\sigma \in \mathcal{C}_b^\infty(I, \mathbb{C})$ and $\phi \in \mathcal{C}_b^\infty(I, \mathbb{R})$ two smooth functions with bounded derivatives at any order. We assume the following :*

- ***(H1)*** *The function ϕ' does not vanish on $I \setminus \{0\}$ and is of the form $\phi'(y) = y^m \phi_1(y)$ where ϕ_1 is a smooth function such that $\phi_1(0) \neq 0$.*
- ***(H2)*** *The function σ is of the form $\sigma(y) = y^k \sigma_0(y)$ where σ_0 is a smooth function.*

If $k \leq m$ then there exist a constant $C > 0$ independent of h such that, for any $x \in I$ and any $h > 0$ small enough,

$$\left| \int_\alpha^x e^{\frac{i}{h}\phi(y)} \sigma(y) dy \right| \leq C \left(h^{\frac{k+1}{m+1}} \|\sigma_0\|_{L^\infty(I)} + h^{\frac{k+2}{m+1}} \|\sigma_0'\|_{L^\infty(I)} \log\left(\frac{1}{h}\right)^{\delta_{m,k+1}} \right) \tag{A.1}$$

where $\delta_{k,m+1}$ is the Kronecker symbol. Furthermore, regardless of k and m, the following asymptotic expansion holds:

(A.2)

$$e^{-\frac{i}{h}\phi(0)} \int_I \sigma(y) e^{\frac{i}{h}\phi(y)} dy = \frac{2\mu_{k,m}\left(\frac{\varepsilon(\phi)(k+1)}{2(m+1)}\pi\right)}{m+1} \mathbf{\Gamma}\left(\frac{k+1}{m+1}\right) \left(\frac{m+1}{|\phi_1(0)|}\right)^{\frac{k+1}{m+1}} \sigma_0(0) h^{\frac{k+1}{m+1}} + \mathcal{O}\left(h^{\frac{k+2}{m+1}}\right)$$

where $\mathbf{\Gamma}$ *is the Gamma function,* $\varepsilon(\phi)$ *is* $\mathrm{sgn}(\phi_1(0))$ *and* $\mu_{k,m}$ *is defined in* (2.5) *replacing* v_m *by* $\phi_1(0)$*. When* k *and* m *are both odd the value of the previous integral is, with error* $\mathcal{O}\left(h^{\frac{k+3}{m+1}}\right)$,

$$\text{(A.3)} \qquad \frac{2\mu_{k+1,m}\left(\frac{\varepsilon(\phi)(k+2)}{2(m+1)}\pi\right)}{m+1}\mathbf{\Gamma}\left(\frac{k+2}{m+1}\right)\left(\frac{m+1}{|\phi_1(0)|}\right)^{\frac{k+2}{m+1}}\left[\sigma_0'(0)-\frac{2(2k+1)\phi_1'(0)}{(m+2)\,|\phi_1(0)|}\sigma_0(0)\right]h^{\frac{k+2}{m+1}}.$$

Using a Taylor expansion in x near the critical point $x=0$ of the phase, we reduce the proof to a computation of a Fresnel-type integral $\displaystyle\int y^k e^{\frac{i}{h}y^{m+1}}dy$ via complex analysis, and a careful examination of the contribution of that integral in a $\mathcal{O}\left(h^{1/m+1}\right)$-neighborhood of $x=0$.

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