Inverse Scattering on the Metric Graph for Graphene

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1 Introduction

We denote by $\Gamma = (\mathcal{V}, \mathcal{E})$ the graph, where $\mathcal{V}$ is the vertex set and $\mathcal{E}$ is the edge set, respectively. We assume that Γ is periodic. We omit the definition of the general periodic graph Γ, as we only treat the hexagonal lattice whose definition will be given later. We can regard the edge as a segment; for convenience sake, we identify $\boldsymbol{e} \in \mathcal{E}$ as a closed interval $[0, 1]$:

$$\boldsymbol{e} = \{(1-x)\,\boldsymbol{e}(0) + x\boldsymbol{e}(1);\, x \in [0,1]\}\,, \quad \boldsymbol{e}(0), \boldsymbol{e}(1) \in \mathcal{V}, \tag{1}$$

which means that $\boldsymbol{e}$ can be regarded as a directed edge. The metric graph is the graph equipped with one-dimemsional Schrödinger operator $-d^2/dx^2 + q_{\boldsymbol{e}}(x)$ on each edge $\boldsymbol{e} \in \mathcal{E}$ for a real-valued function $q_{\boldsymbol{e}}(x) \in L^2(0,1)$ with suitable boundary conditions on every vertices. We assume that $q_{\boldsymbol{e}}(x)$ is symmetric on any $\boldsymbol{e} \in \mathcal{E}$, i.e., $q_{\boldsymbol{e}}(x) = q_{\boldsymbol{e}}(1-x)$, and $q_{\boldsymbol{e}} = 0$ except for a finite number of edges. If $q_{\boldsymbol{e}} = 0$ for all edges, the Schrödinger operator is the metric Laplacian on $\mathcal{E}$. It is worth mentioning that the choice of direction of each edge does not affect the results.

Let $u = \{u_{\boldsymbol{e}}\}_{\boldsymbol{e}\in\mathcal{E}}$, where $u_{\boldsymbol{e}}(x)$ is defined on $\boldsymbol{e}$, $0 \le x \le 1$. We define the Hilbert space $L^2(\mathcal{E}) = \oplus_{\boldsymbol{e}\in\mathcal{E}} L^2_{\boldsymbol{e}}$, where $L^2_{\boldsymbol{e}} = L^2(0,1)$, equipped with the following inner product:

$$(u, v)_{L^2(\mathcal{E})} = \sum_{e\in\mathcal{E}} (u_{\boldsymbol{e}}, v_{\boldsymbol{e}})_{L^2(0,1)} = \sum_{\boldsymbol{e}\in\mathcal{E}} \int_0^1 u_{\boldsymbol{e}}(x)\overline{v_{\boldsymbol{e}}(x)}\,dx.$$

The Sobolev spaces $H^s(\mathcal{E})$ on $\mathcal{E}$, $s \in \mathbb{R}$, are defined in a similar way:

$$H^s(\mathcal{E}) = \oplus_{\boldsymbol{e}\in\mathcal{E}} H^s_{\boldsymbol{e}},$$

where $H^s_{\boldsymbol{e}} = H^s(0,1)$.

For $u = \{u_{\boldsymbol{e}}\}_{\boldsymbol{e}\in\mathcal{E}}$, we define the normal derivative $\partial_{\boldsymbol{e}} u(p)$ at the boundaries $p = \boldsymbol{e}(0), \boldsymbol{e}(1) \in \mathcal{V}$ as follows:

$$\partial_{\boldsymbol{e}} u(p) = \begin{cases} u'_{\boldsymbol{e}}(0), & \text{if } p = \boldsymbol{e}(0), \\ -u'_{\boldsymbol{e}}(1), & \text{if } p = \boldsymbol{e}(1), \end{cases} \quad \text{where } u'_{\boldsymbol{e}} = \frac{du_{\boldsymbol{e}}}{dx}.$$

For $u = \{u_{\boldsymbol{e}}\}_{\boldsymbol{e}\in\mathcal{E}}$, the Kirchhoff condition is given as follows:

(K-1) u is continuous at any vertex $v \in \mathcal{V}$,

(K-2) $\sum_{\boldsymbol{e}\in\mathcal{E}} \partial_{\boldsymbol{e}} u(v) = 0$ at any vertex $v \in \mathcal{V}$.

The Kirchhoff Laplacian $H_{\mathcal{E}}^{(0)}$ is a metric Laplacian which satisfies the Kirhhoff condition:

$$H_{\mathcal{E}}^{(0)} u = \{-u''_{\boldsymbol{e}}\}_{\boldsymbol{e}\in\mathcal{E}}\,, \quad \text{where } u''_{\boldsymbol{e}} = \frac{d^2}{dx^2} u_{\boldsymbol{e}},$$

$$\mathrm{Dom}(H_{\mathcal{E}}^{(0)}) = \left\{u \in H^2(\mathcal{E});\, u \text{ satisfies (K-1) and (K-2)}\right\}.$$

It is easy to see that $H_{\mathcal{E}}^{(0)}$ is self-adjoint on $L^2(\mathcal{E})$.

For the Kirchhoff Schrödinger operator on the metric graph, we impose the following assumption on the potential $q = \{q_{\boldsymbol{e}}\}_{\boldsymbol{e}\in\mathcal{E}}$:

(Q-1) $q_{\boldsymbol{e}} \in L^2(0,1)$ is real-valued for every $\boldsymbol{e} \in \mathcal{E}$,

(Q-2) $q_{\boldsymbol{e}} = 0$ except for a finite number of edges in $\mathcal{E}$,

(Q-3) $q_{\boldsymbol{e}}$ is symmetric, i.e., $q_{\boldsymbol{e}}(1-x) = q_{\boldsymbol{e}}(x)$ for every $\boldsymbol{e} \in \mathcal{E}$.

Then the Kirhihoff Schrödinger operator $H_{\mathcal{E}}$ is defined as follows:

$$H_{\mathcal{E}} u = \{-u''_{\boldsymbol{e}} + q_{\boldsymbol{e}} u_{\boldsymbol{e}}\}_{\boldsymbol{e}\in\mathcal{E}} \quad \text{with } \mathrm{Dom}(H_{\mathcal{E}}) = \mathrm{Dom}(H_{\mathcal{E}}^{(0)}),$$

where $q = \{q_{\boldsymbol{e}}\}_{\boldsymbol{e}\in\mathcal{E}}$ satisfies (Q-1), (Q-2) and (Q-3). One can easily see that $H_{\mathcal{E}}$ is self-adjoint on $L^2(\mathcal{E})$; moreover, $\sigma_{ac}(H_{\mathcal{E}}) = \sigma_{ac}(H_{\mathcal{E}}^{(0)})$ (see [3]).

2 Schrödinger Operators on the Hexagonal Metric Graph

2.1 Hexagonal Lattice

Fron now on, we consider the hexagonal lattice $\Gamma = (\mathcal{V}, \mathcal{E})$ which is a two dimensional periodic graph. Let us define $\mathcal{V}$ and $\mathcal{E}$ as follows.

Take points $p^{(1)} = (1,0)$ and $p^{(2)} = (2,0)$, and the vectors $\boldsymbol{v}_1 = (3/2, -\sqrt{3}/2)$ and $\boldsymbol{v}_2 = (3/2, \sqrt{3}/2)$. Put $\boldsymbol{v}(n) = n_1\boldsymbol{v}_1 + n_2\boldsymbol{v}_2$, where $n = (n_1, n_2) \in \mathbb{Z}^2$. Then the vertex set $\mathcal{V}$ of the hexagonal lattice is defined by

$$\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2, \quad \mathcal{V}_j = \left\{p^{(j)} + \boldsymbol{v}(n);\, n \in \mathbb{Z}^2\right\},\ j = 1, 2.$$

Take the edges $\boldsymbol{e}_j = (\boldsymbol{e}_j(0), \boldsymbol{e}_j(1))$, $j = 1, 2, 3$, as follows:

$$\boldsymbol{e}_1 = (p^{(2)}, (5/2, -\sqrt{e}/2)),\ \boldsymbol{e}_2 = (p^{(2)}, (5/2, \sqrt{3}/2)),\ \boldsymbol{e}_3 = (p^1, p^{(2)}).$$

Put $\boldsymbol{e}_j + [n] = (\boldsymbol{e}_j(0) + \boldsymbol{v}(n), \boldsymbol{e}_j(1) + \boldsymbol{v}(n))$, $j = 1, 2, 3$. The edge set $\mathcal{E}$ of the hexagonal lattice is defined by

$$\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3, \quad \mathcal{E}_j = \left\{\boldsymbol{e}_j + [n];\, n \in \mathbb{Z}^2\right\},\ j = 1, 2, 3.$$

The fundamental domain $\boldsymbol{F}_\Gamma$ of the hexagonal lattice is defined by

$$\boldsymbol{F}_\Gamma = \left\{y_1\boldsymbol{v}_1 + y_2\boldsymbol{v}_2;\, y_1, y_2 \in [0,1)\right\},$$

which is shown as the gray area in Fig.1 of the hexagonal lattice.

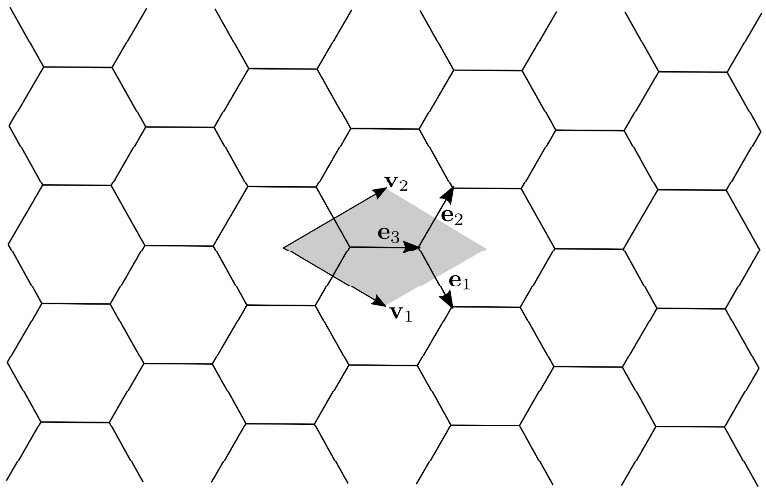

Figure 1: The hexagonal lattice and its fundamental domain

2.2 Fourier Transform

Denote the flat torus by $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$. We define the fundamental graph $\Gamma_* = (\mathcal{V}_*, \mathcal{E}_*)$ of the hexagonal lattice Γ, whose vertex set is $\mathcal{V}_* = \left\{p_*^{(1)}, p_*^{(2)}\right\}$ and the edge set is $\mathcal{E}_* = \{\boldsymbol{e}_{1*}, \boldsymbol{e}_{2*}, \boldsymbol{e}_{3*}\}$, see Fig.2.

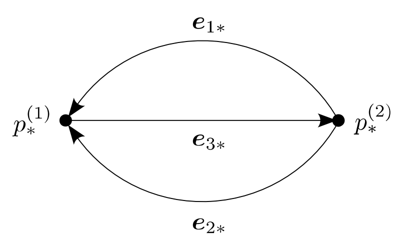

Figure 2: The fundamental graph of the hexagonal lattice

We define the unitary operator $\mathcal{U}_\mathcal{E} = (\mathcal{U}_{\mathcal{E},1}, \mathcal{U}_{\mathcal{E},2}, \mathcal{U}_{\mathcal{E},3}) : L^2(\mathcal{E}) \to L^2(\mathbb{T}^2 \times \mathcal{E}_*)$ as follows: for $f = \{f_{\boldsymbol{e}}\}_{\boldsymbol{e}\in\mathcal{E}} = \cup_{j=1}^3 \left\{f_{\boldsymbol{e}_j+[n]}\right\}_{n\in\mathbb{Z}^2}$,

$$(U_{\mathcal{E},j} f)(\xi, x) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}^2} e^{-in\cdot\xi} f_{\boldsymbol{e}_j+[n]}(x), \quad (\xi, x) \in \mathbb{T}^2 \times [0,1].$$

The adjoint operator of $\mathcal{U}_{\mathcal{E},j}^*$ is

$$\left(U_{\mathcal{E},j}^* g\right)_{\boldsymbol{e}_j+[n]}(x) = \frac{1}{2\pi} \int_{\mathbb{T}^2} e^{in\cdot\xi} g(\xi, x)\, d\xi$$

for the function g on $\mathbb{T}^2 \times [0,1]$.

Let us define the index of the edges as follows:

$$\mathrm{Ind}(\boldsymbol{e}) = \begin{cases} (1,0), & \text{if } \boldsymbol{e} = \boldsymbol{e}_{1*}, \\ (0,1), & \text{if } \boldsymbol{e} = \boldsymbol{e}_{2*}, \\ (0,0), & \text{if } \boldsymbol{e} = \boldsymbol{e}_{3*}. \end{cases}$$

We also define

$$\delta_j(v) = \begin{cases} 1, & \text{if } v = \boldsymbol{e}_{j*}(1), \\ 0, & \text{if } v = \boldsymbol{e}_{j*}(0). \end{cases}$$

Put $\hat{H}^{(0)}_{\mathcal{E}_*} = \mathcal{U}_{\mathcal{E}} H^{(0)}_{\mathcal{E}} \mathcal{U}^*_{\mathcal{E}}$ with domain $\mathrm{Dom}(\hat{H}^{(0)}_{\mathcal{E}_*}) = \mathcal{U}_{\mathcal{E}}\, \mathrm{Dom}(H^{(0)}_{\mathcal{E}})$ on $L^2(\mathbb{T}^2 \times \mathcal{E}_*)$. By the Floquet-Bloch theory, $\hat{H}^{(0)}_{\mathcal{E}_*}$ has the direct integral decomposition (cf. [3]):

$$\hat{H}^{(0)}_{\mathcal{E}_*} = \int_{\mathbb{T}^2}^{\oplus} \hat{H}^{(0)}_{\mathcal{E}_*}(\xi)\, d\xi,$$

where for $\hat{u} = \mathcal{U}_{\mathcal{E}} u = \left\{\hat{u}_{\boldsymbol{e}_{j*}}(\xi, x)\right\}_{j=1}^{3}$, $(\xi, x) \in \mathbb{T}^2 \times [0,1]$,

$$\left(\hat{H}^{(0)}_{\mathcal{E}_*}\hat{u}\right)_{\boldsymbol{e_{j*}}}(\xi, x) = -\hat{u}''_{\boldsymbol{e}_{j*}}(\xi, x), \quad \hat{u}''_{\boldsymbol{e_{j*}}} = \frac{\partial^2 \hat{u}_{\boldsymbol{e}_{j*}}}{\partial x^2}(\xi, x).$$

For $\hat{u} = \left\{\hat{u}_{\boldsymbol{e}_{j*}}(\xi, x)\right\}_{j=1}^{3}$, the Kirchhoff condition (K-1) and (K-2) are transfered as

$$e^{-i\delta_j v\, \mathrm{Ind}(\boldsymbol{e}_{j*})\cdot\xi}\hat{u}_{\boldsymbol{e}_{j*}}(\xi, \delta_J(v)) = e^{-i\delta_k(v)\, \mathrm{Ind}(\boldsymbol{e}_{k*})\cdot\xi}\hat{u}_{\boldsymbol{e}_{k*}}(\xi, \delta_k(v)),$$

for a common end point $v \in \mathcal{V}_*$ of $\boldsymbol{e}_{j*}$ and $\boldsymbol{e}_{k*}$, and

$$\sum_{j=1}^{3} (-1)^{\delta_j(v)}\, e^{-i\delta_j(v)\, \mathrm{Ind}(\boldsymbol{e_{j*}})\cdot\xi}\hat{u}'_{\boldsymbol{e}_{j*}}(\xi, \delta_j(v)) = 0, \quad \hat{u}'_{\boldsymbol{e}_{j*}}(\xi, x) = \frac{\partial \hat{u}_{\boldsymbol{e}_{j*}}}{\partial x}(x, x),$$

for $v \in \mathcal{V}_*$.

2.3 Spectrum of Metric Laplacian

Let us define the twisted discrete Laplacian $\Delta_{\mathcal{V}_*}(\xi)$ on $\mathcal{V}_*$ as follows:

$$\left(\Delta_{\mathcal{V}_*}(\xi)\psi\right)(v) = \frac{1}{3}\sum_{(v,v')\in\mathcal{E}_*} e^{-i(-1)^{\delta_j(v')}\, \mathrm{Ind}(\boldsymbol{e}_{j*})\cdot\xi}\psi(v'), \quad \psi \in \ell^2(\mathcal{V}_*).$$

The eigenvalue problem of $\hat{H}^{(0)}_{\mathcal{E}_*}$ is closely related to that of $-\Delta_{\mathcal{V}_*}(\xi)$ on $\ell^2(\mathcal{V}_*)$. The equation $-\Delta_{\mathcal{V}_*}(\xi)\psi = \mu(\xi)\psi$ holds if and only if

$$\begin{bmatrix} -\mu(\xi) & -p(\xi) \\ -\overline{p(\xi)} & -\mu(\xi) \end{bmatrix} \begin{bmatrix} \psi(p^{(1)}_*) \\ \psi(p^{(2)}_*) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where for $\xi = (\xi_1, \xi_2) \in \mathbb{T}^2$,

$$p(\xi) = \frac{1}{3}\left(1 + e^{-i\xi_1} + e^{i\xi_2}\right).$$

Then we have

$$\sigma(-\Delta_{\mathcal{V}_*}(\xi)) = \{\pm |p(\xi)|\}.$$

Here we note that

$$|p(\xi)| = \frac{1}{3}\sqrt{3 + 2\cos\xi_1 + 2\cos\xi_2 + 2\cos(\xi_1 - \xi_2)} \in [0,1].$$

Put $\mu_1(\xi) = -|p(\xi)|$ and $\mu_2(\xi) = |p(\xi)|$. The corresponding normalized eigenfunctions are

$$a_k(\xi) = \frac{1}{\sqrt{2}\,|p(\xi)|}\begin{bmatrix} (-1)^{k-1} p(\xi) \\ |p(\xi)| \end{bmatrix}, \quad k = 1,2.$$

The spectrum of $H_{\mathcal{E}_*^{(0)}}(\xi)$ is well-known (cf. [3], [4]). Define

$$z_{j,k}(\xi) = \begin{cases} z_k(\xi) + \pi j, & \text{if } j \text{ is even,} \\ (\pi - z_k(\xi)) + \pi j, & \text{if } j \text{ is odd,} \end{cases}$$

where

$$z_k(\xi) = \arccos(-\mu_k(\xi)) \in [0,\pi], \quad k = 1,2.$$

Then, we have

$$\hat{H}^{(0)}_{\mathcal{E}_*}(\xi) = \hat{H}^D_{\mathcal{E}_*}(\xi) \oplus \hat{H}^V_{\mathcal{E}_*}(\xi),$$

$$\hat{H}^D_{\mathcal{E}_*}(\xi) = \sum_{j\geq 0} (\pi j)^2\, \hat{P}^D_J(\xi), \quad \hat{H}^V_{\mathcal{E}_*}(\xi) = \sum_{j\geq 0,\ k\in\{1,2\}} z_{j,k}(\xi)^2 \hat{P}^V_{j,k}(\xi),$$

where $\hat{P}^D_{j,k}(\xi)$ and $\hat{P}^V_{j,k}(\xi)$ are projections to eigenvalues $(\pi j)^2$ and $z_{j,k}(\xi)^2$. Therefore, we have

$$\sigma_{\mathrm{ac}}(H^{(0)}_{\mathcal{E}}) = \sigma_{\mathrm{ac}}(H_{\mathcal{E}}) = \cup_{j\geq 0, k\in\{1,2\}}[\lambda^-_{j,k}, \lambda^+_{j,k}] = \cup[0,\infty),$$

where

$$\lambda^{\pm}_{j,k} = \begin{cases} \left(z^{\pm}_k + \pi j\right)^2, & \text{if } j \text{ is even,} \\ \left(\pi - z^{\mp}_k + \pi j\right)^2, & \text{if } j \text{ is odd.} \end{cases}$$

Moreover, we have the eigenvalues of infinite multiplicities:

$$\sigma_{\mathrm{fb}}(H^{(0)}_{\mathcal{E}}) = \left\{(\pi j)^2;\, j = 1,2,\ldots\right\}.$$

The spectrum of $H^{(0)}_{\mathcal{E}}$ is shown in Fig.3.

For $\lambda^-_{j,k} < \lambda < \lambda^+_{j,k}$, it holds that $0 < \left(\cos\sqrt{\lambda}\right)^2 < 1$. We define the Fermi surface

$$M(\cos\sqrt{\lambda}) = \left\{\xi \in \mathbb{T}^2;\, |p(\xi)|^2 = \left(\cos\sqrt{\lambda}\right)^2\right\},$$

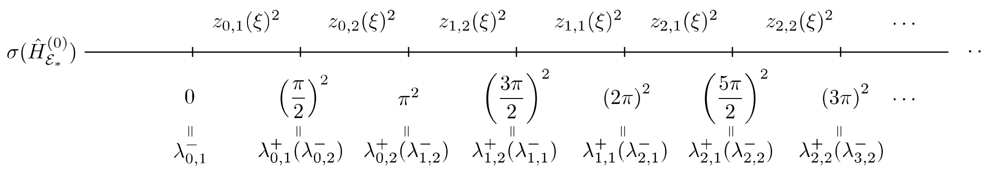

Figure 3: The spectrum of $H^{(0)}_{\mathcal{E}}$.

and

$$M_k(\cos\sqrt{\lambda}) = \left\{\xi \in \mathbb{T}^2;\ \mu_k(\xi) = \cos\sqrt{\lambda}\right\}, \quad k = 1, 2.$$

Let $\mathcal{T}^{(0)} = \sigma_{\rm fb}(H^{(0)}_{\mathcal{E}}) \cup \left\{\lambda > 0;\ \cos\sqrt{\lambda} = 0, \pm 1/3\right\}$. Then,

$$\nabla_\xi \left(|p(\xi)|\right) = \frac{1}{9}\begin{bmatrix} -\sin\xi_1 - \sin(\xi_1 - \xi_2) \\ -\sin\xi_2 + \sin(\xi_1 - \xi_2)\end{bmatrix} \neq 0$$

for $\xi = (\xi_1, \xi_2) \in M(\cos\sqrt{\lambda})$. Therefore, for $\lambda \in \left[\lambda^-_{j,k}, \lambda^+_{j,k}\right] \setminus \mathcal{T}^{(0)}$, $M(\cos\sqrt{\lambda})$ is one dimensional real analytic submanifold of $\mathbb{T}^2$. See Fig.4.

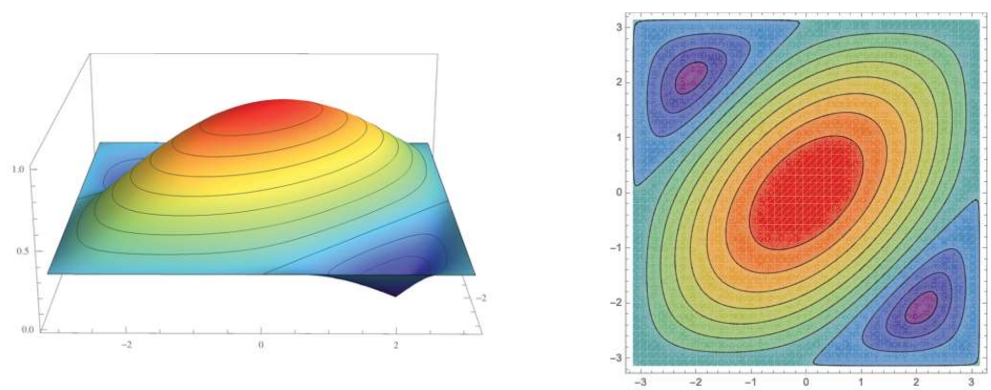

Figure 4: The Fermi surface $M(\cos\sqrt{\lambda})$ is the level set of $|p(\xi)|^2$.

3 Scattering Theory

We define the weighted L^2-norm on $\mathcal{E}$ for $f = \cup_{j=1}^3 \left\{f_{\boldsymbol{e}_j+[n]}\right\}_{n\in\mathbb{Z}^2}$:

$$\|f\|^2_{L^{2,s}(\mathcal{E})} = \sum_{l=1}^{3}\sum_{n\in\mathbb{Z}^2}(1+|n|)^s \left\|f_{\boldsymbol{e}_l+[n]}\right\|^2_{L^2(0,1)}, \quad s \in \mathbb{R}.$$

Then the weighted L^2-space is

$$L^{2,s}(\mathcal{E}) = \left\{f = \cup_{j=1}^3 \left\{f_{\boldsymbol{e}_j+[n]}\right\}_{n\in\mathbb{Z}^2};\ \|f\|^2_{L^{2,s}(\mathcal{E})} < \infty\right\}.$$

We also define the Agmon-Hörmander spaces $\mathcal{B}(\mathcal{E})$ and $\mathcal{B}^*(\mathcal{E})$ by the following norms:

$$\|f\|_{\mathcal{B}(\mathcal{E})} = \sum_{l=1}^{3}\sum_{j=0}^{\infty} r_j^{1/2}\left(\sum_{r_{j-1}\le|n|<r_j} \|f_{\boldsymbol{e}_{l+[n]}}\|^2_{L^2(0,1)}\right)^{1/2}, \quad r_{-1}=0,\ r_j=2^j,\ j\ge 1,$$

$$\|u\|^2_{\mathcal{B}^*(\mathcal{E})} = \sup_{\rho>1}\frac{1}{\rho}\sum_{l=1}^{3}\sum_{|n|<\rho}\|u_{\boldsymbol{e}_{l+[n]}}\|^2_{L^2(0,1)}.$$

We note that for $s>1/2$,

$$L^{2,s}(\mathcal{E})\subset\mathcal{B}(\mathcal{E})\subset L^{2,1/2}(\mathcal{E})\subset L^2(\mathcal{E})\subset L^{2,-1/2}(\mathcal{E})\subset\mathcal{B}^*(\mathcal{E})\subset L^{2,-s}(\mathcal{E}).$$

We define $\mathcal{B}_0^*(\mathcal{E})$ by the totality of $u\in\mathcal{B}^*(\mathcal{E})$ such that

$$\lim_{\rho\to\infty}\frac{1}{\rho}\sum_{l=1}^{3}\sum_{|n|<\rho}\|u_{\boldsymbol{e}_{l+[n]}}\|^2_{L^2(0,1)}=0.$$

The correspoondig function spaces on $\mathbb{T}^2\times\mathcal{E}^*$ is defined by passing them to $\mathcal{U}_{\mathcal{E}}$.

Let $R^{(0)}_{\mathcal{E}}(\kappa) = \left(H^{(0)}_{\mathcal{E}}-\kappa\right)^{-1}$ for $\kappa\in\mathbb{C}\setminus[0,\infty)$. Then the weak* limits $R^{(0)}_{\mathcal{E}}(\lambda\pm i0)=\lim_{\epsilon\downarrow 0}R^{(0)}_{\mathcal{E}}(\lambda\pm i\epsilon)$ exists in $\boldsymbol{B}(\mathcal{B}(\mathcal{E}),\mathcal{B}^*(\mathcal{E}))$ for $\lambda\in(0,\infty)\setminus\mathcal{T}^{(0)}$. $u^{(0)}_{\pm}=R^{(0)}_{\mathcal{E}}(\lambda\pm i0)f$ is said to be outgoing $(+)$ and incoming $(-)$ in view of the singularity expansion on the Fermi surface (cf. [1]).

Let $h_{\boldsymbol{e}}=-d^2/dx^2+q_{\boldsymbol{e}}$ with the Dirichlet boundary condition at $\boldsymbol{e}(0)$ and $\boldsymbol{e}(1)$ for $e\in\mathcal{E}$. Put $\mathcal{T}=\mathcal{T}^{(0)}\cup(\cup_{\boldsymbol{e}\in\mathcal{E}}\sigma(h_{\boldsymbol{e}}))$.

Let $R_{\mathcal{E}}(\kappa)=\left(H_{\mathcal{E}}-\kappa\right)^{-1}$ for $\kappa\in\mathbb{C}\setminus\boldsymbol{R}$. By the resolvent equation, we have $R_{\mathcal{E}}(\kappa)=R^{(0)}_{\mathcal{E}}(\kappa)\left(1-qR^{(0)}_{\mathcal{E}}(\kappa)\right)$. We put $Q_{\mathcal{E}}(\kappa)=1-qR_{\mathcal{E}}(\kappa)$. Then the limiting absorption principle holds, i.e., $R_{\mathcal{E}}(\lambda\pm i0)\in\boldsymbol{B}(\mathcal{B}(\mathcal{E}),\mathcal{B}^*(\mathcal{E}))$ for $\lambda\in(0,\infty)\setminus\mathcal{T}$. We also call $u_{\pm}=R^{(0)}_{\mathcal{E}}(\lambda\pm i0)f$ outgoing $(+)$ and incoming $(-)$.

Then we can prove the following theorem. For the proof, see [2].

Theorem 3.1 (Rellich Type Theorem) *Let $\lambda\in(0,\infty)\setminus\mathcal{T}$. Suppose that $u\in\mathcal{B}(\mathcal{E})$ satisfies $(H_{\mathcal{E}}-\lambda)\,u=0$ and u is outgoing (or incomming). Then $u=0$.*

3.1 Scattering Amplitude

We define the Hilbert spaces $\boldsymbol{h}_k(\lambda)$, $k=1,2$:

$$\boldsymbol{h}_k=\begin{cases} L^2(M_k(\cos\sqrt{\lambda})\times\mathcal{E}_*), & \text{if } \lambda\in(\lambda^-_{j,k},\lambda^+_{j,k}),\ j\ge 0,\\ \{0\}, & \text{otherwise.}\end{cases}$$

We put $\boldsymbol{h}(\lambda)=\boldsymbol{h}_1(\lambda)a_1(\xi)\oplus\boldsymbol{h}_2(\lambda)a_2(\xi)$, where $a_k(\xi)$, $k=1,2$, are the normalized eigenfuncsions of the twisted Laplacian $-\Delta_{\mathcal{V}_*}(\xi)$. The distorted Fourier transformation $\mathcal{F}^{(0)}(\lambda) = \big(\mathcal{F}_1^{(0)}(\lambda), \mathcal{F}_2^{(0)}(\lambda)\big) \in \boldsymbol{B}(\mathcal{B}(\mathcal{E}), \boldsymbol{h}(\lambda))$ associated with $H_{\mathcal{E}}^{(0)}$ is defined by

$$\mathcal{F}_k^{(0)} f = \begin{cases} \hat{P}_{j,k}^{V}(\lambda)\hat{f}\big|_{M_k(\cos\sqrt{\lambda})}, & \text{if } \lambda \in (\lambda_{j,k}^{-}, \lambda_{j,k}^{+}),\ j \geq 0, \\ \{0\}\,, & \text{otherwise.} \end{cases}$$

We note that $\mathcal{F}^{(0)}(\lambda)^* \in \boldsymbol{B}(\boldsymbol{h}(\lambda), \mathcal{B}^*(\mathcal{E}))$. Then we have the following lemma. For the proof, see [2].

Lemma 3.1 *For $\lambda \in (0,\infty) \setminus \mathcal{T}^{(0)}$, we have*

- *$\mathcal{F}^{(0)}(\lambda)$ diagonalizes $H_{\mathcal{E}}^{(0)}$, i.e., $\mathcal{F}^{(0)}(\lambda)\big(H_{\mathcal{E}}^{(0)} - \lambda\big) = 0$,*
- *$\mathcal{F}^{(0)}(\lambda)^*\phi$ for any $\phi \in \boldsymbol{h}(\lambda)$ is a generalized eigenfunction of $H_{\mathcal{E}}^{(0)}$.*

The distorted Fourier transformation $\mathcal{F}^{(\pm)}(\lambda) = \big(\mathcal{F}_1^{(\pm)}(\lambda), \mathcal{F}_2^{(\pm)}(\lambda)\big)$ associated with $H_{\mathcal{E}}$ is defined by

$$\mathcal{F}_k^{(\pm)}(\lambda) = \mathcal{F}_k^{(0)}(\lambda) Q_{\mathcal{E}}(\lambda \pm i0), \quad k = 1, 2.$$

Then we have the following lemma. For the proof, see [2].

Lemma 3.2 *For $\lambda \in (0,\infty) \setminus \mathcal{T}$, we have*

- *$\mathcal{F}^{(\pm)}(\lambda)$ diagonalizes $H_{\mathcal{E}}^{(0)}$, i.e., $\mathcal{F}^{(\pm)}(\lambda)\big(H_{\mathcal{E}} - \lambda\big) = 0$,*
- *$\mathcal{F}^{(\pm)}(\lambda)^*\phi$ for any $\phi \in \boldsymbol{h}(\lambda)$ is a generalized eigenfunction of $H_{\mathcal{E}}$ in $\mathcal{B}^*(\mathcal{E})$.*

Define the scattering amplitude $A(\lambda)$ by

$$A(\lambda) = \mathcal{F}^{(+)} q \mathcal{F}^{(0)}(\lambda)^*, \quad \lambda \in (0,\infty) \setminus \mathcal{T}.$$

Let $P_{\mathrm{ac}}(H_{\mathcal{E}}^{(0)})$ be the projection on the absolutely continuous space of $H_{\mathcal{E}}^{(0)}$. It is well-known that the scattering matrix $S(\lambda)$ associated with the wave operator

$$W_{\pm} = \operatorname*{s-lim}_{t \to \pm\infty} e^{itH_{\mathcal{E}}} e^{-itH_{\mathcal{E}}^{(0)}} P_{\mathrm{ac}}(H_{\mathcal{E}}^{(0)})$$

is written by $S(\lambda) = 1 - 2\pi i A(\lambda)$.

4 Boundary Value Problem

We take the bounded domain $\Omega^i = \big(\overline{\mathcal{V}^i}, \mathcal{E}^i\big)$, $\overline{\mathcal{V}^i} = \mathcal{V}^i \cup \partial\mathcal{V}^i$, as Fig.5:

- $\mathcal{V}^i$ is "the internal points" of Ω^i,
- $\partial\mathcal{V}^i$ is "the boundary points" of Ω^i

Here we impose the additional assumption on the potential $q = \{q_{\boldsymbol{e}}\}_{\boldsymbol{e}\in\mathcal{E}}$:

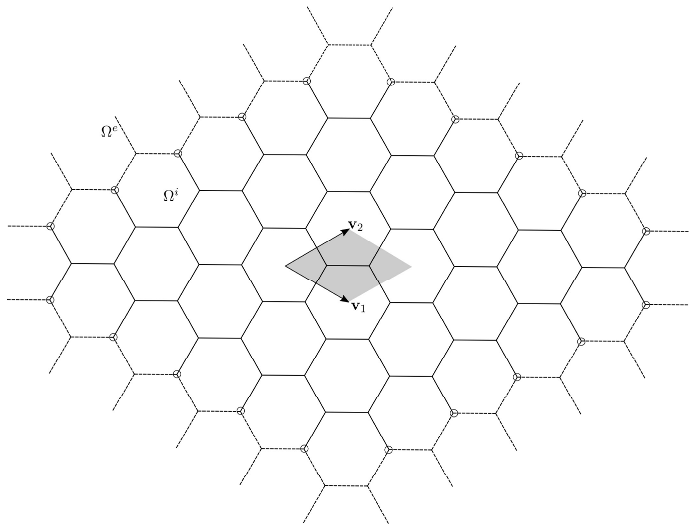

Figure 5: Bounded Domain in the Hexagonal Lattice.

(Q-4) $\mathrm{supp}(q) \subset \{\boldsymbol{e} \subset \mathcal{E}^i;\, \boldsymbol{e}(0), \boldsymbol{e}(1) \in \mathcal{V}^i\}$.

Let us consider the Dirichlet boundary problem on $\mathcal{E}^i$. We define the differential operator $-\Delta^i_{\mathcal{E}} + q$ on Ω^i by

$$\left(-\Delta^i_{\mathcal{E}} + q\right) u = \{-u''_{\boldsymbol{e}} + q_{\boldsymbol{e}} u_{\boldsymbol{e}}\}_{\boldsymbol{e} \in \mathcal{E}^i},$$

with

$$\mathrm{Dom}(-\Delta^i_{\mathcal{E}} + q) = \left\{u \in H^2(\mathcal{E}^i);\, u \text{ satisfies (K-1) and (K-2)},\ u\big|_{\partial\mathcal{V}^i} = 0\right\}.$$

Let $\sigma_{\mathrm{D}}(-d^2/dx^2 + q_{\boldsymbol{e}})$ be the Dirichlet eigenvalues of $-d^2/dx^2 + q_{\boldsymbol{e}}$ on $\boldsymbol{e}$. We put

$$\mathcal{T} = \cup_{\boldsymbol{e} \in \mathcal{E}^i} \sigma_D(-d^2/dx^2 + q_{\boldsymbol{e}}).$$

We note that $\mathcal{T}$ is a discrete set. We also note that there may exist Dirichlet eigenfunctions of $H_{\mathcal{E}} = -\Delta^i_{\mathcal{E}} + q$ supported on cycles of edges (ex. the eigenfunctions associated with flat band spectrum $\sigma_{\mathrm{fb}}(H^{(0)}_{\mathcal{E}})$). These Dirichlet eigenvalues of $H_{\mathcal{E}}$ are embedded in the continuous spectrum $\sigma_{\mathrm{cont}}(H_{\mathcal{E}}) = [0, \infty)$.

For $\lambda \in \mathbb{C} \setminus \sigma_{\mathrm{D}}(-\Delta^i_{\mathcal{E}} + q)$ and $f \in \ell^2(\partial\mathcal{V}^i)$, there exists a unique solution $u = \{u_{\boldsymbol{e}}\}_{\boldsymbol{e} \in \mathcal{E}^i}$ of the following boundary value problem:

$$\left(-\Delta^i_{\mathcal{E}} + q\right) u = 0 \text{ in } \mathcal{E}^i, \quad u = f \text{ on } \partial\mathcal{V}^i.$$

We define the Dirichlet-Neumann (D-N) map $\Lambda^i_{\mathcal{E}}(\lambda)$ as follows: for $f \in \ell^2(\partial\mathcal{V}^i)$,

$$\left(\Lambda^i_{\mathcal{E}}(\lambda) f\right)(p) = -\partial_{\boldsymbol{e}} u(p), \quad p \in \partial\mathcal{V}^i,\ \boldsymbol{e} \in \mathcal{E}_p \cap \mathcal{E}^i,$$

where u is the solution of the above boundary value problem.

Taking into account of the Kirchhoff conditions at $v \in \mathcal{V}^i$, one can obtain the Green's formula by integration by parts:

$$\sum_{e \in \mathcal{E}^i} \int_0^1 \left(\left(\Delta_{\mathcal{E}}^i u\right)_{e} \cdot \overline{u_{e}} - u_{e} \cdot \overline{\left(\Delta_{\mathcal{E}}^i u\right)_{e}} \right) dx$$
$$= \sum_{p \in \partial \mathcal{V}^i} \left(\left(\Lambda_{\mathcal{E}}^i(\lambda) f\right) \cdot \overline{f(p)} - f(p) \cdot \overline{\left(\Lambda_{\mathcal{E}}^i(\lambda) f\right)(p)} \right).$$

We remark that $\Lambda_{\mathcal{E}}^i(\lambda) f$ is regarded as the outer normal derivative in this sense.

By using the D-N map $\Lambda_{\mathcal{E}}^i(\lambda)$, one can construct the outgoing solution u to $(H_{\mathcal{E}} - \lambda) u = 0$. If the scattering amplitudes $A^{q_1}(\lambda)$ and $A^{q_2}(\lambda)$ for q_1 and q_2 coincide, the outgoing solutions satisfiy $u^{q_1} - u^{q_2} \in \mathcal{B}_0^*(\mathcal{E})$. By using the Rellich type uniequeness theorem, we have $u^{q_1} = u^{q_2}$. Then the D-N map for q_1 and q_2 coincide. Therefore we have

Lemma 4.1 *Suppose that q_1 and q_2 satisfies (Q-1) - (Q-4). Then $A^{q_1} = A^{q_2}$ if and only if $\Lambda_{\mathcal{E}}^{q_1,i}(\lambda) = \Lambda_{\mathcal{E}}^{q_2,i}(\lambda)$.*

For the proof of the lemma, see [2].

Then we have the following main theorem.

Theorem 4.1 *Suppose that we know the scattering amplitude $A(\lambda)$ associated with the Schrödinger operator $H_{\mathcal{E}}$ for $\lambda \in I$, where I is an open interval in $(0, \infty) \setminus \mathcal{T}$. Then we can determine the potential $q = \{q_{e}\}_{e \in \mathcal{E}}$ uniquely.*

In view of the Lemma 4.1, we only have to derive a reconstruction from $\Lambda_{\mathcal{E}}^i(\lambda)$ to determine $q = \{q_{e}\}_{e \in \mathcal{E}^i}$.

Inverse Boundary Value Problem: Given $\Lambda_{\mathcal{E}}^i(\lambda)$ for $\lambda \in \mathbb{C} \setminus \sigma_{\mathrm{D}}(-\Delta_{\mathcal{E}}^i + q)$, can we determine the potential $q = \{q_{e}\}_{\mathcal{E}^i}$?

4.1 Inverse Spectral Problem on Finite Interval

Let us consider the Dirichlet problem on $[0, 1]$:

$$-y'' + qy = \kappa y \text{ in } (0,1), \quad y(0) = y(1) = 0. \tag{2}$$

Let $\kappa(q) = \{\kappa_1, \kappa_2, \dots\}$ be all the Dirichlet eigenvalues of (2). If q is real-valued and symmetric, then $\kappa(q)$ and q is one-to-one by the Borg's theorem (cf. [5]).

Theorem 4.2 *(Borg's theorem) Let $E = \left\{q \in L^2(0,1);\, q(1-x) = q(x)\right\}$. Then $E : q \mapsto \kappa(q)$ is one-to-one.*

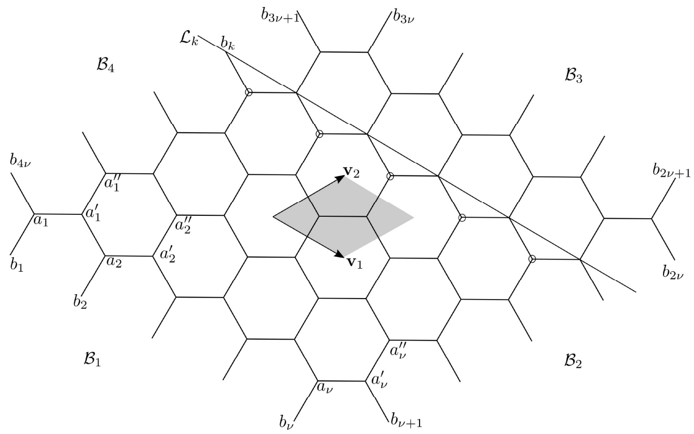

Figure 6: Labeling of the boundary $\partial\mathcal{V}^i = \cup_{j=1}^4 \mathcal{B}_j$

4.2 Special Solution ofo Cauchy Problem on Ω

Split the boundary of Ω^i into parts $\partial\mathcal{V}^i = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$, where

$$\mathcal{B}_1 = \{b_1, \ldots, b_\nu\}, \quad \mathcal{B}_2 = \{b_{\nu+1}, \ldots, b_{2\nu}\},$$

$$\mathcal{B}_3 = \{b_{2\nu+1}, \ldots, b_{3\nu}\}, \quad \mathcal{B}_2 = \{b_{3\nu+1}, \ldots, b_{4\nu}\}.$$

See Fig.6. We can construct a special solution of the Cauchy problem by the D-N map, which grows to a paticular direction.

We identify $f \in \ell^2(\partial\mathcal{V}^i)$ with $f = [f(b_1), \ldots, f(b_{4\nu})]^T \in \mathbb{C}^{4\nu}$. So we can regard the D-N map $\Lambda^i_{\mathcal{E}}(\lambda)$ as $4\nu\times 4\nu$ matrix. We write $f(\mathcal{B}_j) = \left[f(b_{(j-1)\nu+1}), \ldots, f(b_{j\nu})\right]^T$. Let $\Lambda(\lambda; \mathcal{B}_j, \mathcal{B}_k)$ be a $\nu\times\nu$ submatrix of the D-N map such that $\Lambda(\lambda; \mathcal{B}_j, \mathcal{B}_k) f(\mathcal{B}_k) \in \ell^2(\mathcal{B}_j)$. Then we split $\Lambda^i_{\mathcal{E}}(\lambda)$ into block matrix expression:

$$\Lambda^i_{\mathcal{E}}(\lambda) = \begin{bmatrix} \Lambda(\lambda;\mathcal{B}_1,\mathcal{B}_1) & \Lambda(\lambda;\mathcal{B}_1,\mathcal{B}_2) & \cdots & \Lambda(\lambda;\mathcal{B}_1,\mathcal{B}_4) \\ \Lambda(\lambda;\mathcal{B}_2,\mathcal{B}_1) & \Lambda(\lambda;\mathcal{B}_2,\mathcal{B}_2) & \cdots & \Lambda(\lambda;\mathcal{B}_2,\mathcal{B}_4) \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda(\lambda;\mathcal{B}_4,\mathcal{B}_1) & \Lambda(\lambda;\mathcal{B}_4,\mathcal{B}_2) & \cdots & \Lambda(\lambda;\mathcal{B}_4,\mathcal{B}_4) \end{bmatrix}.$$

Suppose $f(\mathcal{B}_j) = 0$, $j = 1, 2, 4$ and $\left(\Lambda^i_{\mathcal{E}}(\lambda) f\right)(\mathcal{B}_1) = 0$. Then we can construct the solusions on all the edges $\boldsymbol{e} \in \mathcal{E}$ uniquely as $u_{\boldsymbol{e}} = 0$ from $\mathcal{B}_1$ to $\mathcal{B}_3$. See Fig.7. We note that $\lambda \notin \cup_{\boldsymbol{e}\in\mathcal{E}^i} \sigma_{\mathrm{D}}(-d^2/dx^2 + q_{\boldsymbol{e}})$. Thus we proved the following lemma. For the details, see [2].

Lemma 4.2 *If* $\Lambda \in \mathcal{C} \setminus \left(\sigma_D(-\Delta^i_{\mathcal{E}} + q) \cup \mathcal{T}\right)$*, then the matrices* $\Lambda(\lambda; \mathcal{B}_1, \mathcal{B}_3)$ *and* $\Lambda(\lambda; \mathcal{B}_2, \mathcal{B}_4)$ *are retular.*

As a consequence of the Lemma 4.2, we have the following fact.

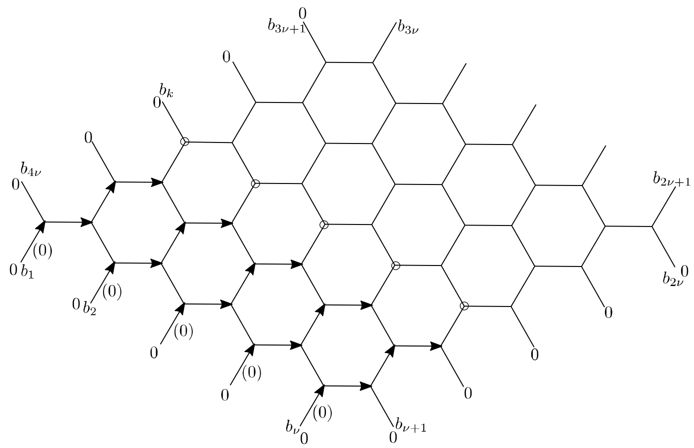

Figure 7: Construction of the solutions for the Cauchy problem.

Corollary 4.1 *Let $\lambda \in \mathbb{C} \setminus \left(\sigma_D(-\Delta^i_{\mathcal{E}} + q) \cup \mathcal{T}\right)$. If $g(\mathcal{B}_1) \in \mathbb{C}^\nu$ and $f(\mathcal{B}_j) \in \mathbb{C}^\nu$, $j = 1, 2, 4$, are given, then there exists a unieque $f(\mathcal{B}_3) \in \mathbb{C}^\nu$ such that*

$$\left(\Lambda^i_{\mathcal{E}}(\lambda) f\right)(\mathcal{B}_1) = g(\mathcal{B}_1).$$

Moreover, we have

$$f(\mathcal{B}_3) = \Lambda(\lambda; \mathcal{B}_1, \mathcal{B}_3)^{-1} \left(g(\mathcal{B}_1) - \sum_{j=1,2,4} \Lambda(\lambda; \mathcal{B}_1, \mathcal{B}_j) f(\mathcal{B}_j) \right). \tag{3}$$

In vew of the corollary, $f(\mathcal{B}_3)$ is determined by the D-N matrix $\Lambda^i_{\mathcal{E}}(\lambda)$ and the given vectors $g(\mathcal{B}_1)$ and $f(\mathcal{B}_j)$, $j = 1, 2, 4$. See Fig.8

In the same way, if $g(\mathcal{B}_2) \in \mathbb{C}^\nu$ and $f(\mathcal{B}_j) \in \mathbb{C}^\nu$, $j = 1, 2, 3$, are given, then there exists a unieque $f(\mathcal{B}_4) \in \mathbb{C}^\nu$ such that

$$f(\mathcal{B}_4) = \Lambda(\lambda; \mathcal{B}_4, \mathcal{B}_2)^{-1} \left(g(\mathcal{B}_2) - \sum_{j=1,2,3} \Lambda(\lambda; \mathcal{B}_2, \mathcal{B}_j) f(\mathcal{B}_j) \right), \tag{4}$$

which means that $f(\mathcal{B}_4)$ is determined by the D-N matrix $\Lambda^i_{\mathcal{E}}(\lambda)$ and the given vectors $g(\mathcal{B}_2)$ and $f(\mathcal{B}_j)$, $j = 1, 2, 3$.

Then we have the following theorem.

Theorem 4.3 *Let $I \subset (0, \infty) \setminus \left(\sigma_D(-\Delta^i_{\mathcal{E}} + q) \cup \mathcal{T}\right)$ be an open interval. If the D-N map $\Lambda^i_{\mathcal{E}}(\lambda)$ is given for any $\lambda \in I$, then the potential $q = \{q_{\boldsymbol{e}}\}_{\boldsymbol{e} \in \mathcal{E}^i}$ is determined uniequely.*

The reconstruction procedure consists of several setps. At Step 1, we put the Cauchy data $f(\mathcal{B}_j)$, $j = 1, 2, 3$, and $g(B_2)$ as in Fig.9. Then, using the formula (4), we get $f(\mathcal{B}_3)$. Moreover, we have the following:

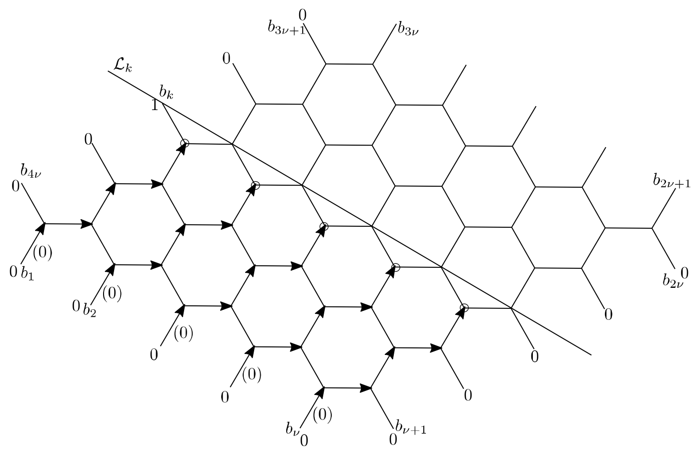

Figure 8: the unieque special solution of the Cauchy problem.

- Since the D-N map is given, we can compute the value $u(\Diamond)$ at $\Diamond \in \mathcal{V}^i$.
- $u(\circ) = 0$ by the special solution of the Cauchy problem.
- Consider $-u_{\boldsymbol{e}}'' + q_{\boldsymbol{e}} u_{\boldsymbol{e}} = \lambda u_{\boldsymbol{e}}$ on the edge $\otimes$. Put $\boldsymbol{e}(0) = \circ$. Then we obtain the fundamental solution $u_{\boldsymbol{e}}(x, \lambda)$, satisfying $u_{\boldsymbol{e}}(0, \lambda) = 0$, $u_{\boldsymbol{e}}'(0, \lambda) = 1$.
- By the analytic continuation of $\Lambda_{\mathcal{E}}^i(\lambda)$ with respect to $\lambda \in \mathbb{C}$, we know all the zeros of $u_{\boldsymbol{e}}(1, \lambda)$ (i.e., all the Dirichlet eigenvalues on $\boldsymbol{e}$).
- By the Borg's theorem, we can determine $q_{\boldsymbol{e}}$ on $\otimes$.

To be more specific, there are two types of edges $\boldsymbol{e}$ of $\otimes 1$ and $\otimes 2$ in Fig.9 whose reconstruction procedures slightely differ.

Step 1 on $\boldsymbol{e}$ of $\otimes 1$:

Let $\boldsymbol{e} = \otimes 1$, and $\boldsymbol{e}(0) = \circ$, $\boldsymbol{e}(1) = \Diamond$. Then $u_{\boldsymbol{e}}$ is the solution of

$$-u_{\boldsymbol{e}}'' + q_{\boldsymbol{e}} u_{\boldsymbol{e}} = \lambda u_{\boldsymbol{e}} \text{ in } (0,1), \quad u_{\boldsymbol{e}}(0) = 0,\ u_{\boldsymbol{e}}'(0) = f_1(\lambda),\ u_{\boldsymbol{e}}(1) = g_1(\lambda),$$

where $f_1(\lambda)$ and $g_1(\lambda)$ are analytic for λ. Put $\psi_{\boldsymbol{e}} = u_{\boldsymbol{e}}/f_1(\lambda)$. Then $\psi_{\boldsymbol{e}}$ is the solution of

$$-\psi_{\boldsymbol{e}}'' + q_{\boldsymbol{e}} \psi_{\boldsymbol{e}} = \lambda \psi_{\boldsymbol{e}} \text{ in } (0,1), \quad \psi_{\boldsymbol{e}}(0) = 0,\ \psi_{\boldsymbol{e}}'(0) = 1,\ \psi_{\boldsymbol{e}}(1) = \frac{g_1(\lambda)}{f_1(\lambda)}.$$

By the analytic continuation of $f_1(\lambda)/g_1(\lambda)$, we know all the zeros of $\psi_{\boldsymbol{e}}(1)$ which are Dirichlet eigenvalues on $\boldsymbol{e}$ of $\otimes 1$. By the Borg's theorem, we can determine $q_{\boldsymbol{e}}$ on $\boldsymbol{e}$ of type $\otimes 1$.

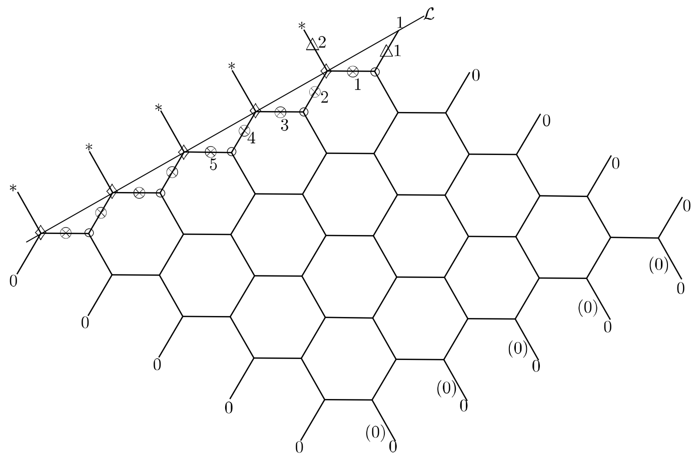

Figure 9: The reconstruction procedure of Step 1.

Step 1 on $\boldsymbol{e}$ of $\otimes 2$:

Let $\boldsymbol{e} = \otimes 1$, and $\boldsymbol{e}(0) = \circ$, $\boldsymbol{e}(1) = \Diamond$. Then $u_{\boldsymbol{e}}$ is the solution of

$$-u_{\boldsymbol{e}}'' + q_{\boldsymbol{e}} u_{\boldsymbol{e}} = \lambda u_{\boldsymbol{e}} \text{ in } (0,1), \quad u_{\boldsymbol{e}}(0) = 0,\ u_{\boldsymbol{e}}(1) = g_2(\lambda),\ u_{\boldsymbol{e}}'(1) = f_2(\lambda),$$

where $f_2(\lambda)$ and $g_2(\lambda)$ are analytic for λ. Put $\psi_{\boldsymbol{e}} = u_{\boldsymbol{e}}/f_2(\lambda)$. Then $\psi_{\boldsymbol{e}}$ is the solution of

$$-\psi_{\boldsymbol{e}}'' + q_{\boldsymbol{e}} \psi_{\boldsymbol{e}} = \lambda \psi_{\boldsymbol{e}} \text{ in } (0,1), \quad \psi_{\boldsymbol{e}}(0) = 0,\ \psi_{\boldsymbol{e}}(1) = \frac{g_2(\lambda)}{f_2(\lambda)},\ \psi_{\boldsymbol{e}}'(1) = 1.$$

By the analytic continuation of $f_2(\lambda)/g_2(\lambda)$, we know all the zeros of $\psi_{\boldsymbol{e}}(1)$ which are Dirichlet eigenvalues on $\boldsymbol{e}$ of $\otimes 2$. By the Borg's theorem, we can determine $q_{\boldsymbol{e}}$ on $\boldsymbol{e}$ of type $\otimes 2$.

We can repeat the above procedures for $\boldsymbol{e}$ of $\otimes$j, $j = 3, 4, \ldots$, right below the line $\mathcal{L}$ in Fig.9.

At Step 2, using the formula (3), we follow the same procedure as Step 1. See Fig.10. Then we can reconstruct the potentials $q_{\boldsymbol{e}}$ for $\boldsymbol{e} = \otimes$ right below the line $\mathcal{L}'$ in Fig.10.

At Step 3, we already know the potential $q_{\boldsymbol{e}}$ for $\boldsymbol{e} = \times$ in Fig.11. Then, using the formula (4) as Step 1, we can reconstruct the potential $q_{\boldsymbol{e}}$ for $\boldsymbol{e} = \otimes$ in Fig.11. Repeating this procedure, we can reconstruct the potential $q_{\boldsymbol{e}}$ for $\boldsymbol{e} = \times$ in Fig.12 right above the line $\mathcal{L}'$ before Step 4.

At Step 4, using the formula (3) as Step 2, we can reconstruct the potential $q_{\boldsymbol{e}}$ for $\boldsymbol{e} = \otimes$ in Fig.12. Repeating this procedure, we can reconstruct the potential $q_{\boldsymbol{e}}$ for $\boldsymbol{e} = \times$ in Fig.13 right above the line $\mathcal{L}$.

So far, we arrive at one size smaller unknown region where the potential $q_{\boldsymbol{e}}$'s are unknown. Then we can restart Step 1 to Step 5 in the smaller region.

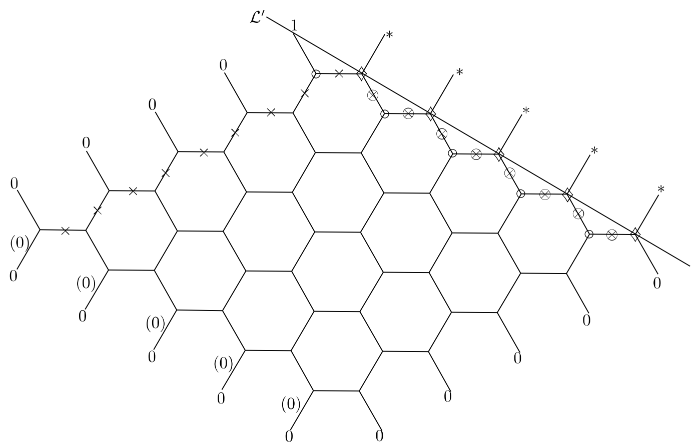

Figure 10: The reconstruction procedure of Step 2.

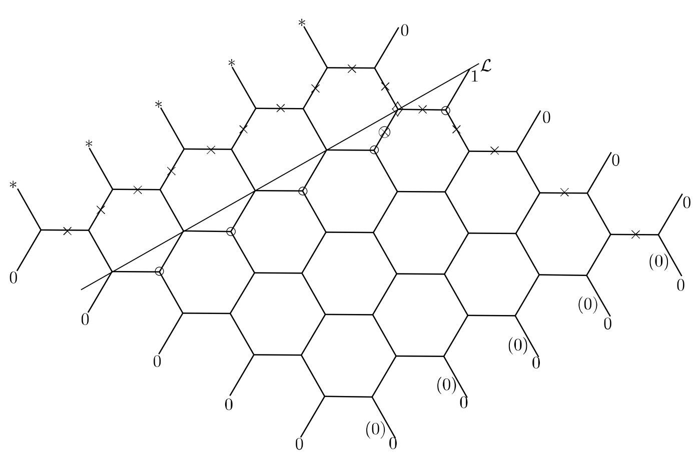

Figure 11: The reconstruction procedure of Step 3.

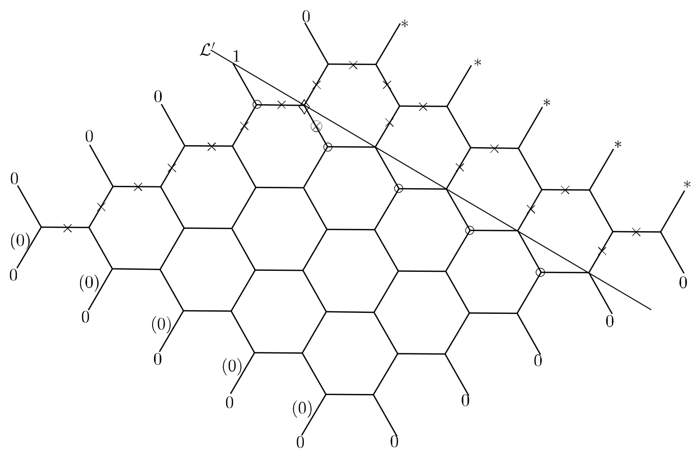

Figure 12: The reconstruction procedure of Step 4.

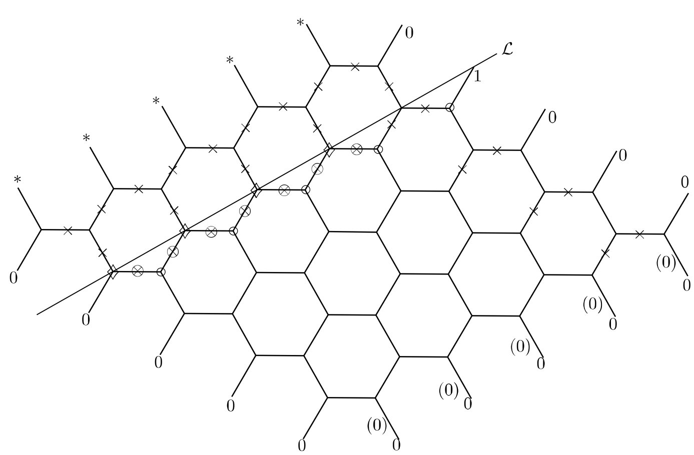

Figure 13: The reconstruction procedure of Step 5.

Finally, we can reconstruct all the pontential $q_{\boldsymbol{e}}$, $\boldsymbol{e} \in \mathcal{E}^i$. For the details of the reconstruction procesure, see [2].

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