# On the inverse scattering for time-decaying harmonic oscillators

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## 1 Introduction

The harmonic oscillator as a self-adjoint operator acting on $L^2(\mathbb{R}^n)$ is given by

$$-\hbar^2\Delta/(2m) + m\omega^2 x^2/2, \tag{1.1}$$

were $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is the position of the particle and $\Delta = \sum_{j=1}^n \partial_{x_j}^2$ is the Laplacian. We also denoted the Plank constant, mass of the particle and frequency by $\hbar = h/(2\pi)$, $m > 0$ and $\omega > 0$ respectively. In the following, by the suitable scale conversion and coordinate rotation, we let $\hbar$ and $m$ be $\hbar = m = 1$ without loss of generality, and write such that

$$H_0 = p^2/2 + \omega^2 x^2/2, \tag{1.2}$$

where $p$ is the momentum operator $-\mathrm{i}\nabla = -\sqrt{-1}(\partial_{x_1}, \ldots, \partial_{x_n})$. In the quantum system governed by the harmonic oscillator (1.2), it is well known that the particle makes bound states only and can not make scattering states. However, if $\omega^2$ is replaced by the time-dependent coefficient

$$k(t) = \begin{cases} \omega^2 & \text{if} \quad |t| < r_0, \\ \sigma/t^2 & \text{if} \quad |t| \geqslant r_0. \end{cases} \tag{1.3}$$

for $0 < \sigma < 1/4$, $\omega > 0$ and $r_0 > 0$, it is proved by [11] that the particle can generates scattering states . We define the time-dependent free Hamiltonian

$$H_0(t) = p^2/2 + k(t)x^2/2 \tag{1.4}$$

and perturbed full Hamiltonian $H(t) = H_0(t) + V$, where $V = V(x)$ is the real-valued function that satisfies Assumption 1.1. In this report, we assume that the space dimension $n \geqslant 2$ and introduce the result of the inverse scattering [9]. By applying the time-dependent method invented by Enss-Weder [4], we can prove that the scattering operator which is defined by the wave operators determines potential function $V$ uniquely.

The assumptions for the potential $V$ are quite important in scattering theory. We state the details of these assumptions below. We use the following notations. The bracket of $x$ has the definition $\langle x \rangle = \sqrt{1+|x|^2}$, $A \lesssim B$ means that there exists a positive constant $C$ such that $A \leqslant CB$, and $\|\cdot\|$ and $(\cdot,\cdot)$ denote the $L^2(\mathbb{R}^n)$-norm and scalar product of $L^2(\mathbb{R}^n)$ respectively. We also define

$$0 < \lambda = (1-\sqrt{1-4\sigma})/2 < 1/2. \tag{1.5}$$

**Assumption 1.1.** *The potential function $V$ is decomposed into a bounded part and a singular part,*

$$V(x) = V^{\mathrm{bdd}}(x) + V^{\mathrm{sing}}(x). \tag{1.6}$$

*The real-valued function $V^{\mathrm{bdd}} \in L^\infty(\mathbb{R}^n)$ satisfies*

$$|V^{\mathrm{bdd}}(x)| \lesssim \langle x \rangle^{-\rho} \tag{1.7}$$

*almost everywhere, with $x \in \mathbb{R}^n$ and $\rho > 1/(1-\lambda)$. The real-valued function $V^{\mathrm{sing}} \in L^q(\mathbb{R}^n)$ is compactly supported, where $q$ satisfies*

$$\infty > q \begin{cases} = 2 & \text{if} \quad n \leqslant 3, \\ > n/2 & \text{if} \quad n \geqslant 4. \end{cases} \tag{1.8}$$

By virtue of Yajima [22, Theorem 6 and Remark (a)], the existence of the propagators uniquely generated by $H_0(t)$ and $H(t)$ is guaranteed under Assumption 1.1. We denote these propagators by $U_0(t,s)$ and $U(t,s)$, respectively. The wave operators

$$W^{\pm} = \operatorname*{s-lim}_{t\to\pm\infty} U(t,0)^* U_0(t,0) \tag{1.9}$$

then exist by [11, Theorem 1] and the scattering operator is defined such that

$$S(V) = (W^+)^* W^-. \tag{1.10}$$

Although the existence of (1.9) is proved by [11, Theorem 1] only for $V = V^{\mathrm{bdd}}$, we can immediately prove the existence of (1.9) for $V = V^{\mathrm{bdd}} + V^{\mathrm{sing}}$ using propagation estimates [11, Proposition 2].

The main theorem ([9, Theorem 1.5]) in this report is the following.

**Theorem 1.2.** *Let $V_1$ and $V_2$ satisfy Assumption 1.1. If $S(V_1) = S(V_2)$, then $V_1 = V_2$ holds.*

Since the Enss–Weder time-dependent method was invented, a lot of authors proved the uniqueness of the interaction potentials for various quantum systems. The results for external electric fields; [1], [2], [3], [8], [14], [15], [17] and [21], for repulsive Hamiltonians; [6], [10] and [16], for fractional and relativistic Laplacians; [7] and [12], and for non-linear Schrödinger equations and the Hartree-Fock equations; [18], [19] and [20].

## 2 Reconstruction theorem

By the definition of $k(t)$, $H_0(t) \equiv H_0$ is a time-independent harmonic oscillator for $|t| < r_0$. By the well-known Mehler formula [13, Theorem 5.29], the time evolution for $0 < |t| < r_0$ and $\omega t \notin (\pi/2)\mathbb{Z}$ can be written such that

$$e^{-\mathrm{i}tH_0} = \mathrm{i}^{n/2} e^{-\mathrm{i}\omega \tan \omega t x^2/2} D(\cos \omega t) e^{-\mathrm{i} \tan \omega t p^2/(2\omega)}, \tag{2.1}$$

where $D$ denotes the dilation $D(t)\phi(x) = (\mathrm{i}t)^{-n/2}\phi(x/t)$ for $\phi \in L^2(\mathbb{R}^d)$. On the other hand, $U_0(t,s)$ also has the convenient factorizations for $t, s \geqslant r_0$ or $t, s \leqslant -r_0$, that were proved by [11, Proposition 1]. We define

$$\tilde{U}_0(t) = e^{\mathrm{i}\lambda x^2/(2t)} e^{-\mathrm{i}\lambda \log t A} e^{-\mathrm{i}t^{1-2\lambda}p^2/(2(1-2\lambda))} \tag{2.2}$$

if $t \geqslant r_0$ and

$$\tilde{U}_0(t) = e^{\mathrm{i}\lambda x^2/(2t)} e^{-\mathrm{i}\lambda \log(-t) A} e^{\mathrm{i}(-t)^{1-2\lambda}p^2/(2(1-2\lambda))} \tag{2.3}$$

if $t \leqslant -r_0$, where $A = (p \cdot x + x \cdot p)/2$. Then $U_0(t,s) = \tilde{U}_0(t)\tilde{U}_0(s)^*$ holds for $t, s \geqslant r_0$ or $t, s \leqslant -r_0$. We additionally define $\tilde{U}_0(t) = e^{-itH_0}$ if $|t| < r_0$. The following strong limits

$$\tilde{W}^{\pm} = \operatorname*{s-lim}_{t \to \pm\infty} U(t,0)^* \tilde{U}_0(t) \tag{2.4}$$

exist because (1.9) exist and we define

$$\tilde{S}(V) = (\tilde{W}^+)^* \tilde{W}^-. \tag{2.5}$$

Noting that $W^{\pm} = \tilde{W}^{\pm}\tilde{U}_0(s_{\pm})^* U_0(s_{\pm}, 0)$ for $s_+ \geqslant r_0$ and $s_- \leqslant -r_0$, we easily find that $S$ and $\tilde{S}$ the relation

$$S(V) = U_0(s_+, 0)^* \tilde{U}_0(s_+) \tilde{S}(V) \tilde{U}_0(s_-)^* U_0(s_-, 0), \tag{2.6}$$

and that $S(V_1) = S(V_2)$ is equivalent to $\tilde{S}(V_1) = \tilde{S}(V_2)$.

The following reconstruction theorem ([9, Theorem 3.1]) and Plancherel formula associated with the Radon transform (see [5, Theorem 2.17 in Chap.1]), yield the proof of Theorem 1.2 as in the proof of [4, Theorem 1.1].

**Theorem 2.1.** *Let $\Phi_0 \in \mathscr{S}(\mathbb{R}^n)$ such that $\mathscr{F}\Phi_0 \in C_0^\infty(\mathbb{R}^n)$. For $v \in \mathbb{R}^n$, its normalization is $\hat{v} = v/|v|$. Let $\Phi_v = e^{\mathrm{i}v\cdot x}\Phi_0$ and $\Psi_v$ have the same properties. Then*

$$\lim_{|v| \to \infty} |v| (\mathrm{i}(\tilde{S}(V) - 1)\Phi_v, \Psi_v) = \int_{-\infty}^{\infty} (V(x + \hat{v}t)\Phi_0, \Psi_0) \mathrm{d}t \tag{2.7}$$

*holds.*

The key estimate for the proof of Theorem 2.1 is following Lemmas 2.2 ([9, Lemmas 2.3 and 3.2]).

**Lemma 2.2.** *Let $\Phi_v$ be as in Theorem 2.1. Then*

$$\int_{-\infty}^{\infty} \|V(x)\tilde{U}_0(t)\Phi_v\|\mathrm{d}t = O(|v|^{-1}) \tag{2.8}$$

*holds as* $|v| \to \infty$.

To analyze the time-evolution by $U_0(t,0)$ for all $t \in \mathbb{R}$ directly is difficult but can be overcome by pursuing the evolution of $e^{-\mathrm{i}\tan\omega t p^2/(2\omega)}$ if $|t| < r_0$ and $e^{\mp\mathrm{i}|t|^{1-2\lambda}p^2/(2(1-2\lambda))}$ if $|t| \geqslant r_0$.

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