The criterion for $\mathcal{A}$-simple singularities of plane curves and its applications

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1 Introduction

In this paper, we provide criteria for all $\mathcal{A}$-simple singularities. It is known that the $\mathcal{A}$-simple singularities of plane curve germs $\mathbb{R}, 0 \to \mathbb{R}^2, 0$ are classified as follows.

Fact 1.1 ([1, Theorem 3.8])**.** The following are representatives of the $\mathcal{A}$-simple germs $\phi : \mathbb{R}, 0 \longrightarrow \mathbb{R}^2, 0$:

Type	Normal form of ϕ
A_{2k}	(t^2, t^{2k+1})
E_{6k}	$(t^3, t^{3k+1} + \varepsilon_p t^{3(k+p)+2})$, $0 \le p \le k-2$; (t^3, t^{3k+1})
E_{6k+2}	$(t^3, t^{3k+2} + \varepsilon_{p+1} t^{3(k+p)+4})$, $0 \le p \le k-2$; (t^3, t^{3k+2})
W_{12}	$(t^4, t^5 \pm t^7)$, (t^4, t^5)
$W^{\#}_{1,2q-1}$	$(t^4, t^6 + t^{2q+5})$, $q \ge 1$
W_{18}	$(t^4, t^7 \pm t^9)$, $(t^4, t^7 \pm t^{13})$, (t^4, t^7)

where ε_p is 1 if p is even; ± 1 if p is odd.

Among these, necessary and sufficient criteria are already known for A_2, E_6 (see [6, Theorem 1.3.2, Theorem 1.3.4]), A_4 (see [8, Theorem 1.23]), A_6 (see [5, Theorem A.1]), W_{12} (see [7, Theorem 4.14]). In this paper, we present a unified method that provides criteria for all $\mathcal{A}$-simple singularities.

To this end, we first introduce the curvature parameter and establish a fundamental theorem for curves of finite multiplicity. We then derive criteria for all $\mathcal{A}$-simple singularities.

Finally, as an application of our criteria, we describe parallel curves of regular points and $\mathcal{A}$-simple singularities degenerate into $\mathcal{A}$-simple singularities. In general, the parallel curve of a regular curve develops singularities when it reaches the radius of curvature. It is known that a non-vertex point degenerates into an A_2 singularity, while a first-order vertex degenerates into an E_6 singularity. Using our criteria, we show that a second-order vertex degenerates into a W_{12} singularity. Furthermore, for other $\mathcal{A}$-simple singularities, the parallel curve has the same type of singularity at corresponding points, but degenerates at a certain distance. In most cases, this distance coincides with the radius of curvature of a regular approximation; however, for E_{12} and E_{14} singularities, we show that a different type of degeneration, called equi-multiple degeneration, occurs.

This paper is a summary of joint work with Toshizumi Fukui, presented in [4].

2 Curvature parameter

In this section, we restrict our discussion to plane curves; however, similar arguments can be developed for curves in $\mathbb{R}^n$. See [3] for details.

First, in order to fix notation, we review the case of regular plane curves.

Review 2.1. Let C^∞-map $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a regular curve.

(1) There exists a C^∞ reparameterization $s = s(t)$ such that s is an arc length parameter. Moreover,

$$\frac{d\gamma}{ds}(s) = \boldsymbol{t}(s),$$

where $\boldsymbol{t}$ is the unit tangent vector.

(2) We define the curvature $\hat{\kappa}$ of $\gamma(s)$ by the following differential equation:

$$\frac{d}{ds}\begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n} \end{pmatrix} = \begin{pmatrix} 0 & \hat{\kappa} \\ -\hat{\kappa} & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n} \end{pmatrix}, \tag{2.1}$$

where $\hat{\kappa} = \hat{\kappa}(s)$ is a C^∞ function and $\boldsymbol{n}$ is the unit normal vector.

Next, we introduce the notion of curvature parameter for an irreducible curve germ in $\boldsymbol{R}^2, 0$. To this end, we first define the multiplicity of a map.

Definition 2.2. (Multiplicity) We say that a C^∞-map $\phi : \boldsymbol{R}, 0 \longrightarrow \boldsymbol{R}^2, 0$ is **of multiplicity** m at $t = 0$, if there exists a C^∞-map $\hat{\phi} : \boldsymbol{R}, 0 \longrightarrow \boldsymbol{R}^2, 0$ so that

$$\phi(t) = \frac{t^m}{m!}\hat{\phi}(t), \quad \hat{\phi}(0) \neq 0.$$

Lemma 2.3. Let a C^∞-map $\phi_m : \mathbb{R}, 0 \to \mathbb{R}^2, 0$ be of multiplicity m at $t = 0$.

(1) There exists a C^∞ reparameter $s = s(t)$ such that $\pm s^m/m!$ is an arc length parameter. Moreover, there exists a unit vector field $\boldsymbol{t}$ along the curve ϕ_m such that
$$\frac{d\phi}{ds} = \frac{s^{m-1}}{(m-1)!}\boldsymbol{t}.$$
We call s the **curvature parameter** of ϕ_m.

(2) The curvature κ of $\phi_m(t)$ is given by
$$\kappa(t) = \frac{(m-1)!}{t^{m-1}}\hat{\kappa}(t)$$
where t is a general parameter and $\hat{\kappa}$ is the same function as in (2.1).

Proof. (1) can be proved in the same way as the construction of the arc length parameter in the regular case. (2) follows by a direct computation. □

Thus, the curvature parameter can be regarded as a natural extension of the arc length parameter to singular curves. In particular, by differentiating a curve parametrized by the curvature parameter with respect to s, one obtains the unit tangent vector $\boldsymbol{t}$ and the unit normal $\boldsymbol{n}$. Moreover, (2) shows that the curvature of a singular curve can be expressed in terms of the curvature $\hat{\kappa}$ of a regular curve sharing the same frame.

By using the above results, we can establish the following fundamental theorem for plane curves of multiplicity m.

Theorem 2.4. Let $\hat{\kappa}(s)$ be a C^∞ function. Then there exists a plane curve of multiplicity m whose curvature parameter is s and whose curvature is given by $\dfrac{(m-1)!}{s^{m-1}}\hat{\kappa}(s)$. Moreover, such a curve is uniquely determined up to rotations and translations.

Proof. As in the regular case, the existence of a unit tangent vector $\boldsymbol{t}(s)$ follows from the fundamental theorem of the theory of curves applied to $\hat{\kappa}(s)$. Define $\phi_m(s)$ by

$$\phi_m(s) = \int_0^s \frac{s^{m-1}}{(m-1)!}\boldsymbol{t}\,ds. \tag{2.2}$$

Then $\phi_m(s)$ gives the desired plane curve of multiplicity m. □

3 Curvature Criteria of $\mathcal{A}$-simple singularities

First, we express $\hat{\kappa}(s)$ as the following power series:

$$\hat{\kappa}(s) = \sum_{i \geq 0} \hat{\kappa}_i \frac{s^i}{i!} = \hat{\kappa}_0 + \hat{\kappa}_1 s + \hat{\kappa}_2 \frac{s^2}{2!} + \hat{\kappa}_3 \frac{s^3}{3!} + \cdots .$$

Next, by solving the differential equation (2.1), we obtain $\boldsymbol{t}(s)$. Then, we construct a singular curve $\phi_2(s)$ of multiplicity 2 with an arbitrary curvature by (2.2). Finally, by applying suitable coordinate changes to $\phi_2(s)$ and transforming it into the normal form of an A_{2k} singularity, we obtain the following criterion.

Remark 3.1. In the case $m = 1$, the expression for $\phi_1(s)$ coincides with Bouquet's formula.

Theorem 3.2. Let s be the curvature parameter. Then a plane curve $\gamma(s)$ is $\mathcal{A}$-equivalent to an A_{2k} singularity if and only if it is of multiplicity 2 and satisfies

$$\hat{\kappa}_0 = \hat{\kappa}_2 = \cdots = \hat{\kappa}_{2k-4} = 0 \quad \text{and} \quad \hat{\kappa}_{2k-2} \neq 0.$$

Example 3.3. $\gamma(t) = (t^4, t^2 + t^5)$ is $\mathcal{A}$-equivalent to $A_6 : (t^2, t^7)$.

Proof. By computing the arc length and the curvature, we obtain

$$L(t) = \int_0^t \sqrt{(2t + 5t^4)^2 + 16t^6}\, dt = t^2 + t^5 + o(t^5)$$

$$\kappa(t) = \frac{4t^3(5t^3 - 4)}{((2t + 5t^4)^2 + 16t^6)^{3/2}} = -2 + \frac{35}{2}t^3 + 12t^4 + o(t^4).$$

Next, we determine a reparameterization such that

$$L(t(s)) = \frac{s^2}{2!} + o(s^5)$$

For instance, we set

$$t(s) = t_0 s \left(1 + t_1 \frac{s}{2} + t_2 \frac{s^2}{3!} + t_3 \frac{s^3}{4!} + \cdots \right) \quad (t_0 \neq 0)$$

and determine the coefficients $(t_0), t_1, t_2, \ldots$ successively. In this case, we obtain

$$\left(t_0^2 = \frac{1}{2}\right), \ t_1 = t_2 = 0, \ t_3 = -12t_0^3, \ \ldots$$

Hence,

$$\kappa(t(s)) = \frac{1}{s}\left(-2s + 420t_0^3\frac{s^4}{4!} + 1440t_0^4\frac{s^5}{5!} + o(s^5)\right)$$

Therefore, $\hat{\kappa}_0 = \hat{\kappa}_2 = 0,\ \hat{\kappa}_4 = 420t_0^3 \neq 0$, which implies that γ is $\mathcal{A}$-equivalent to A_6. □

By carrying out computations similar to those for the A_{2k} singularities for ϕ_3 and ϕ_4, we can obtain analogous criteria for the remaining $\mathcal{A}$-simple singularities.

To describe these criteria in a concise form, we introduce the following notation:

$$\theta(s) = \int_0^s \hat{\kappa}(u)\ du, \quad \theta(s) = \sum_{i\geq 1} \theta_i \frac{s^i}{i!} = \theta_1 s + \theta_2\frac{s^2}{2!} + \theta_3\frac{s^3}{3!} + \cdots ,$$

In other words $\theta_{i+1} = \hat{\kappa}_i$.

Theorem 3.4. $\phi_m(s)$ defines

$$\begin{array}{ll}
\text{(i) } A_{2k} & \Longleftrightarrow m = 2,\ \theta_i = 0,\ (i \not\equiv 0 \mod 2,\ i < 2k-1);\ \theta_{2k-1} \neq 0, \\
\text{(ii}-1\text{) } E_{6k} & \Longleftrightarrow m = 3,\ \theta_i = 0,\ (i \not\equiv 0 \mod 3,\ i < 3k-2);\ \theta_{3k-2} \neq 0, \\
\text{(ii}-2\text{) } E_{6k+2} & \Longleftrightarrow m = 3,\ \theta_i = 0,\ (i \not\equiv 0 \mod 3,\ i < 3k-1);\ \theta_{3k-1} \neq 0, \\
\text{(iii}-1\text{) } W_{12} & \Longleftrightarrow m = 4,\ \theta_1 \neq 0, \\
\text{(iii}-2\text{) } W^{\#}_{1,2q-1} & \Longleftrightarrow m = 4,\ \theta_1 = 0,\ \theta_2 \neq 0,\ \theta_i = 0, \\
 & \qquad\qquad\qquad (i \not\equiv 0 \mod 2,\ i < 2q+1),\ \theta_{2q+1} \neq 0, \\
\text{(iii}-3\text{) } W_{18} & \Longleftrightarrow m = 4,\ \theta_1 = \theta_2 = 0,\ \theta_3 \neq 0.
\end{array}$$

Furthermore, by solving (2.1) for $\boldsymbol{n}(s)$, we immediately obtain the following corollary.

Corollary 3.5. $\phi_m(s)$ defines a front singularity if and only if $m \geq 2$ and $\theta_1 \neq 0$.

Proof. A curve $\gamma(s)$ is a front singularity if and only if $\boldsymbol{n}'(s) \neq 0$, which is equivalent to $\theta_1 \neq 0$. Because $\boldsymbol{n}(s) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} \theta_1 \\ 0 \end{pmatrix} s + \cdots$. □

4 Parallel Curves

We consider the parallel curves of $\phi_m(t)$, (2.2), defined by $\phi_m^\delta(t) = \phi_m(t) + \delta\boldsymbol{n}$ where δ is a non-zero constant. We assume that t is a curvature parameter of ϕ_m.

Lemma 4.1. The multiplicity m^δ of the parallel $\phi_m^\delta(t)$ at $t=0$ is given by the following:

$$m^\delta = \begin{cases} \text{ord } \theta, & \text{if ord } \theta < m; \\ m, & \text{if ord } \theta > m, \text{ or ord } \theta = m, \ \delta \neq \theta_m^{-1}; \\ \min\{i : \theta_i \neq 0, i > \text{ord } \theta\}, & \text{if ord } \theta = m, \ \delta = \theta_m^{-1}. \end{cases}$$

Proof. This is a consequence of the following computation:

$$\begin{aligned}(\phi_m^\delta)'(t) =& \Big(\frac{t^{m-1}}{(m-1)!} - \delta\kappa\Big)\boldsymbol{t} \\ =& \Big(\frac{t^{m-1}}{(m-1)!} - \delta \sum_{i \geq \text{ord } \theta} \theta_i \frac{t^{i-1}}{(i-1)!}\Big)\boldsymbol{t} \\ =& \Big((1-\delta\theta_m)\frac{t^{m-1}}{(m-1)!} - \delta \sum_{i \geq \text{ord } \theta, \ i \neq m} \theta_i \frac{t^{i-1}}{(i-1)!}\Big)\boldsymbol{t}.\end{aligned}$$

□

Example 4.2 (Conditions of degeneracy)**.** .

(1) In the case $m = 1$, the parallel curve of a regular curve at distance δ develops a singularity if and only if $\delta = 1/\theta_1$.

(2) In the case $m = 2$, the singularity further degenerates if and only if $\theta_1 = 0$ and $\delta = 1/\theta_2$.

(3) In the case $m = 3$, the singularity further degenerates if and only if $\theta_1 = \theta_2 = 0$ and $\delta = 1/\theta_3$.

(4) In the case $m = 4$, the singularity further degenerates if and only if $\theta_1 = \theta_2 = \theta_3 = 0$ and $\delta = 1/\theta_4$.

Under these conditions, by computing the curvature of the parallel curve, we obtain the degeneracy behavior of singularities.

4.1 The case of multiplicity 1

Theorem 4.3. The parallel curve at distance δ becomes singular when $\delta = 1/\theta_1$. Moreover, under this condition, the following hold:

(1) If the curve has no vertex (i.e. $\theta_2 \neq 0$) at the point under consideration, then the parallel curve degenerates to one having an $A_2 : (t^2, t^3)$ singularity.

(2) If the curve has a first-order vertex (i.e. $\theta_2 = 0,\ \theta_3 \neq 0$) at the point under consideration, then the parallel curve degenerates to one having an $E_6 : (t^3, t^4)$ singularity.

(3) If the curve has a second-order vertex (i.e. $\theta_2 = \theta_3 = 0,\ \theta_4 \neq 0$) at the point under consideration, then the parallel curve degenerates to one having a $W_{12} : (t^4, t^5),\ (t^4, t^5 \pm t^7)$ singularity.

(4) In general, if the curve has a $(n-2)$-th order vertex at the point under consideration, then the parallel curve degenerates to one having a front singularity of multiplicity n.

Remark 4.4. The cases of A_2 and E_6 have already been treated in [2] and [8], respectively. The results from W_{12} onward appear to be new.

4.2 The case of multiplicity 2

Theorem 4.5. In the case of multiplicity 2, the parallel curve degenerates when $\theta_1 = 0$ and $\delta = 1/\theta_2$. Moreover, under this condition, the parallel curve degenerates as follows:

(1) If the curve has an $A_4 : (t^2, t^5)$ singularity, then the parallel curve degenerates to one having an $E_8 : (t^3, t^5)$ singularity.

(2) Let $k \geq 3$. If the curve has an $A_{2k} : (t^2, t^{2k+1})$ singularity and $\theta_4 \neq 0$, then the parallel curve degenerates to one having a $W^{\#}_{1,2k-5} : (t^4, t^6 + t^{2k+1})$ singularity.

(3) Let $k \geq 3$. If the curve has an $A_{2k} : (t^2, t^{2k+1})$ singularity and $\theta_4 = 0$, then the parallel curve degenerates to one having a singularity of multiplicity at least 5 (that is, a non-$\mathcal{A}$-simple singularity).

Corollary 4.6. Under the same assumptions as above, let $0 \leq p \leq k-3$. Then, for A_{2k}, the multiplicity m^δ satisfies

$$m^\delta = \begin{cases} 2p+4, & \text{if } \theta_{2q+4} = 0\ (q = 0, 1, \ldots, p-1) \text{ and } \theta_{2p+4} \neq 0; \\ 2k-1, & \text{if } \theta_{2q+4} = 0\ (0 \leq q \leq k-3). \end{cases}$$

This theorem shows that the type of an A_{2k}-singularity is determined by the coefficients θ_i with $i \not\equiv 0 \mod 2$, whereas the multiplicity m^δ is determined by those with $i \equiv 0 \mod 2$.

4.3 The case of multiplicity 3

Theorem 4.7. In the case of multiplicity 3, the parallel curve degenerates when $\theta_1 = \theta_2 = 0$ and $\delta = 1/\theta_3$. Moreover, under this condition, the parallel curve degenerates as follows:

(1) If the curve has an $E_8 : (t^3, t^5)$ singularity, then the parallel curve degenerates to one having a $W_{18} : (t^4, t^7), (t^4, t^7 \pm t^9), (t^4, t^7 \pm t^{13})$ singularity.

(2) Let $k \geq 2$. If the curve has an E_{6k} or E_{6k+2} singularity, then the parallel curve degenerates to one having a singularity of multiplicity at least 5 (that is, a non-$\mathcal{A}$-simple singularity).

Corollary 4.8. Under the same assumptions as above, let $0 \leq p \leq k-3$. Then, for E_{6k} (resp. E_{6k+2}), the multiplicity m^δ satisfies

$$m^\delta = \begin{cases} 3p+6, & \text{if } \theta_{3q+6} = 0 \ (q = 0, 1, \cdots, p-1), \\ & \qquad\qquad \text{and } \theta_{3q+6} \neq 0 \ (q = p); \\ 3k-2 \ (\text{resp. } 3k-1), & \text{if } \theta_{3q+6} = 0 \ (0 \leq q \leq k-3). \end{cases}$$

This theorem shows that the type of E type singularity is determined by the coefficients θ_i with $i \not\equiv 0 \mod 3$, whereas the multiplicity m^δ is determined by those with $i \equiv 0 \mod 3$.

In contrast to the degenerations considered so far, it turns out that a degeneration in which the type of singularity changes while preserving multiplicity 3 occurs only in the following cases.

Theorem 4.9. Let $k \geq 2$. Then the following hold:

(1) In the case of an E_{6k} singularity:
If the curve has a $(t^3, t^{3k+1} + \varepsilon_{k-2} t^{6k-4})$ singularity, then the parallel curve degenerates to one having (t^3, t^{3k+1}) singularity, where

$$\delta^{-1} = \theta_3 - \frac{2(6k-7)!}{(3k-2)!(3k-3)!} \frac{\theta_{3k-2}^2}{\theta_{6k-7}}.$$

(2) In the case of an E_{6k+2} singularity:
If the curve has a $(t^3, t^{3k+2} + \varepsilon_{k-1} t^{6k-2})$ singularity, then the parallel curve degenerates to one having (t^3, t^{3k+2}) singularity, where

$$\delta^{-1} = \theta_3 - \frac{2(6k-5)!}{(3k-1)!(3k-2)!} \frac{\theta_{3k-1}^2}{\theta_{6k-5}}.$$

We call this special type of degeneration an **equi-multiple degeneration**.

Example 4.10. In the case $k = 2$.

(1) For $E_{12} : (t^3, t^7 + t^8),\ (t^3, t^7)$ singularity, if the curve has a $(t^3, t^7 + t^8)$ singularity, then the parallel curve degenerates to one having (t^3, t^7) singularity, where
$$\delta^{-1} = \theta_3 - \frac{5}{3}\frac{\theta_4^2}{\theta_5}.$$

(2) For $E_{14} : (t^3, t^8 \pm t^{10}),\ (t^3, t^8)$ singularity, if the curve has a $(t^3, t^8 \pm t^{10})$ singularity, then the parallel curve degenerates to one having (t^3, t^8) singularity, where
$$\delta^{-1} = \theta_3 - \frac{7}{2}\frac{\theta_5^2}{\theta_7}.$$

4.4 The case of multiplicity 4

Theorem 4.11. For singularities W_{12}, $W^{\#}_{1,2q-1}$, and W_{18}, the parallel curves do not exhibit any degeneration of singularities, including those to non-$\mathcal{A}$-simple singularities.

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