Relative version of Herbert's multiple-point formula

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1 Introduction

Multiple points of differentiable maps are classical objects. Starting with the work of H. Whitney, they have been deeply studied by many researchers and from many geometric aspects. One of classical works is R. J. Herbert's cohomological enumerative formula for multiple-point loci of generic immersions [Her75]. In this report, we show a relative version of Herbert's formula and give an application to Ekholm–Szűcs' formula for Ekholm's Vassiliev-type linking invariant for generic immersions [Ekh01, ES03].

Here are some remarks.

(1) A special case and an essential observation for the relative version of Herbert's formula already appeared in the work of Ekholm–Szűcs [ES03]. This report contributes in that a general statement and its precise proof are given, and that the linking invariant formula is generalized with simplifying its proof.

(2) Ekholm's linking invariant has recently been applied to the topology of complex surface singularities in $\mathbb{C}^3$ and associated immersions [PT23, PS24, GP24], and also to a formula for the Haefliger invariant of high-codimensional knots [GT25].

(3) The present study is a part of the author's challenge to establish a relative version of Thom polynomial theory. The relative version of Herbert's formula is a candidate of relative Thom polynomials for real multi-singularities. See [Tan26] for the detail.

Organization

The rest of this report is organized as follows. In §2, we briefly recall some notions of immersions and Herbert's formula. In §3, we give the precise statement and proof of the relative version of Herbert's multiple-point formula. In §4, we apply the formula to Ekholm's linking invariant.

Throughout this report, all manifolds and maps are assumed to be of class C^∞. The boundary of an oriented manifold is oriented by the outward normal first convention.

Acknowledgement

This work was supported by JSPS KAKENHI Grant Number 25KJ0480.

2 Herbert's formula

Let M be a compact m-manifold, N an n-manifold, and $f\colon M \looparrowright N$ a generic (i.e., self-transverse) immersion. Let $r = n - m$ denote the codimension of f.

For an integer $k \geqslant 1$, we define the *multiple-point loci* of f by

$$\begin{aligned} N_k &= N_k(f) = \{y \in N \mid \# f^{-1}(y) = k\} \subset N, \\ M_k &= M_k(f) = f^{-1}(N_k) \qquad\qquad\qquad \subset M. \end{aligned}$$

Then their closures $\overline{N_k} \subset N$, $\overline{M_k} \subset M$ are locally of the forms of the union of some normally crossing linear subspaces. Hence, they admit the Whitney stratifications

$$\overline{N_k} = \bigcup_{i=k}^{\infty} N_i, \quad \overline{M_k} = \bigcup_{i=k}^{\infty} M_i$$

and forms Whitney stratified cycles in the sense of Goresky [Gor81]. Note that their normal bundles consist of copies of the fiber of the normal bundle ν_f of f.

Then we have two types of cohomology classes as follows. Here, the coefficient ring is chosen to be $\mathbb{Z}$ if ν_f is oriented and r is even, or $\mathbb{Z}_2$ otherwise.

- The *multiple-point loci classes*:

$$n_k = \mathrm{Dual}[\overline{N_k}] \in H^{kr}(N), \quad m_k = \mathrm{Dual}[\overline{M_k}] \in H^{(k-1)r}(M).$$

- The *normal Euler class* of f:

$$e(\nu_f) = \mathrm{Dual}[s^{-1}(M)] \in H^r(M),$$

  where $s\colon M \to \nu_f$ is a section which is transverse to the zero section. This is independent of the choice of s.

In terms of these classes, Herbert's formula is stated as follows.

Theorem 2.1 ([Her75]). *Let $k \geqslant 1$ be an integer. Then*

$$m_{k+1} = f^* n_k - e(\nu_f) \smile m_k \in H^{kr}(M).$$

Remark 2.2. See [Ron80, LS07] for alternative proofs of Theorem 2.1. See also [Ron73, Kle77, Ron84, Rim02, Ohm24] for a generalization for singular maps and complex analytic case.

The following is immediate by solving the above recursive equation.

Corollary 2.3. *Let $k \geqslant 1$ be an integer. Then*

$$m_{k+1} = f^* n_k + (-1)^k e(\nu_f)^k \in H^{kr}(M).$$

3 Relative version of Herbert's formula

Again, let M be a compact m-manifold, N an n-manifold, and $f\colon M \looparrowright N$ a generic (i.e., self-transverse) immersion. Let $r = n - m$ denote the codimension of f. Under additional assumptions on the boundary, the basic cohomology classes in the previous section are upgraded into relative versions as follows.

First, we assume that f satisfies that $f^{-1}(\partial N) = \partial M$, and put

$$f_\partial = f|_{\partial M}^{\partial N}\colon \partial M \looparrowright \partial N.$$

We also assume that f is transverse to ∂N, i.e., the restriction of f to a collar neighborhood of ∂M is of the form $f_\partial \times \mathrm{id}_{[0,\varepsilon]}$. Then we obtain the following.

- If $M_{k+1}(f_\partial) = \varnothing$, then the multiple-point loci of f are disjoint from the boundary. Therefore, we obtain relative classes:

$$n_k^{\mathrm{rel}} = \mathrm{Dual}[\overline{N_k}] \in H^{kr}(N, \partial N), \quad m_k^{\mathrm{rel}} = \mathrm{Dual}[\overline{M_k}] \in H^{(k-1)r}(M, \partial M).$$

- If f_∂ admits a nowhere-vanishing normal vector field θ, then the normal Euler class of f *relative to* θ is defined:

$$e(\nu_f|\theta) = \mathrm{Dual}[s^{-1}(M)] \in H^r(M, \partial M),$$

  where $s\colon M \to \nu_f$ is a section which is transverse to the zero section and $s|_{\partial M} = \theta$. This is independent of the choice of s.

Then a relative version of Theorem 2.1 is stated as follows.

Theorem 3.1. *Let $k \geqslant 2$ be an integer. Assume that $M_{k+1}(f_\partial) = \varnothing$ and that f_∂ admits a nowhere-vanishing normal vector field θ. Then*

$$m_{k+1}^{rel} = f^* n_k^{rel} - e(\nu_f|\theta) \smile m_k^{rel} \in H^{kr}(M, \partial M).$$

We also have the following modified version.

Theorem 3.2. *Let $k \geqslant 1$ be an integer. Assume that $M_k(f_\partial) = \varnothing$ and that f_∂ admits a nowhere-vanishing normal vector field θ. Then*

$$m_{k+1}^{rel} = \mathrm{Dual}[g^{-1}(\overline{N_k})] - e(\nu_f|\theta) \smile m_k \in H^{kr}(M, \partial M),$$

where $g\colon (M, \partial M) \looparrowright (N, \partial N)$ is a perturbation of f which is obtained by pushing f into the θ-direction near ∂M.

Note that although m_k is an absolute cohomology class, the product $e(\nu_f|\theta) \smile m_k$ is a relative one. We also note that Theorems 3.1 and 3.2 hold even in the case relative to a submanifold $S \subset M$ which does not meet ∂M.

Before proofs of these theorems, we see an immediate consequence. The following is the relative version of Corollary 2.3.

Corollary 3.3. *Under the same setup in Theorem 3.2,*

$$m_{k+1}^{rel} = \mathrm{Dual}[g^{-1}(\overline{N_k})] + (-1)^k e(\nu_f|\theta)^k \in H^{kr}(M, \partial M).$$

Proof. By Theorem 3.2, we have

$$m_{k+1}^{\mathrm{rel}} = \mathrm{Dual}[g^{-1}(\overline{N_k})] - e(\nu_f|\theta) \smile m_k.$$

On the other hand, by Theorem 2.1, we have

$$m_k = (-1)^{k-1} e(\nu_f)^{k-1}.$$

Therefore,

$$\begin{aligned} e(\nu_f|\theta) \smile m_k &= e(\nu_f|\theta) \smile (-1)^{k-1} e(\nu_f)^{k-1} \\ &= e(\nu_f|\theta) \smile (-1)^{k-1} e(\nu_f|\theta)^{k-1} \\ &= (-1)^{k-1} e(\nu_f|\theta)^k. \end{aligned}$$

□

We give the proof of Theorems 3.1 and 3.2, which are due to Herbert's original one and Ekholm–Szűcs' argument [Her75, ES03].

3.1 Proof of Theorem 3.1

Fix a Riemannian metric d_2 on N and induce d_1 on M by the immersion f. Let $B_1(x; r)$ denote the open ball in M centered at x and of radius $r > 0$ with respect to d_1. The notation is similar for d_2.

Since M is compact and f is an immersion, there is a positive number $\eta > 0$ such that for any $x \in M$, f is an embedding on the open ball $B_1(x; 3\eta)$. Furthermore, there is another positive number $\delta > 0$ such that for any $x, x' \in M$, if $d_2(f(x), f(x')) < \delta$, then it holds either $d_1(x, x') < \eta$ or $d_1(x, x') > 2\eta$. Hereafter, we fix such numbers.

We choose the normal bundle ν_f to $f\colon M \looparrowright N$ so that it is compatible with the immersion $f_\partial = f|_{\partial M}^{\partial N}\colon \partial M \looparrowright \partial N$. Let s_0 denote the zero-section of ν_f and $\bar{f}\colon \nu_f \to TN$ the natural bundle homomorphism. We take a tubular neighborhood T of $s_0(M)$ in $E(\nu_f)$ so that the map $\exp = \exp_N \circ \bar{f}|_T\colon T \to N$ is an immersion which is an embedding on $T \cap E(\nu_f|_{B_1(p;\eta)})$ for each $p \in M$. We also choose a section $s\colon M \to T$ so that

(1) s is transverse to $s_0(M)$;

(2) $s|_{\partial M \cup S}$ coincides with the given normal vector field θ up to scaling;

(3) the map $g = \exp \circ s\colon M \to N$ is an immersion and transverse to N_i for any $1 \leqslant i \leqslant k$, and satisfies that $d_2(f(x), g(x)) < \delta$ for any $x \in M$.

In the following, we will prove the formula showing

$$g^* n_k^{\mathrm{rel}} = e(\nu_f|\theta) \smile m_k^{\mathrm{rel}} + m_{k+1}^{\mathrm{rel}}.$$

First, set

$$M_{k+1}^g = g^{-1}(N_k) = \{x \in M \mid g(x) = f(x_1) \text{ for some } x_1 \in M_k\}.$$

Since $\overline{N_k}$ does not meet ∂N and g is transverse to any stratum of $\overline{N_k}$, the closure $\overline{M_{k+1}^g} = g^{-1}(\overline{N_k})$ is also a stratified cycle and carries the class $g^* n_k^{\mathrm{rel}} \in H^*(M, \partial M)$ as its dual. We also parameterize $\overline{M_{k+1}^g}$ by the manifold

$$\widetilde{M}_{k+1}^g = \{(x, [x_1, \dots, x_k]) \in M \times \widetilde{N}_k \mid g(x) = f(x_1)\}$$

and the projection $\pi\colon M \times \widetilde{N}_k \to M$. It is clear that $\pi(\widetilde{M}_{k+1}^g) = \overline{M_{k+1}^g}$.

Now, we consider subsets

$$O_{\mathrm{near}} = \{(x, [x_1, \dots, x_k]) \in M \times \widetilde{N}_k \mid d_1(x, x_i) < \eta \text{ for some } i\},$$
$$O_{\mathrm{far}} = \{(x, [x_1, \dots, x_k]) \in M \times \widetilde{N}_k \mid d_1(x, x_i) > 2\eta \text{ for any } i\}$$

of $M \times \widetilde{M}_k$. Obviously, these are open and disjoint. Furthermore, since g was a δ-approximation to f and by the choice of δ, we have

$$\widetilde{M}_{k+1}^g \subset O_{\mathrm{near}} \cup O_{\mathrm{far}}.$$

Putting

$$X_{\mathrm{near}} = \pi(\widetilde{M}_{k+1}^g \cap O_{\mathrm{near}}), \quad X_{\mathrm{far}} = \pi(\widetilde{M}_{k+1}^g \cap O_{\mathrm{far}}),$$

which are clopen subsets of $\overline{M_{k+1}^g}$, we have the decomposition

$$\overline{M_{k+1}^g} = X_{\mathrm{near}} \cup X_{\mathrm{far}}.$$

Then we will show that the duals of X_{near} and X_{far} are $e(\nu_f) \smile m_k^{\mathrm{rel}}$ and m_{k+1}^{rel}, respectively.

Claim 1. *One has $X_{\mathrm{near}} = s^{-1}(s_0(M)) \cap \overline{M_k}$. Moreover, each intersection $s^{-1}(s_0(M)) \cap M_i$ ($i \geqslant k$) is transverse.*

Proof. Let $x \in X_{\mathrm{near}}$. There is $[x_1, \dots, x_k] \in \widetilde{M}_k$ such that $(x, [x_1, \dots, x_k]) \in \widetilde{M}_{k+1}^g \cap O_{\mathrm{near}}$. Without loss of generality, assume that $d_1(x, x_1) < \eta$. Since $g(x) = f(x_1)$ and exp was an embedding $T \cap E(\nu_f|_{B_1(p;\eta)})$, we have that $s(x) = s_0(x_1)$ and hence $x = x_1$. This means that

$$X_{\mathrm{near}} = \{x \in M \mid g(x) = f(x), x \in \overline{M_k}\},$$

which completes the proof of the former assertion.

We show the latter assertion. Fix an integer $i \geqslant k$. Let $x \in s^{-1}(s_0(M)) \cap M_i$ and put $y = f(x) \in N_i$. Then we have distinct points $x, x_2, \dots, x_i \in f^{-1}(y)$. Since g is transverse to $N_i = f(M_i)$, we have

$$dg_x(T_x M) + df_x(T_x M_i) = T_y W.$$

Restricting this equality to $df_x(T_xM)$,

$$df_x(T_x s^{-1}(s_0(M))) + df_x(T_xM_i) = df_x(T_xM).$$

However, since df_x was injective,

$$T_x s^{-1}(s_0(M)) + T_xM_i = T_xM.$$

This means that the intersection $s^{-1}(s_0(M)) \cap M_i$ is transverse. □

By the condition (2) for s, the set $s^{-1}(s_0(M))$ carries the class $\bar{e}$ as its dual. Furthermore, $\overline{M_k}$ carries the class m_k^{rel} as its dual. Therefore, the set X_{near} carries the class $e(\nu_f) \smile m_k^{\mathrm{rel}} \in H^{kr}(M, \partial M)$ as its dual for the $\mathbb{Z}_2$-coefficient case. Let us consider the integer coefficient case, i.e., ν_f is cooriented and r is even.

Claim 2. *The set X_{near} carries the class $e(\nu_f) \smile m_k^{rel} \in H^{kr}(M, \partial M)$ as its dual without sign ambiguity.*

Proof. We show the coorientation of $X_{\mathrm{near}} \subset \overline{M_{k+1}^g} = g^{-1}(\overline{N_k})$ in M as transverse pullback coincides with that of $s^{-1}(s_0(M)) \cap \overline{M_k}$ in M as transverse intersection.

For a cooriented subspace V of a vector space W, let $\mathrm{Co}(V \subset W)$ denote its coorientation. Then for $x \in M_{k+1}^g \cap X_{\mathrm{near}}$,

$$\begin{aligned}
&\mathrm{Co}(T_xM_{k+1}^g && \subset T_xM)\\
=&\mathrm{Co}(T_{g(x)}N_{k-1} && \subset T_{g(x)}N)\\
=&\mathrm{Co}(dg_x(T_xM) \cap T_{g(x)}N_k && \subset dg_x(T_xM))\\
=&\mathrm{Co}(df_x(T_xM) \cap dg_x(T_xM) \cap T_{g(x)}N_k && \subset dg_x(T_xM))\\
=&\mathrm{Co}(df_x(T_xM) \cap dg_x(T_xM) \cap T_{g(x)}N_k && \subset df_x(T_xM))\\
=&\mathrm{Co}(df_x(T_x s^{-1}(s_0(M))) \cap df_x(T_xM_k) && \subset df_x(T_xM))\\
=&\mathrm{Co}(T_x s^{-1}(s_0(M)) \cap T_xM_k && \subset T_xM).
\end{aligned}$$

□

We also observe X_{far}.

Claim 3. *The set X_{far} is homologous to $\overline{M_{k+1}}$ as a stratified cycle of* $\mathrm{Int}\, M$.

Proof. Let $x \in X_{\mathrm{far}}$. Then there is $[x_1, \ldots, x_k] \in \widetilde{M}_k$ such that $(x, [x_1, \ldots, x_k]) \in \widetilde{M}_{k+1}^g \cap O_{\mathrm{far}}$. Since $d_1(x, x_i) > 2\eta$, the points $x, x_1, \ldots, x_k$ are distinct. This means that

$$X_{\mathrm{far}} = \left\{ x \in M \;\middle|\; \begin{array}{l} \text{there are } x_1, \ldots, x_k \in M \text{ such that} \\ g(x) = f(x_1) = \cdots = f(x_k) \\ \text{and } x, x_1, \ldots, x_k \text{ are distinct} \end{array} \right\}.$$

We define the homotopy

$$H\colon M \times I \to N; \quad H(x,t) = h_t(x) = \exp(\rho(t) \cdot s(x)),$$

between $h_0 = f$ and $h_1 = g$, where $I = [0,1]$ and $\rho\colon I \to I$ is a function such that

$$\rho([0,1/3]) = 0, \quad \rho([2/3,1]) = 1.$$

Then consider the set

$$C = \bigcup_{t\in I} \pi(\widetilde{M}^{h_t}_{k+1} \cap O_{\mathrm{far}}) \times \{t\} \subset M \times I,$$

which is a clopen subset of $H^{-1}(\overline{N_k})$. Now, we deform the stratification $f(M) = \bigcup_{i=1}^{\infty} N_i$ so that H is transverse to it. Then C forms a cobordism between X_{far} and $\overline{M_{k+1}}$. $\square$

Since $\overline{M_{k+1}}$ does not meet ∂M by assumption, the cycle X_{far} carries the class $m_{k+1} \in H^{kr}(M, \partial M)$ as its dual. This completes the proof of Theorem 3.1.

3.2 Proof of Theorem 3.2

We just mimic the above proof. Notice that the section s (the perturbation g of f) can be chosen sufficiently small so that the set M^g_{k+1} does not meet ∂M by dimensional reason. This means that the absolute class $f^* n_k \in H^{kr}(M)$ admits the lift $\mathrm{Dual}[g^{-1}(N_k)] \in H^{kr}(M, \partial M)$.

4 Application to Ekholm's linking invariant

We briefly recall Ekholm's work. See [Ekh01, §4] for details. Let $k \geqslant 1, r \geqslant 2$ be integers, V a closed $(kr-1)$-manifold, and $\iota\colon V \looparrowright \mathbb{R}^{(k+1)r-1}$ a generic immersion. First, the following holds by dimensional reason.

Lemma 4.1. *The k-fold loci $M_k(\iota) \subset V$ and $N_k(\iota) \subset \mathbb{R}^{(k+1)r-1}$ form closed submanifolds. Furthermore, the restriction of ι to $M_k(\iota)$ admits a nowhere-vanishing normal vector field.*

Definition 4.2 ([Ekh01, §4.5])**.** Let θ be a nowhere-vanishing normal vector field of the restriction of ι to $M_k(\iota)$. Define a normal vector field Θ on $N_k(\iota) \subset \mathbb{R}^{(k+1)r-1}$ by

$$\Theta(y) = \exp \circ\, \theta(x_1) + \cdots + \exp \circ\, \theta(x_k) \quad (y \in N_k(\iota), \iota^{-1}(y) = \{x_1, \cdots, x_k\}),$$

where $\exp\colon T \to \mathbb{R}^{(k+1)r-1}$ is the immersion of the tubular neighborhood of ι. Push $N_k(\iota)$ into the Θ-direction and let $N_k(\iota)^\theta$ denote its result submanifold. Then we define the *linking invariant of ι with respect to θ* to be the linking number of $\iota(V)$ and $N_k(\iota)^\theta$ in $\mathbb{R}^{(k+1)r-1}$:

$$L_k(\iota)_\theta = \mathrm{lk}_{\mathbb{R}^{(k+1)r-1}}(\iota(V), N_k(\iota)^\theta) \in R.$$

Hereafter, let R denote the ring $\mathbb{Z}$ if V is oriented and r is even, or $\mathbb{Z}_2$ otherwise. We additionally assume that $H_{r-1}(V;R) = H_r(V;R) = 0$ so we have natural isomorphisms

$$H_{r-1}(\partial E(\nu_\iota); R) \cong H_{r-1}(S^{r-1}; R) \cong R.$$

Definition 4.3 ([Ekh01, §4.6])**.** Consider the class $[\theta(M_k(\iota))] \in H_{r-1}(\partial E(\nu_\iota); R)$. Let it also denote the number in R corresponding by the above isomorphisms. Then we define the *linking invariant* of ι to be

$$L_k(\iota) = L_k(\iota)_\theta - [\theta(M_k(\iota))] \in R.$$

Lemma 4.4 ([Ekh01, Lemmata 4.15 and 4.17])**.** *The quantity $L_k(\iota)$ is independent of the choice of θ and invariant under regular homotopy through generic immersions.*

Now, we show the following.

Theorem 4.5. *Let M be a compact kr-manifold with boundary such that*

$$H_{r-1}(\partial M; R) = H_r(\partial M; R) = 0.$$

Let $f\colon (M, \partial M) \looparrowright (\mathbb{R}_+^{(k+1)r}, \mathbb{R}^{(k+1)r-1})$ be a generic immersion. Assume that $\iota := f_\partial$ admits a nowhere-vanishing normal vector field of θ. Then

$$L_k(\iota)_\theta = (k+1)\#N_{k+1}(f) - (-1)^k\langle e(\nu|\theta)^k, [M, \partial M]\rangle \in R.$$

where $\#N_{k+1}(f)$ is the algebraic number of $N_{k+1}(f)$.

Proof. First, the k-fold locus of f does not meet the boundary by dimensional reason. Applying Corollary 3.3 to f with normal frame θ, we have

$$m_{k+1} = \mathrm{Dual}[g^{-1}(\overline{N_k})] + (-1)^k e(\nu_f|\theta)^k,$$

where $g\colon (M, \partial M) \looparrowright (\mathbb{R}_+^{(k+1)r}, \mathbb{R}^{(k+1)r-1})$ is a perturbation of f into the θ-direction near ∂M. Evaluating both hand sides by the fundamental class $[M, \partial M]$,

$$(k+1) \cdot \#N_{k+1} = \langle \mathrm{Dual}[g^{-1}(\overline{N_k})], [M, \partial M]\rangle + (-1)^k\langle e(\nu_f|\theta)^k, [M, \partial M]\rangle.$$

Therefore, it suffices to show that

$$\langle \mathrm{Dual}[g^{-1}(\overline{N_k})], [M, \partial M]\rangle = L_k(\iota)_\theta.$$

One can see that

$$\begin{aligned}\langle \mathrm{Dual}[g^{-1}(\overline{N_k})], [M, \partial M]\rangle &= \#(f^\theta)^{-1}(\overline{N_k(f)}) \\ &= \mathrm{I}_{\mathbb{R}_+^{(k+1)r}}(f^\theta(M), \overline{N_k(f)}) \\ &= \mathrm{lk}_{\mathbb{R}^{(k+1)r-1}}(f^\theta(\partial M), N_k(\iota)),\end{aligned}$$

where $\mathrm{I}_X(Y, Z)$ denotes the algebraic number of the intersection of chains Y and Z in X. Now, we push $f^\theta(\partial M)$ back to $f(\partial M) = \iota(\partial M)$ and also push $N_k(\iota)$ forward to the $(-\Theta)$-direction. These pushings can be simultaneously realized by an ambient isotopy of $\mathbb{R}^{(k+1)r-1}$. Hence,

$$\begin{aligned}\langle \mathrm{Dual}[g^{-1}(\overline{N_k})], [M, \partial M]\rangle &= \mathrm{lk}_{\mathbb{R}^{(k+1)r-1}}(\iota(\partial M), N_k(\iota)^{-\theta}) \\ &= L_k(\iota)_{-\theta},\end{aligned}$$

where $N_k(\iota)^\theta$ is the result of the pushing $N_k(\iota)$. To complete the proof, we see that

$$L_k(\iota)_{-\theta} = L_k(\iota)_\theta.$$

By the assumption $H_{r-1}(V) = H_r(V) = 0$, we have

$$L_k(\iota)_{-\theta} = L_k(\iota)_\theta - ([\theta(M_k(\iota))] - [-\theta(M_k(\iota))])$$

in $H_{r-1}(S^{r-1}; R)$. Furthermore, the term

$$[\theta(M_k(\iota))] - [-\theta(M_k(\iota))] = [\theta(M_k(\iota))] - (-1)^r[\theta(M_k(\iota))].$$

vanishes. For, it is obvious in the unoriented case and r is even in the oriented case. This completes the proof. □

When $k = 2l$ and $r = 2$, Theorem 3.2 directly deduces Ekholm–Szűcs' formula. Let V be a closed oriented $(4l-1)$-manifold and $\iota\colon V \looparrowright \mathbb{R}^{4l+1}$ a generic immersion.

Lemma 4.6. *If V is 2-connected, then there always exists a nowhere-vanishing normal vector field θ of ι and it is homotopically unique. Therefore, $L_{2l}(\iota)_\theta$ is independent of the choice of θ.*

By this reason, $L_{2l}(\iota)_\theta$ will be denoted by $L_{2l}(\iota)$ below. Furthermore, one can show the following.

Lemma 4.7 ([ES03, §8.1, Claim (a) in the proof of Theorem 1.3])**.** *There is a positive integer d such that the disjoint union of d copies of ι is null-cobordant as an oriented immersion. That is, $d \cdot \iota\colon d \cdot V \looparrowright \mathbb{R}^{4l+1}$ bounds a generic immersion $f\colon M \looparrowright \mathbb{R}^{4l+2}_+$ of a connected compact oriented $4l$-manifold M.*

We have the following, combining the above and Theorem 4.5.

Corollary 4.8 ([ES03, Theorem 1.3])**.** *Let V be a 2-connected closed oriented $(4l-1)$-manifold and $\iota\colon V \looparrowright \mathbb{R}^{4l+1}$ a generic immersion. Choose a positive integer d, a connected compact oriented $4l$-manifold M, and a generic immersion $f\colon M \looparrowright \mathbb{R}^{4l+2}_+$ such that $\partial M = d \cdot V$ and f is bounded by $d \cdot \iota\colon d \cdot V \looparrowright \mathbb{R}^{4l+1}$. Then*

$$L_{2l}(\iota) = \frac{1}{d} \cdot \left\{ -(2l+1)\#N_{2l+1}(f) + \langle e(\nu_f|\theta)^{2l}, [M, \partial M]\rangle \right\} \in \mathbb{Z},$$

where θ is an arbitrary nowhere-vanishing normal vector field of ι.

Remark 4.9. In [ES03], Theorem 3.2 was used to show the invariance of the quantity

$$(2l+1)\#N_{2l+1}(f) - \langle e(\nu_f|\theta)^{2l}, [M, V]\rangle + d \cdot L_{2l}(\iota) \in \mathbb{Z}$$

up to cobordism. Then it was shown that this quantity is zero using a fact on cobordism group, and in turn, Corollary 4.8.

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