Scattering theory and an index theorem on the radial part of $\mathrm{SL}(2,\mathbb{R})$

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Abstract

In this conference proceeding's paper we briefly present the spectral and scattering theory of the Casimir operator acting on radial functions in $L^2\big(\mathrm{SL}(2,\mathbb{R})\big)$. After a suitable decomposition, these investigations consist in studying a family of differential operators acting on the half-line. This material is mainly borrowed from the joint paper [3] with H. Inoue to which we refer for the details . This work has been a first attempt to connect group theory, special functions, scattering theory, C^*-algebras, and Levinson's theorem.

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Contents

1 Origin / motivation

We consider the group $G=\mathrm{SL}(2,\mathbb{R})$, and let $\mathscr{H}:=L^2(G)\equiv L^2(G,\mathrm{d}g)$ endowed with a left Haar measure $\mathrm{d}g$. For $m,n\in\mathbb{Z}$ with $m-n\in 2\mathbb{Z}$, we define the subspace $\mathscr{H}_{m,n}\subset\mathscr{H}$ by the condition

$$f\in\mathscr{H}_{m,n}\Longleftrightarrow f(u_{\theta_1}\,g\,u_{\theta_2})=\mathrm{e}^{i(m\theta_1+n\theta_2)}f(g)$$

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for $g \in G$ and $\theta_1, \theta_2 \in [0, 2\pi)$, and where $u_\theta := \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$. Then, one has

$$\mathscr{H} \cong \bigoplus_{m,n\in\mathbb{Z}, m-n\in 2\mathbb{Z}} \mathscr{H}_{m,n} \cong \bigoplus_{m,n\in\mathbb{Z}, m-n\in 2\mathbb{Z}} L^2(\mathbb{R}_+).$$

Consider now the Casimir operator Ω on $\mathrm{SL}(2,\mathbb{R})$ and set $H = -(\Omega + 1)$ for its realization in $\mathscr{H}$ as a second order differential operator. Then, through the same reduction and unitary transformations one gets

$$H \cong \bigoplus_{m,n\in\mathbb{Z}, m-n\in 2\mathbb{Z}} \Big[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{m^2 + n^2 - 1 - 2mn\cosh(2x)}{\sinh(2x)^2} \Big].$$

By setting $\mu := \frac{|m-n|}{2}$ and $\nu := \frac{|m+n|}{2}$, and using hyperbolic function identities one gets the differential operators $D_{\mu,\nu}$ acting on $\mathbb{R}_+$ and given by

$$D_{\mu,\nu} := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \underbrace{\Big(\mu^2 - \frac{1}{4}\Big)\frac{1}{\sinh(x)^2\cosh(x)^2} + (\mu^2 - \nu^2)\frac{1}{\cosh(x)^2}}_{:= V_{\mu,\nu}(x)}.$$

These expressions are well defined for any $\mu, \nu \geq 0$, and we shall consider this generality in the sequel. Observe that the potential is defined by the function $V_{\mu,\nu} : \mathbb{R}_+ \to \mathbb{R}$ satisfying

- $V_{\mu,\nu}(x) \sim \big(\mu^2 - \frac{1}{4}\big)\frac{1}{x^2}$ near 0,
- $V_{\mu,\nu}(x) \sim 4\big(\mu^2 - \nu^2\big)\mathrm{e}^{-2x}$ near $+\infty$.

Due to the singularity of the potential at 0, there exist several self-adjoint realizations in $L^2(\mathbb{R}_+)$, see for example [1]. For our purpose, we fix the self-adjoint realization defined by

$$\mathrm{dom}(H_{\mu,\nu}) := \Big\{ f \in \mathrm{dom}(D^{\max}_{\mu,\nu}) \mid \exists c \in \mathbb{C} \text{ s.t. } f(x) - c x^{\frac{1}{2}+\mu} \in \mathrm{dom}(D^{\min}_{\mu,\nu}) \text{ near } 0 \Big\},$$
$$H_{\mu,\nu} := D^{\max}_{\mu,\nu}\big|_{\mathrm{dom}(H_{\mu,\nu})}$$

where $\mathrm{dom}(D^{\min}_{\mu,\nu})$ and $\mathrm{dom}(D^{\max}_{\mu,\nu})$ denote the standard minimal and maximal domains for the operator $D_{\mu,\nu}$.

2 Aims of the study

Let us now enumerate the aims of these investigations.

- Study the spectral and the scattering theory for the self-adjoint realization $H_{\mu,\nu}$ in $L^2(\mathbb{R}_+)$,
- Look at the interactions between scattering theory, special functions, and C^*-algebras,
- Establish an index theorem for the operator $H_{\mu,\nu}$,
- Get a topological Levinson's theorem in representation theory, and look for its meaning, as inspired by [2] ?

3 Spectral theory

We briefly sketch the spectral theory for the operator $H_{\mu,\nu}$. For $\zeta \in \mathbb{C} \setminus \mathbb{R}$ with $\Re(\zeta) > 0$, consider the equation

$$-u''(x) + V_{\mu,\nu}(x)u(x) = -\zeta^2 u(x), \qquad x \in \mathbb{R}_+.$$

Its solutions are given by the functions

$$x \mapsto L_{\mu,\nu}(x,\zeta) := \tanh(x)^{\frac{1}{2}+\mu} \cosh(x)^{\zeta} F\big(\alpha - \zeta/2, \beta - \zeta/2; 1+\mu; \tanh(x)^2\big),$$

$$x \mapsto M_{\mu,\nu}(x,\zeta) := \tanh(x)^{\frac{1}{2}-\mu} \cosh(x)^{-\zeta} F\big(1-\alpha + \zeta/2, 1-\beta + \zeta/2; 1+\zeta; \cosh(x)^{-2}\big),$$

with $\alpha := \frac{1+\mu+\nu}{2} > 0$ and $\beta := \frac{1+\mu-\nu}{2} \in \mathbb{R}$, and where $F \equiv {}_2F_1$ is the Gauss hypergeometric function defined for $|z| < 1$ and $c \in \mathbb{C} \setminus \{0,-1,-2,\dots\}$ by

$$F(a,b;c;z) := \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}.$$

The notation Γ denotes the usual Γ-function. The Wronskian for these two solutions can be computed and one gets

$$W_{\mu,\nu}(\zeta) = -\frac{2\Gamma(1+\mu)\Gamma(1+\zeta)}{\Gamma(\alpha+\zeta/2)\Gamma(\beta+\zeta/2)}.$$

The resolvent $R_{\mu,\nu}(z)$ of the operator $H_{\mu,\nu}$ can now be expressed in terms of these functions.

Lemma 3.1 (Resolvent). *For $\zeta \in \mathbb{C}\setminus\mathbb{R}$ with $\Re(\zeta) > 0$, the kernel of the resolvent $R_{\mu,\nu}(-\zeta^2) := (H_{\mu,\nu} + \zeta^2)^{-1}$ is given by*

$$R_{\mu,\nu}(-\zeta^2; x, y) = -\frac{1}{W_{\mu,\nu}(\zeta)} \begin{cases} L_{\mu,\nu}(x,\zeta) M_{\mu,\nu}(y,\zeta) & \text{if } 0 < x < y \\ L_{\mu,\nu}(y,\zeta) M_{\mu,\nu}(x,\zeta) & \text{if } 0 < y < x. \end{cases}$$

In addition, by studying the Wronskian one infers an explicit expression for the number of bound states (eigenvalues) for the operator $H_{\mu,\nu}$. Note that all eigenvalues are located on the half-line $(-\infty, 0)$.

Proposition 3.2 (Number of bound states). *The number of eigenvalues of $H_{\mu,\nu}$ is given by*

$$\#\sigma_{\mathrm{p}}(H_{\mu,\nu}) = \begin{cases} 0 & \text{if } \beta = \frac{1+\mu-\nu}{2} > 0, \\ \left\lceil \frac{\nu-\mu-1}{2} \right\rceil & \text{if } \beta = \frac{1+\mu-\nu}{2} \le 0, \end{cases}$$

where the ceiling function $\lceil\cdot\rceil$ is defined by $\lceil t \rceil := \min\{m \in \mathbb{Z} \mid m \ge t\}$ for $t \in \mathbb{R}$.

The next result is related to the continuous spectrum of the operator $H_{\mu,\nu}$. A limiting absorption principle corresponds to a control of the resolvent of this operator near the real axis. In the statement, the spaces $L^2_{\pm s}(\mathbb{R}_+)$ correspond to the usual weighted Hilbert spaces with the weight defined by $x \mapsto (1+x^2)^{\pm s/2}$. We also introduce three new functions, obtained by a limiting procedure of the functions already introduced, namely

$$\mathcal{W}^{\pm}_{\mu,\nu}(k) := \lim_{\varepsilon \searrow 0} W_{\mu,\nu}\big(\sqrt{-(k^2 \pm i\varepsilon)}\big) = -\frac{2\Gamma(1+\mu)\Gamma(1\mp ik)}{\Gamma(\alpha \mp ik/2)\Gamma(\beta \mp ik/2)},$$

$$\mathcal{L}_{\mu,\nu}(x,k) := \tanh(x)^{\frac{1}{2}+\mu} \cosh(x)^{\mp ik} F\big(\alpha \pm ik/2, \beta \pm ik/2; 1+\mu; \tanh(x)^2\big),$$

$$\mathcal{M}^{\pm}_{\mu,\nu}(x,k) := \tanh(x)^{\frac{1}{2}-\mu} \cosh(x)^{\pm ik} F\big(1-\alpha \mp ik/2, 1-\beta \mp ik/2; 1\mp ik; \cosh(x)^{-2}\big).$$

Note that the expressions for $\mathcal{L}_{\mu,\nu}(x,k)$ is independent of the $\pm$-sign chosen in the r.h.s.

Proposition 3.3 (Limiting absorption principle)**.** *For $k > 0$ and $-\zeta^2 = k^2 \pm i\varepsilon$ the limits $\lim_{\varepsilon \searrow 0} R_{\mu,\nu}(k^2 \pm i\varepsilon) =: R_{\mu,\nu}(k^2 \pm i0)$ exist in the sense of operators from $L^2_s(\mathbb{R}_+)$ to $L^2_{-s}(\mathbb{R}_+)$ for any $s > \frac{1}{2}$, uniformly in k on each compact subset of $\mathbb{R}_+$. In addition, their kernels are given by*

$$R_{\mu,\nu}(k^2 \pm i0; x, y) = -\frac{1}{\mathcal{W}^{\pm}_{\mu,\nu}(k)} \begin{cases} \mathcal{L}_{\mu,\nu}(x,k)\mathcal{M}^{\pm}_{\mu,\nu}(y,k) & \text{if } 0 < x < y, \\ \mathcal{L}_{\mu,\nu}(y,k)\mathcal{M}^{\pm}_{\mu,\nu}(x,k) & \text{if } 0 < y < x. \end{cases}$$

Based on these expressions, one can introduce the notion of spectral density. More precisely, for $k > 0$ we define the spectral density by

$$p_{\mu,\nu}(k^2) := \frac{1}{2\pi i}\big(R_{\mu,\nu}(k^2 + i0) - R_{\mu,\nu}(k^2 - i0)\big)$$

which exists as a bounded operator from $L^2_s(\mathbb{R}_+)$ to $L^2_{-s}(\mathbb{R}_+)$ for any $s > 1/2$, and has kernel

$$\begin{aligned} p_{\mu,\nu}(k^2; x, y) &= \frac{k}{\pi}\frac{1}{|\mathcal{W}^{+}_{\mu,\nu}(k)|^2}\mathcal{L}_{\mu,\nu}(x,k)\mathcal{L}_{\mu,\nu}(y,k) \\ &= \frac{1}{2k}\mathcal{F}^{-}_{\mu,\nu}(x,k)\mathcal{F}^{+}_{\mu,\nu}(y,k) \\ &= \frac{1}{2k}\mathcal{F}^{+}_{\mu,\nu}(x,k)\mathcal{F}^{-}_{\mu,\nu}(y,k), \end{aligned}$$

with

$$\mathcal{F}^{\pm}_{\mu,\nu}(x,k) := -2^{\pm ik}\sqrt{\frac{2}{\pi}}\frac{k}{\mathcal{W}^{\mp}(k)}\mathcal{L}_{\mu,\nu}(x,k).$$

Even if the spectral density is uniquely defined, let us mention that the expressions for $\mathcal{F}^{\pm}_{\mu,\nu}(x,k)$ are not unique, and a suitable choice has been made.

4 Generalized Fourier kernels

We now start the investigations on the scattering theory. We firstly derive a new expression for $\mathcal{F}^{-}_{\mu,\nu}(x,k)$ for $x, k \in \mathbb{R}_+$:

$$\begin{aligned} &\mathcal{F}^{-}_{\mu,\nu}(x,k) \\ &= \frac{2^{-ik}}{\sqrt{2\pi}}k\frac{\Gamma(\alpha - ik/2)\Gamma(\beta - ik/2)}{\Gamma(1+\mu)\Gamma(1-ik)}\tanh(x)^{\frac{1}{2}+\mu}\cosh(x)^{ik}F\big(\alpha - ik/2, \beta - ik/2; 1+\mu; \tanh(x)^2\big) \\ &= 2^{-ik}k\sqrt{\frac{1}{2\pi}}\frac{\Gamma(\alpha - ik/2)\Gamma(\beta - ik/2)}{\Gamma(1+\mu)\Gamma(1-ik)}\tanh(x)^{\frac{1}{2}+\mu}\cosh(x)^{2\alpha}F\big(\alpha - ik/2, \alpha + ik/2; 1+\mu; -\sinh(x)^2\big) \\ &= -i\big\{\mathcal{F}_{\mu,\nu}(x,k)\sigma_{\mu,\nu}(k) - \overline{\mathcal{F}_{\mu,\nu}(x,k)}\big\} \end{aligned}$$

with

$$\mathcal{F}_{\mu,\nu}(x,k) := \frac{1}{\sqrt{2\pi}}\tanh(x)^{\frac{1}{2}+\mu}(\mathrm{e}^{x} + \mathrm{e}^{-x})^{ik}F\big(\alpha - ik/2, \beta - ik/2; 1 - ik; \cosh(x)^{-2}\big)$$

and

$$\sigma_{\mu,\nu}(k) := \frac{\Gamma(\alpha - ik/2)\Gamma(\beta - ik/2)\Gamma(1 + ik/2)\Gamma(1/2 + ik/2)}{\Gamma(\alpha + ik/2)\Gamma(\beta + ik/2)\Gamma(1 - ik/2)\Gamma(1/2 - ik/2)}.$$

Note that several relations involving the hypergeometric function have been used in this computation. We also concentrate only on $\mathcal{F}^-_{\mu,\nu}(x,k)$ but similar formulas exist for $\mathcal{F}^+_{\mu,\nu}(x,k)$. Note also that the function $\sigma_{\mu,\nu}$ is defined by continuity at some singular points, and that it takes values in the set of complex numbers of modulus 1. In fact, the only dangerous factors are $\Gamma(\beta \mp ik/2)$ for $k \searrow 0$.

Let us now state two results about the asymptotic behavior of this function.

Lemma 4.1. *Locally uniformly in $k > 0$ one has as $x \to \infty$*

$$\mathcal{F}^-_{\mu,\nu}(x,k) = \frac{-i}{\sqrt{2\pi}}\Big(\mathrm{e}^{ikx}\sigma_{\mu,\nu}(k) - \mathrm{e}^{-ikx}\Big) + O(\mathrm{e}^{-2x}).$$

In the next statement, we use the notation J_μ for the usual Bessel function. Note that the first statement is rather well-known [6], and has been studied extensively in [4].

Lemma 4.2. *For any fixed $x, k \in \mathbb{R}_+$ one has*

$$\lim_{\epsilon \searrow 0} \mathcal{F}^-_{\mu,\nu}(\epsilon x, k/\epsilon) = \mathrm{e}^{-i\frac{\pi}{2}(\mu-\frac{1}{2})}\sqrt{\frac{2}{\pi}}\mathcal{J}_\mu(xk) \equiv \mathrm{e}^{-i\frac{\pi}{2}(\mu-\frac{1}{2})}\sqrt{kx}\,J_\mu(xk),$$

$$\lim_{\epsilon \to \infty} \mathcal{F}^-_{\mu,\nu}(\epsilon x, k/\epsilon) = \frac{-i}{\sqrt{2\pi}}\Big(\mathrm{e}^{ikx}\sigma_{\mu,\nu}(0) - \mathrm{e}^{-ikx}\Big)$$

with

$$\sigma_{\mu,\nu}(0) = \begin{cases} 1 & \text{if } \beta \notin -\mathbb{N}, \\ -1 & \text{if } \beta \in -\mathbb{N}. \end{cases} \tag{1}$$

Clearly, among all possible values for $\beta \in \mathbb{R}$, the case $\beta \in -\mathbb{N}$ corresponds to the exceptional case, while the case $\beta \notin -\mathbb{N}$ corresponds to the generic situation.

For $f \in L^2(\mathbb{R}_+)$ and $k > 0$ we can now define the generalized Fourier transforms by the relations

$$[\mathcal{F}^\pm_{\mu,\nu} f](k) := \int_0^\infty \overline{\mathcal{F}^\pm_{\mu,\nu}(x,k)} f(x)\,\mathrm{d}x = \int_0^\infty \mathcal{F}^\mp_{\mu,\nu}(x,k) f(x)\,\mathrm{d}x,$$

and the Møller wave operators

$$W_\pm(H_{\mu,\nu}, H_{\mathrm{D}}) := \big(\mathcal{F}^\pm_{\mu,\nu}\big)^* \mathcal{F}_{\mathrm{D}}.$$

with $\big(\mathcal{F}_{\mathrm{D}} f\big)(k) := \sqrt{\frac{2}{\pi}}\int_0^\infty \sin(kx) f(x)\,\mathrm{d}x$ the Fourier sine transform. Here D refers to the Dirichlet Laplacian H_{D}. The following statement makes the link between the various stationary expressions introduced so far, and the time dependent version of scattering theory.

Proposition 4.3. *The following equalities hold:*

$$W_\pm(H_{\mu,\nu}, H_{\mathrm{D}}) = \operatorname*{s-lim}_{t \to \pm\infty} \mathrm{e}^{itH_{\mu,\nu}}\mathrm{e}^{-itH_{\mathrm{D}}}.$$

In addition,

$$S_{\mu,\nu} := W_+(H_{\mu,\nu}, H_{\mathrm{D}})^* W_-(H_{\mu,\nu}, H_{\mathrm{D}}) = \sigma_{\mu,\nu}\big(\sqrt{H_{\mathrm{D}}}\big).$$

The following equalities are well known:

$$\operatorname*{s-lim}_{t \to -\infty} \mathrm{e}^{itH_{\mathrm{D}}}\, W_-(H_{\mu,\nu}, H_{\mathrm{D}})\,\mathrm{e}^{-itH_{\mathrm{D}}} = 1 \tag{2}$$

$$\operatorname*{s-lim}_{t \to +\infty} \mathrm{e}^{itH_{\mathrm{D}}}\, W_-(H_{\mu,\nu}, H_{\mathrm{D}})\,\mathrm{e}^{-itH_{\mathrm{D}}} = S_{\mu,\nu}. \tag{3}$$

We shall supplement them with other relations of the same type. For that purpose, let $\{U_\tau\}_{\tau\in\mathbb{R}}$ denote the dilation group in $L^2(\mathbb{R}_+)$ acting on $f \in L^2(\mathbb{R}_+)$ and for $x \in \mathbb{R}_+$ as

$$\big[U_\tau f\big](x) = \mathrm{e}^{\tau/2} f(\mathrm{e}^\tau x), \tag{4}$$

Then, motivated by Lemma 4.2, it is natural to look at the following limits:

$$\begin{aligned}\big[U_{-\tau} W_-(H_{\mu,\nu}, H_{\mathrm{D}}) U_\tau f\big](x) &= \int_0^\infty \mathcal{F}^-_{\mu,\nu}\big(\mathrm{e}^{-\tau}x, \mathrm{e}^\tau k\big)[\mathcal{F}_{\mathrm{D}} f](k)\,\mathrm{d}k \\ &\overset{?}{\longrightarrow} \big[\mathrm{e}^{-i\frac{\pi}{2}(\mu-\frac{1}{2})}\mathcal{F}_\mu \mathcal{F}_{\mathrm{D}} f\big](x) \qquad \text{as } \tau \to \infty, \qquad (5)\\ &\overset{?}{\longrightarrow} \begin{cases} [1f](x) & \text{if } \beta \notin -\mathbb{N}, \\ \big[i\mathcal{F}_{\mathrm{N}}\mathcal{F}_{\mathrm{D}} f\big](x) & \text{if } \beta \in -\mathbb{N}, \end{cases} \qquad \text{as } \tau \to -\infty, \qquad (6)\end{aligned}$$

with

$$\big(\mathcal{F}_\mu f\big)(k) := \sqrt{\frac{2}{\pi}} \int_0^\infty \mathcal{J}_\mu(kx) f(x)\,\mathrm{d}x$$

and

$$\big(\mathcal{F}_{\mathrm{N}} f\big)(k) := \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(kx) f(x)\,\mathrm{d}x.$$

In the first expression, $\mathcal{J}_\mu$ denotes the Bessel function for dimension 1, as introduced in [1, App. A.4] and defined by

$$\mathcal{J}_\mu(x) := \sqrt{\frac{\pi x}{2}}\, J_\mu(x).$$

The transformation $\mathcal{F}_\mu$ corresponds to a Hankel transform, while $\mathcal{F}_{\mathrm{N}}$ denotes the Fourier cosine transform. The pointwise convergences mentioned in Lemma 4.2 do not allow us to deduce a useful convergences in the above expressions. For that reason, we mention these convergences as motivations for the operators defined by the r.h.s. In the next section, these operators are going to play an important role.

5 C*-algebras and an index theorem

Let us denote by A the generator of the dilation group introduced in (4). Then, the following equalities can be proved:

$$\begin{aligned} i\mathcal{F}_{\mathrm{N}}\mathcal{F}_{\mathrm{D}} &= -\tanh(\pi A) + i\cosh(\pi A)^{-1}, \\ \mathcal{F}_\mu \mathcal{F}_{\mathrm{D}} &= \frac{\Gamma\big(\frac{\mu+1}{2} - i\frac{A}{2}\big)\,\Gamma\big(\frac{3}{4} + i\frac{A}{2}\big)}{\Gamma\big(\frac{\mu+1}{2} + i\frac{A}{2}\big)\,\Gamma\big(\frac{3}{4} - i\frac{A}{2}\big)} =: \vartheta_{\mu,\mathrm{D}}(A). \end{aligned}$$

In addition, if we set $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ one readily observes that the following properties hold:

- $-\tanh(\pi\cdot) + i\cosh(\pi\cdot)^{-1} \in C\big([-\infty, +\infty]; \mathbb{T}\big)$,
- $\vartheta_{\mu,\mathrm{D}} \in C\big([-\infty, +\infty]; \mathbb{T}\big)$ with $\vartheta_{\mu,\mathrm{D}}(-\infty) = \mathrm{e}^{i\frac{\pi}{2}(\mu-\frac{1}{2})}$, $\vartheta_{\mu,\mathrm{D}}(+\infty) = \mathrm{e}^{-i\frac{\pi}{2}(\mu-\frac{1}{2})}$,
- $\sigma_{\mu,\nu} \in C\big([0, +\infty]; \mathbb{T}\big)$ with $\sigma_{\mu,\nu}(+\infty) = \mathrm{e}^{-i\pi(\mu-\frac{1}{2})}$, $\sigma_{\mu,\nu}(0)$ provided in (1).

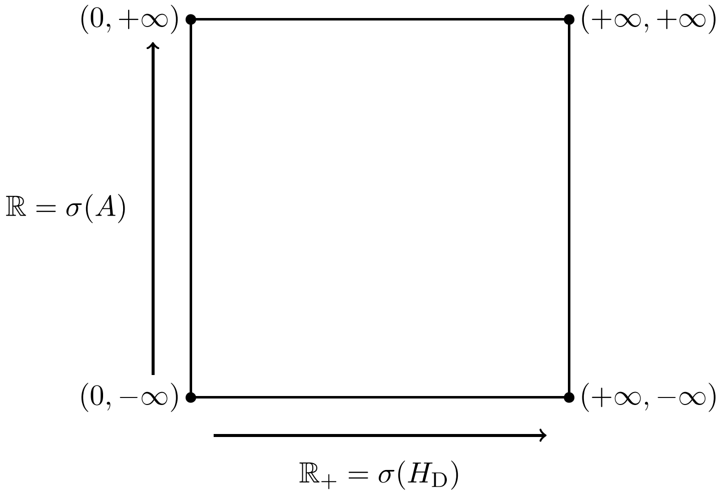

Figure 1: Representation of $\mathscr{E}$

Thus, the first two functions have limits at $\pm\infty$, while the last one has limits at 0 and at $+\infty$. Furthermore, they all take values in the set of complex numbers of modulus 1.

Based on these observations, let us introduce a C^*-subalgebra of $\mathscr{B}\big(L^2(\mathbb{R}_+)\big)$:

$$\mathscr{E} := C^*\Bigg(\eta_i(A)\,\psi_i(H_{\rm D}) \mid \eta_i \in C\big([-\infty,+\infty]\big), \psi_i \in C\big([0,+\infty]\big)\Bigg).$$

This C^*-algebra can be thought as an algebra of functions living on the square $\square$ represented in Figure 1.

Clearly, the three operators $-\tanh(\pi A) + i\cosh(\pi A)^{-1}$, $\vartheta_{\mu,{\rm D}}(A)$, and $\sigma_{\mu,\nu}\big(\sqrt{H_{\rm D}}\big)$ belong to the algebra $\mathscr{E}$. Let us now state a conjecture, which has been proved for several other scattering models [5], but which could not be proved yet for this model. The difficulty is coming from the two convergences (5) and (6) which are not fully understood.

Conjecture 5.1. *For any $\mu, \nu \geq 0$ one has $W_-(H_{\mu,\nu}, H_{\rm D}) \in \mathscr{E}$.*

This conjecture can also be represented on the square, as illustrated in Figure 2. In addition, if we let $\Lambda_1, \Lambda_2, \Lambda_3$ and Λ_4 denote the restrictions on the boundaries of the square, then the relations (2), (3), (5), and (6) lead to the following identifications for $s \in \mathbb{R}$ and $k \in \mathbb{R}_+$:

$$\Lambda_1(s) := \begin{cases} 1 & \text{if } \beta \notin -\mathbb{N} \\ -\tanh(\pi s) + i\cosh(\pi s)^{-1} & \text{if } \beta \in -\mathbb{N}, \end{cases}$$

$$\Lambda_2(k) := \sigma_{\mu,\nu}(k),$$

$$\Lambda_3(s) := \mathrm{e}^{-i\frac{\pi}{2}(\mu-\frac{1}{2})}\vartheta_{\mu,{\rm D}}(s)$$

$$\Lambda_4(k) := 1.$$

Then, the main result of this construction reads:

Theorem 5.2. *For any $\mu, \nu \geq 0$ the function $\Lambda_{\mu,\nu} := \big(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4\big)$ is a continuous function on the boundary of the square and takes values in $\mathbb{T}$. In addition, the following equality holds:*

$$\mathrm{Wind}(\Lambda_{\mu,\nu}) = \#\sigma_{\rm p}(H_{\mu,\nu}) = -\,\mathrm{ind}\big(W_-(H_{\mu,\nu}, H_{\rm D})\big), \tag{7}$$

where the l.h.s. denotes the winding number of the function $\xi \mapsto \Lambda(\xi)$, while the r.h.s. denotes (minus) the index of the Fredholm operator $W_-(H_{\mu,\nu}, H_{\rm D})$. This Fredholm index is also equal to (minus) the number of bound states of $H_{\mu,\nu}$.

Let us emphasize that the proof is based on an explicit computation, but the statement would hold automatically if the conjecture is proved, with K-theoretic arguments. In the results presented above, scattering theory, special functions, and this index theorem complement and stimulate each other. On the other hand, what can we infer from (7) about the representation theory of $\mathrm{SL}(2,\mathbb{R})$ is not known yet.

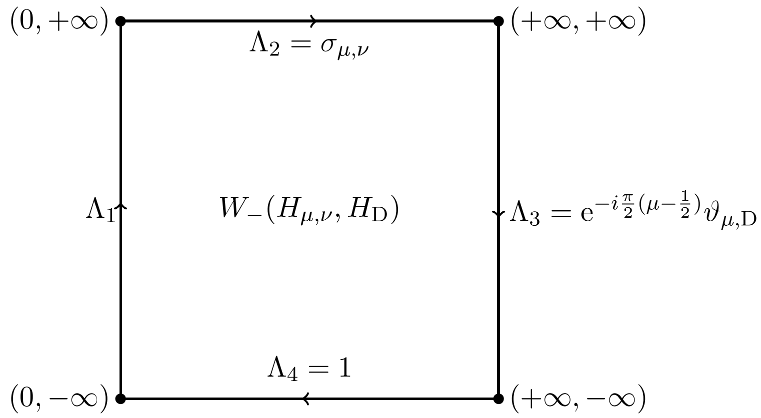

Figure 2: $W_-(H_{\mu,\nu}, H_{\rm D}) \in \mathscr{E}$ and its restriction on the boundaries

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