

SNAPSHOT PROBLEM FOR THE WAVE EQUATION

TOMOYUKI KAKEHI

ABSTRACT. Let us call a smooth solution $u(t, x)$ of the wave equation on \mathbb{R}^n a *wave*. Suppose that we take several snapshots of a wave u at several different times. Then we arrive at the question: Can we reconstruct a wave u from those snapshot data? Roughly speaking, our answer to the above question is as follows. If we take 2 snapshots at $t = 0$ and $t = 1$, then we can uniquely determine the wave at any integer time. If we take 3 snapshots at $t = 0, 1$ and $t = \alpha$, and if the 3 snapshot data satisfy a certain compatibility condition and α is an irrational number and not a Liouville number, then we obtain a unique solution of the wave equation, namely, we can determine the wave uniquely at any time. We also consider a similar problem for the shifted wave equation on the sphere \mathbb{S}^n . This article is a brief summary of the joint work with Jens Christensen, Fulton Gonzalez, and Jue Wang. (See [CGKW24].)

1. INTRODUCTION

In this article, we consider the uniqueness and the existence of the solution to the wave equation

$$\partial_t^2 u(t, x) - \Delta u(t, x) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n,$$

with the condition

$$u|_{t=t_1} = f_1, \dots, u|_{t=t_m} = f_m.$$

Here t_1, \dots, t_m are m given times and f_1, \dots, f_m are m given smooth functions on \mathbb{R}^n . Our question is as follows.

When does the above equation have a unique solution?

As is well known, the motion of a wave u is described by the wave equation $\partial_t^2 u - \Delta u = 0$. Suppose that we take several snapshots $u|_{t=t_1} = f_1, \dots, u|_{t=t_m} = f_m$ of the wave at $t = t_1, \dots, t = t_m$. Then we arrive at the problem.

Can one find the solution to the wave equation with given snapshots?

For m given times t_1, \dots, t_m and for m given functions f_1, \dots, f_m on \mathbb{R}^n , let us consider the following problem

$$(SP) \quad \begin{cases} \partial_t^2 u(t, x) - \Delta u(t, x) = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\ u|_{t=t_1} = f_1, \dots, u|_{t=t_m} = f_m. \end{cases}$$

We call the above problem (SP) **the snapshot problem for the wave equation** and the set of m functions $\{f_1, \dots, f_m\}$ **the snapshot data** of (SP). In addition, we sometimes call the snapshot problem with m snapshots **the m -snapshot problem**.

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Department of Mathematics, University of Tsukuba.

Throughout the article, we assume that $f_1, \dots, f_m \in C^\infty(\mathbb{R}^n)$. However we do not assume any decay condition on the snapshot data $\{f_1, \dots, f_m\}$. It is because any solution to the wave equation has the finite propagation property.

As is well known, the Fourier transform sometimes plays an important role in the study of wave equations. However we note that since each function f_j ($1 \leq j \leq m$) of the snapshot data does not necessarily decrease at infinity, we can no longer apply the Fourier transform to each f_j . This is one of the difficulties of the snapshot problem.

Let us first consider the case $m = 2$. The snapshot problem for the wave equation was first studied by F. John.

In his book “Plane Waves and Spherical Means Applied to Partial Differential Equations” [Joh55], he considers the snapshot problem in the case $n = 3$ and $m = 2$.

For simplicity, we may assume that $t_1 = 0$, $t_2 = 1$ and let $u|_{t=0} = f_0$, $u|_{t=1} = f_1$. Then he showed the following.

Theorem 1.1 (F. John). *While the solution of the snapshot problem is not unique, the solution at any integer time is uniquely determined by the snapshot data $\{f_0, f_1\}$.*

For the details, see Page 118 of the above book.

2. THE 2-SNAPSHOT PROBLEM

Next, let us consider the 2-snapshot problem for general n . As in the John’s case, we may assume that $t_1 = 0$, $t_2 = 1$.

First of all, we see easily that the uniqueness for the 2-snapshot problem does not hold. In fact, if we take any $\zeta \in \mathbb{C}^n$ with $\zeta \cdot \zeta = \pi^2$ and put

$$u(t, x) = \sin(\pi t) e^{i\zeta \cdot x},$$

then $u(t, x)$ satisfies $u(0, x) = u(1, x) = 0$ and it also satisfies the wave equation $\partial_t^2 u - \Delta u = 0$.

What’s interesting is that even though the uniqueness does not hold, the solution u at any integer time is uniquely determined.

Let $u|_{t=0} = f_0$, $u|_{t=1} = f_1$. In addition, let us define a compactly supported distribution Ψ_k ($k \in \mathbb{Z}$) by

$$\widehat{\Psi}_k(\xi) = \frac{\sin(k|\xi|)}{\sin|\xi|} = U_{k-1}(\cos|\xi|) \quad \xi \in \mathbb{R}^n,$$

where U_k denotes the Chebyshev polynomial of the second kind. (Here $U_{-1} = 0$ and if $k < -1$ define U_k by $U_{k-1} = -U_{-k-1}$.) Then we have the following.

Theorem 2.1 ([CGKW24]). *Suppose that u be a solution of the snapshot problem with the snapshot data $\{f_0, f_1\}$. For any integer k , let $u_k = u(k, \cdot)$ be the snapshot of u at $t = k$. Then u_k is explicitly determined by the formula*

$$u_k = f_1 * \Psi_k - f_0 * \Psi_{k-1}.$$

In the above formula, $*$ denotes convolution.

For the proof and related results, see Section 2 of [CGKW24].

In the same way as above, $k \in \mathbb{Z}$ and for $s \in \mathbb{R}$, we define a compactly supported distribution $\Psi_{k,s}$ by

$$\widehat{\Psi}_{k,s}(\xi) = \frac{\sin(ks|\xi|)}{\sin(s|\xi|)} = U_{k-1}(\cos(s|\xi|)) \quad \xi \in \mathbb{R}^n.$$

Suppose that u is a solution of the wave equation. For $T \in \mathbb{R}$, let $u_T = u(T, \cdot)$ be the snapshot of u at $t = T$.

Then by time-translating and scaling, we have

Corollary 2.2. *For $a, b \in \mathbb{R}$ with $a \neq b$ and for any integer k , we have*

$$u_{a+k(b-a)} = u_b * \Psi_{k,b-a} - u_a * \Psi_{k-1,b-a}.$$

3. UNIQUENESS FOR THE 3-SNAPSHOT PROBLEM

Next, let us consider the uniqueness of the snapshot problem in the case $m = 3$.

By time-translating and scaling (t, x) , we may assume that $t_1 = 0$, $t_2 = 1$, $t_3 = \alpha$, where $\alpha \neq 0, 1$. So let us assume that

$$u|_{t=0} = u|_{t=1} = u|_{t=\alpha} = 0.$$

By the theorem and the corollary above, for any integers ℓ, k , we have

$$u_k = u_1 * \Psi_k - u_0 * \Psi_{k-1} = 0, \quad u_{\ell\alpha} = u_\alpha * \Psi_{\ell,\alpha} - u_0 * \Psi_{\ell-1,\alpha} = 0.$$

Again by the above corollary,

$$u_{k+\ell\alpha} = u_{\ell\alpha} * \Psi_{1,\alpha} - u_k * \Psi_{0,\ell\alpha} = 0.$$

If α is an irrational number, the set $\{k + \ell\alpha \mid k, \ell \in \mathbb{Z}\}$ is dense in \mathbb{R} . So by density argument, $u \equiv 0$. As a result, we obtain the following.

Theorem 3.1 (Uniqueness for the 3-snapshot problem [CGKW24]). *If α is an irrational number, then a solution u of the wave equation is uniquely determined by its snapshots $u|_{t=0}, u|_{t=1}, u|_{t=\alpha}$,*

Again by time-translating and scaling (t, x) , we have

Corollary 3.2. *If the ratio $\frac{t_3 - t_1}{t_2 - t_1}$ is an irrational number, a solution u of the wave equation is uniquely determined by its snapshots $u|_{t=t_1}, u|_{t=t_2}, u|_{t=t_3}$,*

4. EXISTENCE THEOREM FOR THE 3-SNAPSHOT PROBLEM

We assume that $t_1 = 0$, $t_2 = 1$, $t_3 = \alpha$, where $\alpha \neq 0, 1$.

Let us consider the snapshot problem.

$$(SP) \quad \begin{cases} \partial_t^2 u(t, x) - \Delta u(t, x) = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\ u|_{t=0} = f_0, u|_{t=1} = f_1, u|_{t=\alpha} = f_\alpha. \end{cases}$$

We note that the above (SP) is overdetermined. In fact, for two given smooth functions f_0, g_0 , the Cauchy problem for the wave equation

$$(CP) \quad \begin{cases} \partial_t^2 u(t, x) - \Delta u(t, x) = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\ u|_{t=0} = f_0, \partial_t u|_{t=0} = g_0. \end{cases}$$

has a unique solution. So we need a compatibility condition on three functions f_0, f_1 and f_α . As is well known, the solution to the Cauchy problem (CP) is given by

$$u(t, x) = (f_0 * S'_t)(x) + (g_0 * S_t)(x),$$

where S_t and S'_t are compactly supported distributions on \mathbb{R}^n whose Fourier transforms are given respectively by

$$(4.1) \quad \widehat{S}_t(\xi) = \frac{\sin(t|\xi|)}{|\xi|}, \quad \widehat{S}'_t(\xi) = \cos(t|\xi|).$$

By the above formula,

$$f_1 = u|_{t=1} = f_0 * S'_1 + g_0 * S_1, \quad f_\alpha = u|_{t=\alpha} = f_0 * S'_\alpha + g_0 * S_\alpha.$$

Namely, we have

$$g_0 * S_1 = f_1 - f_0 * S'_1, \quad g_0 * S_\alpha = f_\alpha - f_0 * S'_\alpha.$$

If there is a solution of (SP), then f_0 , f_1 and f_α satisfy

$$(f_1 - f_0 * S'_1) * S_\alpha = g_0 * S_1 * S_\alpha = g_0 * S_\alpha * S_1 = (f_\alpha - f_0 * S'_\alpha) * S_1.$$

Therefore, we have

Proposition 4.1 (Compatibility condition on the snapshot data). *If the snapshot problem (SP) with the snapshot data $\{f_0, f_1, f_\alpha\}$ has a solution, then f_0 , f_1 and f_α satisfy the following compatibility condition.*

$$(f_1 - f_0 * S'_1) * S_\alpha = (f_\alpha - f_0 * S'_\alpha) * S_1.$$

Let us go back to the snapshot problem (SP).

$$(SP) \quad \begin{cases} \partial_t^2 u(t, x) - \Delta u(t, x) = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\ u|_{t=0} = f_0, u|_{t=1} = f_1, u|_{t=\alpha} = f_\alpha. \end{cases}$$

Our question is as follows.

When does the snapshot problem (SP) have a unique solution?

As we stated before, we need to assume the following.

- (1) α is an irrational number.
- (2) f_0 , f_1 and f_α satisfy the compatibility condition.

$$(f_1 - f_0 * S'_1) * S_\alpha = (f_\alpha - f_0 * S'_\alpha) * S_1.$$

Question: **Are these two conditions sufficient?**

Let us now state our main theorem.

Theorem 4.2 ([CGKW24]). *Let α be an irrational number and not a Liouville number.*

*Suppose that f_0 , f_1 and $f_\alpha \in C^\infty(\mathbb{R}^n)$ satisfy the compatibility condition $(f_1 - f_0 * S'_1) * S_\alpha = (f_\alpha - f_0 * S'_\alpha) * S_1$. Then the snapshot problem*

$$(SP) \quad \begin{cases} \partial_t^2 u(t, x) - \Delta u(t, x) = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\ u|_{t=0} = f_0, u|_{t=1} = f_1, u|_{t=\alpha} = f_\alpha. \end{cases}$$

has a unique solution.

In addition to the above two conditions, we have to assume that α is not a Liouville number.

From now on, we will explain the definition and some properties of a Liouville number.

5. WHAT IS A LIOUVILLE NUMBER?

Definition 5.1. A Liouville number is a real number x with the property that, for every positive integer m , there exists a pair of integers (p_m, q_m) with $q_m > 1$ such that

$$\left| x - \frac{p_m}{q_m} \right| < \frac{1}{q_m^m}.$$

By definition, a Liouville number is a real number which is very closely approximated by a sequence of rational numbers. The following is a famous example given by Liouville.

$$x = \sum_{m=1}^{\infty} 10^{-m!}$$

Properties of Liouville numbers. Let \mathcal{L} be the set of Liouville numbers. The following are known in number theory.

Known facts

- (1) Any Liouville number is a transcendental number.
- (2) \mathcal{L} is an uncountable set and dense in \mathbb{R} .
- (3) The Lebesgue measure of \mathcal{L} is zero.

Remark 5.2. If unfortunately you take the third snapshot at a Liouville number time, then you have to take at least one more snapshot in order to get a solution to the wave equation. But such a bad luck rarely occurs because the Lebesgue measure of the set of Liouville numbers is zero.

6. OUTLINE OF THE PROOF

We have the following diagram of correspondence.

Solutions of the wave equation $\partial_t^2 u(t, x) - \Delta u(t, x) = 0$

$$\begin{cases} u|_{t=0} = f_0, \\ \partial_t u|_{t=0} = g_0 \end{cases} \quad \Downarrow \quad \Uparrow \quad u = f_0 * S'_t + g_0 * S_t$$

Cauchy data $\{f_0, g_0\}$

$$\begin{cases} f_1 = f_0 * S'_1 + g_0 * S_1, \\ f_\alpha = f_0 * S'_\alpha + g_0 * S_\alpha \end{cases} \quad \Downarrow \quad \Uparrow \quad ?$$

Snapshot data $\{f_0, f_1, f_\alpha\}$ with the compatibility condition

The key idea is to show that the mapping from the space of Cauchy data to the space of snapshot data with the compatibility condition is an isomorphism (fill the ?-part in the above diagram) under the assumption that α is not a Liouville number.

6.1. Injectivity of the mapping $\{f_0, g_0\} \mapsto \{f_0, f_1, f_\alpha\}$. First, we consider the injectivity of the mapping

$$\text{Cauchy data } \{f_0, g_0\} \mapsto \text{Snapshot data } \{f_0, f_1, f_\alpha\}.$$

Recall that we are assuming that α is irrational. If $f_0 = f_1 = f_\alpha = 0$, namely if $u|_{t=0} = u|_{t=1} = u|_{t=\alpha} = 0$, then by the uniqueness theorem we have the solution $u = 0$ and therefore, $f_0 = u|_{t=0} = 0$, $g_0 = \partial_t u|_{t=0} = 0$.

6.2. **Surjectivity of the mapping** $\{f_0, g_0\} \mapsto \{f_0, f_1, f_\alpha\}$. Next, we consider the surjectivity of the mapping

$$\text{Cauchy data } \{f_0, g_0\} \mapsto \text{Snapshot data } \{f_0, f_1, f_\alpha\}.$$

Let us recall the compatibility condition

$$(f_1 - f_0 * S'_1) * S_\alpha = (f_\alpha - f_0 * S'_\alpha) * S_1.$$

Our objective is to find g_0 such that

$$g_0 * S_1 = f_1 - f_0 * S'_1, \quad g_0 * S_\alpha = f_\alpha - f_0 * S'_\alpha.$$

In fact, such g_0 satisfies $\partial_t u|_{t=0} = g_0$.

Now let

$$\varphi_1 := f_1 - f_0 * S'_1, \quad \varphi_\alpha := f_\alpha - f_0 * S'_\alpha.$$

Then our objective is to find g_0 such that $g_0 * S_1 = \varphi_1$ and $g_0 * S_\alpha = \varphi_\alpha$ under the condition that $\varphi_1 * S_\alpha = \varphi_\alpha * S_1$.

Let us admit the following theorem.

Theorem 6.1. *Let α be an irrational number and not a Liouville number. Then there exist compactly supported distributions Φ_1 and Φ_α such that $\Phi_1 * S_1 + \Phi_\alpha * S_\alpha = \delta_0$, where δ_0 denotes the Dirac delta function.*

Using the above Φ_1 and Φ_α , let $g_0 = \varphi_1 * \Phi_1 + \varphi_\alpha * \Phi_\alpha$. Then

$$\begin{aligned} g_0 * S_1 &= (\varphi_1 * \Phi_1 + \varphi_\alpha * \Phi_\alpha) * S_1 \\ &= \varphi_1 * \Phi_1 * S_1 + \varphi_\alpha * \Phi_\alpha * S_1 \quad (\text{by } \varphi_1 * S_\alpha = \varphi_\alpha * S_1) \\ &= \varphi_1 * (\Phi_1 * S_1 + \Phi_\alpha * S_\alpha) \\ &= \varphi_1 * \delta_0 = \varphi_1, \quad (\text{by the above theorem}). \end{aligned}$$

Similarly we have $g_0 * S_\alpha = \varphi_\alpha$, which completes the proof of the surjectivity.

Let us go back to the diagram of the correspondence.

Solutions of the wave equation $\partial_t^2 u(t, x) - \Delta u(t, x) = 0$

$$\begin{cases} u|_{t=0} = f_0, \\ \partial_t u|_{t=0} = g_0 \end{cases} \quad \Downarrow \quad \Uparrow \quad u = f_0 * S'_t + g_0 * S_t$$

Cauchy data $\{f_0, g_0\}$

$$\begin{cases} f_1 = f_0 * S'_1 + g_0 * S_1, \\ f_\alpha = f_0 * S'_\alpha + g_0 * S_\alpha \end{cases} \quad \Downarrow \quad \Uparrow \quad ?$$

Snapshot data $\{f_0, f_1, f_\alpha\}$ with the compatibility condition

The ?-part is given by the following..

Solutions of the wave equation $\partial_t^2 u(t, x) - \Delta u(t, x) = 0$

$$\begin{cases} u|_{t=0} = f_0, \\ \partial_t u|_{t=0} = g_0 \end{cases} \quad \Downarrow \quad \Uparrow \quad u = f_0 * S'_t + g_0 * S_t$$

Cauchy data $\{f_0, g_0\}$

$$\begin{cases} f_1 = f_0 * S'_1 + g_0 * S_1, \\ f_\alpha = f_0 * S'_\alpha + g_0 * S_\alpha \end{cases} \quad \Downarrow \quad \Uparrow \quad \begin{cases} g_0 = (f_1 - f_0 * S'_1) * \Phi_1 \\ \quad + (f_\alpha - f_0 * S'_\alpha) * \Phi_\alpha \end{cases}$$

Snapshot data $\{f_0, f_1, f_\alpha\}$ with the compatibility condition

In particular, the above shows that the mapping

$$\text{Cauchy data } \{f_0, g_0\} \mapsto \text{Snapshot data } \{f_0, f_1, f_\alpha\}.$$

is an isomorphism. Therefore, the snapshot problem (SP) has a unique solution.

Remark 6.2. *The proof of Theorem 6.1 uses a deep result on compactly supported distributions by Ehrenpreis. For the details, see his book “Fourier analysis in several complex variables” [Ehr70].*

7. WHAT HAPPENS IF α IS A LIOUVILLE NUMBER?

Let us consider the case α is a Liouville number.

In this case, the mapping

$$\text{Cauchy data } \{f_0, g_0\} \mapsto \text{Snapshot data } \{f_0, f_1, f_\alpha\}.$$

is no longer surjective, and as a result, we see that there exist f_0, f_1, f_α such that the snapshot problem (SP) with the snapshot data $\{f_0, f_1, f_\alpha\}$ does not have a solution.

We will prove it by contradiction. So we assume that the mapping

$$\text{Cauchy data } \{f_0, g_0\} \mapsto \text{Snapshot data } \{f_0, f_1, f_\alpha\}.$$

is surjective.

For a compactly supported distribution T on \mathbb{R}^n , we define a convolution operator $C_T : C^\infty(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^n)$ by

$$C_T f(x) = f * T(x) = \int_{\mathbb{R}^n} f(x-y) dT(y), \quad x \in \mathbb{R}^n, f \in C^\infty(\mathbb{R}^n).$$

Then we have

Theorem 7.1 (Ehrenpreis, [Ehr60]). *$C_T : C^\infty(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^n)$ is surjective if and only if the Fourier-Laplace transform $\widehat{T}(\zeta)$ of T is slowly decreasing.*

For the definition of *slowly decreasing*, see the above paper [Ehr60] by Ehrenpreis. Roughly speaking, if $\widehat{T}(\zeta)$ decays polynomially at infinity, then $\widehat{T}(\zeta)$ is slowly decreasing.

Recall that the compactly supported distribution S_t is defined by

$$\widehat{S}_t(\xi) = \frac{\sin(t|\xi|)}{|\xi|}.$$

So the Fourier-Laplace transform of S_t is given by

$$\widehat{S}_t(\zeta) = \frac{\sin(t\sqrt{\zeta \cdot \zeta})}{\sqrt{\zeta \cdot \zeta}}, \quad \zeta \in \mathbb{C}^n,$$

from which it follows that $\widehat{S}_t(\zeta)$ is slowly decreasing, and thus by the above theorem the convolution operator $C_{S_t} : C^\infty(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^n)$ is surjective.

As in the proof of the existence of the solution, for the snapshot data $\{f_0, f_1, f_\alpha\}$, let

$$\varphi_1 := f_1 - f_0 * S'_1, \quad \varphi_\alpha := f_\alpha - f_0 * S'_\alpha.$$

Then the compatibility condition is given by $\varphi_1 * S_\alpha = \varphi_\alpha * S_1$. Namely $C_{S_\alpha} \varphi_1 = C_{S_1} \varphi_\alpha$.

In addition, the surjectivity of the mapping

$$\text{Cauchy data } \{f_0, g_0\} \mapsto \text{Snapshot data } \{f_0, f_1, f_\alpha\}$$

means that for any pair $(\varphi_1, \varphi_\alpha)$ with $C_{S_\alpha} \varphi_1 = C_{S_1} \varphi_\alpha$, there exists $g_0 \in C^\infty(\mathbb{R}^n)$ such that $C_{S_1} g_0 = \varphi_1$ and $C_{S_\alpha} g_0 = \varphi_\alpha$.

For simplicity, let

$$V = C^\infty(\mathbb{R}^n), \quad S = C_{S_\alpha}, \quad T = C_{S_1}.$$

We note that $TS = ST$ and both $S : V \rightarrow V$ and $T : V \rightarrow V$ are surjective.

Here we have the following lemma.

Lemma 7.2. *Let V be a vector space, and S, T be two surjective linear operators with $ST = TS$. Then the following are equivalent.*

- (i) $S : \text{Ker} T \rightarrow \text{Ker} T$ is surjective.
- (ii) $T : \text{Ker} S \rightarrow \text{Ker} S$ is surjective.
- (iii) For any pair $(v, w) \in V \times V$ with $Sv = Tw$, there is an $x \in V$ such that $Sx = w$ and $Tx = v$.

The proof is straightforward, so we omit it.

Now we apply the above lemma to

$$V = C^\infty(\mathbb{R}^n), \quad S = C_{S_\alpha}, \quad T = C_{S_1}.$$

Then we have $C_{S_\alpha} : \text{Ker} C_{S_1} \rightarrow \text{Ker} C_{S_1}$ is surjective.

On the other hand, by the uniqueness theorem, $\text{Ker} C_{S_1} \cap \text{Ker} C_{S_\alpha} = \{0\}$. Thus $C_{S_\alpha} : \text{Ker} C_{S_1} \rightarrow \text{Ker} C_{S_1}$ is injective.

Moreover, we see easily that $C_{S_\alpha} : \text{Ker} C_{S_1} \rightarrow \text{Ker} C_{S_1}$ is a continuous linear operator.

Therefore, by the open mapping theorem, $C_{S_\alpha}^{-1} : \text{Ker} C_{S_1} \rightarrow \text{Ker} C_{S_1}$ is also a continuous linear operator.

Since α is a Liouville number, there exists a sequence of rational numbers $\{p_k/q_k\}_{k=1}^\infty$ such that

$$\left| \alpha - \frac{p_k}{q_k} \right| \leq \frac{1}{q_k^k}, \quad (k = 1, 2, 3, \dots).$$

This inequality gives

$$0 < |\sin(\pi q_k \alpha)| = |\sin \pi(q_k \alpha - p_k)| \leq \pi |q_k \alpha - p_k| \leq \frac{\pi}{q_k^{k-1}}.$$

Hence

$$\left| (\sin(\pi q_k \alpha))^{-1} \right| > \frac{q_k^{k-1}}{\pi}.$$

We take the sequence $\{f_k\}_{k=1}^\infty$ in $C^\infty(\mathbb{R}^n)$ defined by

$$f_k(x) = \frac{1}{q_k^k} e^{i\pi q_k x_1}, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n, \quad k = 1, 2, 3, \dots.$$

We note here that in general for $\phi_\eta(x) = e^{i\eta \cdot x}$ ($\eta \in \mathbb{R}^n$) we have

$$\begin{aligned} C_{S_t} \phi_\eta(x) &= \phi_\eta * S_t(x) \\ &= \int_{\mathbb{R}^n} e^{i\eta \cdot (x-y)} dS_t(y) = e^{i\eta \cdot x} \int_{\mathbb{R}^n} e^{-i\eta \cdot y} dS_t(y) \\ &= \phi_\eta(x) \widehat{S}_t(\eta) = \phi_\eta(x) \frac{\sin(t|\eta|)}{|\eta|} \end{aligned}$$

By taking $\eta = (\pi q_k, 0, \dots, 0)$, $t = 1$, or α in the above equality, we have

$$\begin{aligned} C_{S_1} f_k(x) &= f_k * S_1(x) = \frac{\sin(\pi q_k)}{\pi q_k} f_k(x) = 0, \\ C_{S_\alpha} f_k(x) &= f_k * S_\alpha(x) = \frac{\sin(\pi q_k \alpha)}{\pi q_k} f_k(x). \end{aligned}$$

Thus $f_k \in \text{Ker} C_{S_1}$ and in addition, we have

$$(\#) \quad |C_{S_\alpha}^{-1} f_k(x)| = \left| \frac{\pi q_k}{\sin(\pi q_k \alpha)} \frac{e^{i\pi q_k x_1}}{q_k^k} \right| > 1, \quad \text{for any } x \in \mathbb{R}^n.$$

On the other hand, we see easily that $f_k \rightarrow 0$ in the topology of $C^\infty(\mathbb{R}^n)$ (and of course in the topology of $\text{Ker} C_{S_1}$).

Since $C_{S_\alpha}^{-1} : \text{Ker} C_{S_1} \rightarrow \text{Ker} C_{S_1}$ is a continuous linear operator, $C_{S_\alpha}^{-1} f_k \rightarrow 0$ in the topology of $\text{Ker} C_{S_1}$. which contradicts with the above inequality $(\#)$.

8. SUMMARY

[1] The 2-snapshot problem.

$$(\text{SP}) \quad \begin{cases} \partial_t^2 u(t, x) - \Delta u(t, x) = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\ u|_{t=0} = f_0, u|_{t=1} = f_1. \end{cases}$$

- (1) The uniqueness for the 2-snapshot problem does not hold.
- (2) However, we can determine the snapshot $u_k = u(k, \cdot)$ of the solution u at any integer time $t = k$ by the formula.

$$u_k = f_1 * \Psi_k - f_0 * \Psi_{k-1},$$

where Ψ_k ($k \in \mathbb{Z}$) is a compactly supported distribution defined by

$$\widehat{\Psi}_k(\xi) = \frac{\sin(k|\xi|)}{\sin|\xi|} = U_{k-1}(\cos|\xi|) \quad \xi \in \mathbb{R}^n.$$

[2] The 3-snapshot problem.

$$(\text{SP}) \quad \begin{cases} \partial_t^2 u(t, x) - \Delta u(t, x) = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\ u|_{t=0} = f_0, u|_{t=1} = f_1, u|_{t=\alpha} = f_\alpha. \end{cases}$$

- (1) If α is an irrational number, the uniqueness for the 3-snapshot problem holds.

- (2) In addition, if α is not a Liouville number and if f_0 , f_1 and f_α satisfy the compatibility condition

$$(f_1 - f_0 * S'_1) * S_\alpha = (f_\alpha - f_0 * S'_\alpha) * S_1.$$

then the 3-snapshot problem has a unique solution.

- (3) If α is a Liouville number, there exist f_0 , f_1 and f_α satisfying the compatibility condition such that the 3-snapshot problem with the snapshot data $\{f_0, f_1, f_\alpha\}$ does not have a solution.

9. SNAPSHOT PROBLEM FOR THE SHIFTED WAVE EQUATION ON \mathbb{S}^n

We consider the snapshot problem for the shifted wave equation on the n -dimensional sphere \mathbb{S}^n with the standard metric.

$$(SP) \quad \begin{cases} \partial_t^2 u(t, x) - \left(\Delta_{\mathbb{S}^n} - \left(\frac{n-1}{2} \right)^2 \right) u(t, x) = 0, & (t, x) \in \mathbb{R} \times \mathbb{S}^n \\ u|_{t=0} = f_0, \quad u|_{t=\alpha} = f_\alpha, \end{cases}$$

where $\Delta_{\mathbb{S}^n}$ denotes the Laplace-Beltrami operator on \mathbb{S}^n . We consider $\Delta_{\mathbb{S}^n}$ to be a non-positive operator. Therefore its j -th eigenvalue is given by $-j(j+n-1)$ ($j = 0, 1, 2, \dots$).

Here we note that in the above snapshot problem (SP) we consider the shifted wave equation

$$\partial_t^2 u(t, x) - \left(\Delta_{\mathbb{S}^n} - \left(\frac{n-1}{2} \right)^2 \right) u(t, x) = 0,$$

instead of the standard wave equation. In fact, from the point of view of harmonic analysis on symmetric spaces, it is more natural to deal with the wave equation with the so-called ρ -shift. See, for example, Helgason [Hel92], Olafsson and Schlichtkrull [OS92].

We assume that $f_0, f_\alpha \in C^\infty(\mathbb{S}^n)$ and that for simplicity $\alpha > 0$.

Let S_t be the fundamental solution of the shifted wave equation on \mathbb{S}^n .

Namely, let S_t be the solution of

$$\begin{cases} \partial_t^2 u(t, x) - \left(\Delta_{\mathbb{S}^n} - \left(\frac{n-1}{2} \right)^2 \right) u(t, x) = 0, & (t, x) \in \mathbb{R} \times \mathbb{S}^n \\ u|_{t=0} = 0, \quad \partial u|_{t=0} = \delta_o, \end{cases}$$

where δ_o denotes the Dirac delta function on \mathbb{S}^n whose support is $o = (1, 0, \dots, 0) \in \mathbb{S}^n$.

We remark that as in the Euclidean space convolution on \mathbb{S}^n is defined in a similar way. So we sometimes write $f * g$ for functions f and g on \mathbb{S}^n . But we skip the definition of the convolution on \mathbb{S}^n .

Let us state when the above 2-snapshot problem has a unique solution.

Theorem 9.1 ([CGKW24]). *The 2-snapshot problem (SP) with the snapshot data $\{f_0, f_\alpha\}$ has a unique solution if and only if the following condition holds:*

$$(\diamond) \quad f_\alpha - f_0 * S'_\alpha \in \text{Image of } \{C_{S_\alpha} : C^\infty(\mathbb{S}^n) \rightarrow C^\infty(\mathbb{S}^n)\},$$

where C_{S_α} is the convolution operator on $C^\infty(\mathbb{S}^n)$ defined by $C_{S_\alpha} f = f * S_\alpha$

Next, we state when the condition (\diamond) holds.

Theorem 9.2 ([CGKW24]). *Let $\beta = \alpha/\pi$*

- (i) If n is odd, Condition (\diamond) holds for all pairs $\{f_0, f_\alpha\}$ if and only if β is irrational and not a Liouville number.
- (ii)] If n is even, Condition (\diamond) holds for all pairs $\{f_0, f_\alpha\}$ if and only if β is one of the following numbers:
- β is a rational number p/q in lowest terms with p odd, or
 - β is irrational and $\beta/2$ is not a Liouville number of odd type.

The above theorem shows that the 2-snapshot problem with the snapshot data $\{f_0, f_\alpha\}$ satisfying some suitable conditions has a unique solution. In this respect, there is a big difference between the snapshot problem for the wave equation on \mathbb{R}^n and that for the shifted wave equation on \mathbb{S}^n . Roughly speaking, this is due to the fact that the solution of the Cauchy problem for the shifted wave equation on \mathbb{S}^n is 2π -periodic in t if n is odd and 4π -periodic in t if n is even. More precisely, if $u(t, x)$ is a solution to the Cauchy problem (CP) with the Cauchy data $\{f_0, g_0\}$

$$(CP) \quad \begin{cases} \partial_t^2 u(t, x) - \left(\Delta_{\mathbb{S}^n} - \left(\frac{n-1}{2} \right)^2 \right) u(t, x) = 0, & (t, x) \in \mathbb{R} \times \mathbb{S}^n \\ u|_{t=0} = f_0, \quad \partial_t u|_{t=0} = g_0, \end{cases}$$

then we have $u(t + 2\pi, x) = u(t, x)$ if n is odd and $u(t + 4\pi, x) = u(t, x)$ if n is even. In particular, we have $u(2\pi, x) = f_0(x)$ if n is odd and $u(4\pi, x) = f_0(x)$ if n is even. Therefore, in the case of \mathbb{S}^n , solving the 2-snapshot problem with the snapshot data $u|_{t=0} = f_0, u|_{t=\alpha} = f_\alpha$ is equivalent to solving the 3-snapshot problem with the snapshot data $u|_{t=0} = f_0, u|_{t=2\pi} = f_0, u|_{t=\alpha} = f_\alpha$ if n is odd or with the snapshot data $u|_{t=0} = f_0, u|_{t=4\pi} = f_0, u|_{t=\alpha} = f_\alpha$ if n is even.

10. SOME REMARKS.

In our paper [CGKW24], we also consider the snapshot problem for the shifted wave equation on noncompact symmetric spaces and obtain the analogous result to Theorem 4.2. More precisely, our setting and our result are given as follows. Let $X = G/K$ be a noncompact symmetric space and let us denote by Δ_X the Laplace-Beltrami operator on X . In addition, let ρ be the half sum of the positive restricted roots of X counted with multiplicity and let $\langle \rho, \rho \rangle$ denote the square of the length of ρ . (For the details of symmetric spaces, see Helgason [Hel01], [Hel00], and [Hel08].) In [CGKW24], we also deal with the snapshot problem for the shifted wave equation on X .

$$(SP) \quad \begin{cases} \partial_t^2 u(t, x) - (\Delta_X + \langle \rho, \rho \rangle) u(t, x) = 0, & (t, x) \in \mathbb{R} \times X, \\ u|_{t=0} = f_0, u|_{t=1} = f_1, u|_{t=\alpha} = f_\alpha. \end{cases}$$

Similarly as in (4.1), we define compactly supported distributions S_t and S'_t on X using the Fourier-Laplace transform on X . Then we have

Theorem 10.1 ([CGKW24]). *Let α be an irrational number and not a Liouville number.*

*Suppose that f_0, f_1 and $f_\alpha \in C^\infty(X)$ satisfy the compatibility condition $(f_1 - f_0 * S'_1) * S_\alpha = (f_\alpha - f_0 * S'_\alpha) * S_1$. Then the snapshot problem*

$$(SP) \quad \begin{cases} \partial_t^2 u(t, x) - \Delta u(t, x) = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\ u|_{t=0} = f_0, u|_{t=1} = f_1, u|_{t=\alpha} = f_\alpha. \end{cases}$$

has a unique solution.

The point is that the surjectivity theorem such as Theorem 7.1 holds for convolution operators on noncompact symmetric spaces. (See our papers [CGK17], [GWK21].) This theorem combined with some theorems on the Fourier transform and the horocycle Radon transform enables us to reduce the argument to the snapshot problem for the wave equation on the maximal abelian subspace corresponding to X .

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UNIVERSITY OF TSUKUBA, 1-1-1 TENNODAI, TSUKUBA, IBARAKI 305-8571 JAPAN
 Email address: kakehi@math.tsukuba.ac.jp