Order Functions at Singularities

Dedicated to the memory of Professor Masahiro Shiota

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This is a view on development of study of order functions of function germs at singularities. Various inequalities on order functions are the main topic. We are also concerned with some properties of function germs related to transcendence theory in later sections. We describe mainly in the category of complex analytic geometry and sometimes switch to other categories. The author apologizes for including a correction of his earlier mistake.

1. Introduction

Usually a geometric object is studied directly using metric and measure. Here, however, we study singularities by means of the ring of function germs defined on them following the manner of algebraists. The vanishing order is the most outstanding property of a function germ, of course next to the value.

In this survey, all fields, rings and algebras are assumed to be commutative with unity 1. Recalling some basic definitions concerning singularities, let us explain the notation.

Let $\mathscr{O}_{n,\xi} := \mathbb{C}\{x_1 - \xi_1, \dots, x_n - \xi_n\} = \mathbb{C}\{x - \xi\}$ denote the algebra of convergent power series centred at $\xi = (\xi_1, \dots, \xi_n)$. This is known to be Noeterian. Let $U \subset \mathbb{C}^n$ be an open subset, $\mathscr{O}_U$ the sheaf of all the germs of holomorphic functions on U. Take an ideal sheaf $\mathscr{I} \subset \mathscr{O}_U$. We assume that $\mathscr{I}$ is coherent, that is, it is locally finitely generated together with its relation sheaf. Then non-trivial locus X of $\mathscr{O}_U/\mathscr{I}$ is an analytic subset of U. The sheaf $\mathscr{O}_X = (\mathscr{O}_U/\mathscr{I})|X$ is just one of germs of the holomorphic functions on X. The pair $(X, \mathscr{O}_X)$ is called a local model of complex space. Patching such local models, we get a complex space.

A "germ" of complex space $(X_\xi, \mathscr{O}_{X,\xi})$ is called a *singularity*. The algebra $\mathscr{O}_{X,\xi} := \mathscr{O}_{n,\xi}/\mathscr{I}_{X,\xi}$ over $\mathbb{C}$ is called the analytic local algebra at $\xi \in X$. Indeed, it is a local ring which means that it has a unique maximal ideal $\mathfrak{m}_\xi := \{f \in \mathscr{O}_{X,\xi} : \tilde{f}(\xi) = 0\}$. Here $\tilde{f}$ is a representative of f defined in a neighbourhood of ξ in X (or in $\mathbb{C}^n$). We will not make a strict distinction below between a function germ (a set germ) and its representative in expressions. If $\sqrt{I_{X,\xi}} = I_{X,\xi}$, then $\mathscr{O}_{X,\xi}$ has no non-zero nilpotent element and we say that $\mathscr{O}_{X,\xi}$ is *reduced*. Replacing $\mathbb{C}$ by $\mathbb{R}$, we have the real analytic category in the same way.

In §2 and §3, we explain basic inequalities for order functions on local domains. They reflect some geometric properties of singularities and closely related to each other. In §4, we introduce the zero estimate function $\theta_\Phi : \mathbb{N} \longrightarrow [0, \infty]$ and its growth order $\alpha_\Phi \in [0, \infty]$ for a subset $\Phi = \{\Phi_1, \Phi_2, \dots, \Phi_m\}$ of an integral domain $\mathscr{O}_{X,\xi}$. It is imported from transcendental number theory and can be used to measure tameness of Φ. In §5, this is used to assert that an affine embedding of a complex manifold is fairly tame at a general point. Theorem 5.3 is such a result, *a little improved* form of [Iz11], Theorem 11.

There remained two subjects. One is Diophantine inequality for function germs, a theorem analogous to that in number theory, see [Re1], [II], [HII].. Another subject is Spallek function introduced in [Iz9], an explicit form of geometric *complementary inequality* (CI_3). There is a stupid mistake of calculus in that paper, which the author wish to correct in another opportunity.

The author is very sorry for a serious mistakes found in the proof of the theorem on local zero estimate in [Iz8], Theorem 1.2 (present Theorem 4.2), in addition to many careless mistakes. The conclusion seems to be valid and a corrected proof is posed at the end of this survey (because it is contained in a book and difficult to correct there).

This survey is rather impressionistic and subjective sometimes because of the shallow knowledge of the author and the extensive subject. It overlaps with "Basic properties of germs of analytic mappings of analytic sets and related topics" [Iz10] largely.

This paper is divided into the following sections:

§1: Introduction
§2: Basic Inequalities on Orders
§3: Other Proofs, Variants and Analogies of the Basic Inequalities
§4: Local Zero Estimates
§5: Local Complexity of an Affine Embedding of a Smooth Manifold
§6: Correction of a Former Paper on Local Zero Estimate
(of Theorem 4.5; [Iz8], Thm. 1.2)

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2. Basic Inequalities on Orders

A. Original problem. Let $(A, \mathfrak{m})$ be a Noetherian local ring. We define *(vanishing) order* of $f \in A$ by

$$\nu(f) = \nu_{\mathfrak{m}}(f) := \sup\{p|\ f \in \mathfrak{m}^p\} \in \{0\} \cup \mathbb{N} \cup \{\infty\}.$$

We may call this the *valuation associated with* $\mathfrak{m}$ as well. If $A = \mathscr{O}_{X,\xi} = \mathbb{C}^n\{x - \xi\}/\mathscr{I}_\xi$ with $\xi \in X \subset \mathbb{C}^n$ and $\nu(f) = p$ for $f \in A$, f is represented by an $\tilde{f} \in \mathbb{C}\{x - \xi\}$ whose lowest non-vanishing homogeneous term is of degree p and not by an $\tilde{f}$ with degree smaller than p.

Let us define a more geometric (or analytic) order as follows. Let T be a subset of a complex space X accumulating at ξ: $\xi \in \overline{T \setminus \{\xi\}}$. We define the *geometric order* $\mu_T(f)$ along T as the supremum of p such that

$$\exists U: \text{ a neighbourhood of } \xi,\ \forall x \in T \cap U: \ |\tilde{f}(x)| \le |x - \xi|^p.$$

By Taylor formula with remainder, for any $T \subset X$, we have always

$$\nu_{\mathfrak{m}}(f) \le \mu_T(f) \quad (\ f \in A). \tag{I_3}$$

The first problem of the author (about 50 years ago) is the following.

If the graph of a smooth function f is sufficiently flat at a point along T, does it imply that the order of f is high at that point?

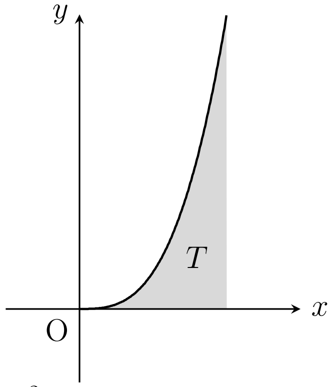
Here, flatness means quick decreasing of the absolute value when the variable approaches ξ. The following is desirable, assuming some fatness of T,

$$(\mathrm{LCI}_3) \qquad \exists a,\ \exists b > 0,\ \forall f:\ \mu_T(f) \leq a \cdot \nu(f) + b.$$

In this paper, (LCI_i) stands for a **linear complementary inequality** to the inequality (I_i) ($i = 0, 1, 2, 3$).

Example 1. We can define two kinds of orders ν and μ above for the real case similarly. Put $T = \{(x, y)|\ 0 < x,\ 0 < y < x^p\} \subset \mathbb{R}^2$ ($p > 1$) and consider the orders at the origin O. Then it is easy to show that $\nu(x) = \mu_T(x) = 1$ and $1 = \nu(y) < \mu_T(y) = p$. We can prove the following inequalities (I_3) and (LCI_3) for a general $f \in \mathbb{C}\{x\}$.

$$\overbrace{\nu(f) \leq \mu_T(f)}^{(\mathrm{I}_3)} \leq p \cdot \nu(f), \qquad \nu(f) \leq \underbrace{\mu_T(f) \leq p \cdot \nu(f)}_{(\mathrm{LCI}_3)}$$



If we replace x^p by the flat function $\exp(-1/x^2)$ in the definition of T above, we have $\nu(y) = 1 < \infty = \mu_T(y)$ and it holds no **complementary inequality** (CI_3). Note that T is open but not subanalytic in this case. Subanalyticity is assumed in Theorem 2.7. The notion of subanalytic set is brought and studied by Hironaka, Gabrièlov about 50 years ago.

At first, the author considered that generalisation of (LCI_3) to singularity is a problem of calculus or metric geometry. However, it has proved to be deep in singular case. Indeed, he needed detours of other complementary inequalities of completely algebraic natures:

$$(\mathrm{LCI}_0) + (\mathrm{LCI}_1) \Longrightarrow (\mathrm{LCI}_2) \Longrightarrow (\mathrm{LCI}_3).$$

There are obvious inequalities (I_i)s among orders. Usually we have their *complementary inequalities* (CI_i)s under some conditions. As a whole:

(CI_i) corresponds to some nice property and (LCI_i), nicer.

B. Reduced order. For an ideal $I \subset R$, we put

$$\nu_I(f) := \sup\{p|\ f \in I^p\}.$$

The first step to the order function is Samuel's **reduced order** [Sa]:

$$\overline{\nu}_I(f) := \lim_{p\to\infty} \nu_I(f^p)/p.$$

Obviously, $\overline{\nu}_I(f^p) = p \cdot \overline{\nu}_I(f)$ ($p \in \mathbb{N}$). He posed the problem whether $\overline{\nu}_I(f) \in \mathbb{Q}$ or not and whether $p \cdot \overline{\nu}_I(f) - \nu_I(f^p)$ ($p \in \mathbb{N}$) is bounded for a reduced Noetherian ring. Rees [Re1] and Nagata [N] solved these problems affirmatively. These are done in 1950s.

It is obvious that

$$(\mathrm{I}_0) \qquad \nu_I(f) \le \overline{\nu}_I(f).$$

Its linear complementary inequality is the following (an answer to Samuel's problem).

Theorem 2.1 (Rees [Re3])**.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring whose completion is reduced (i.e. with no non-zero nilpotent element) and $I \subset A$ an ideal. Then*

$$(\mathrm{LCI}_0) \qquad \exists b > 0,\ \forall f \in A:\ \overline{\nu}_I(f) \le \nu_I(f) + b.$$

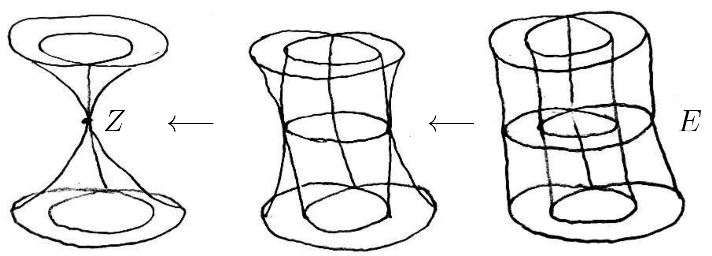
C. Integral dependence and divisorial valuation. Rees [Re1, Re2, Re3] developed the theory of the valuations associated with an ideal and its blowup, summarising as the valuation theorem. We explain this valuation theorem using complex function theory along the geometric lines of Lejeune-Jalaber–Teissier [LT]. This is easy to understand for analysts as the present author. Lejeune-Jalaber and Teissier are not aware of forgoing works of Rees. Their results are not included in the algebraic ones. In particular, some relation of reduced order to Łojasiewicz inequality is treated in [LT], §6, cf. [Ri] for real case also. We do not treat Łojasiewicz inequality here.

Definition 1 (Integral Dependence)**.** Let A be a ring and $I \subset A$ an ideal.

(1) $f \in A$: *integrally dependent on I*
$\iff \exists k \in \mathbb{N},\ \exists a_i \in I^i\ (i = 1, \ldots, k):\ f^k + a_1 f^{k-1} + \cdots + a_k = 0.$
(2) The *integral closure* $\overline{I}$ of I is the set of elements integrally dependent on I. They form an ideal.
(3) Let $(X, \mathscr{O}_X)$ be a complex space. If $\mathscr{I} \subset \mathscr{O}_X$ is a coherent ideal sheaf, the integral closures $\overline{\mathscr{I}_x} \subset \mathscr{O}_X$ $(x \in X)$ form a coherent ideal sheaf of $\mathscr{O}_X$, which we call *integral closure* of $\mathscr{I}$.

We need the blowup of a complex space whose centre is a general closed subspace or a coherent ideal sheaf. Geometric definition of such blowup, namely, definition not using Projan construction, is found in [H2] (real case) or [YFI] (in Japanese).

Let $(X, \mathscr{O}_X)$ be a reduced complex space (namely all the local rings $\mathscr{O}_{X,\xi}$ are reduced) and let $Z \subset X$ be a thin subspace defined by a coherent ideal sheaf $\mathscr{I} \subset \mathscr{O}_X$. Let $P : \tilde{X} \longrightarrow X$ be the blowup of X with centre Z. Then $\tilde{X}$ is also reduced and it can be normalised: $N : Y \longrightarrow \tilde{X}$. The composed morphism $\Pi := N \circ P$ or the complex space Y itself is called a *normalised blowup* with centre Z. The following is an image of Normalised blowup. The centre Z is enlarged to E and the neighbourhood of E become easy to see.



$X \longleftarrow$ blowup of $X \longleftarrow Y$: normalised blowup

It is known that Π is a proper holomorphic map. The singularity of Y is of codimension greater than 1 by normality and the exceptional set $E = \Pi^{-1}(Z) \subset Y$ is locally defined by a non-zero divisor by virtue of blowup. Hence, the inclusion $E \subset Y$ is a smooth hypersurface in a smooth manifold at a general point of E . Let E_i denotes an irreducible component of E with reduced structure. Then we can easily define the *generic vanishing order* $v_i(f)$ along E_i of a pullback of a function germ f around Z.

Theorem 2.2 (Lejune-Jalabert–Teissier [LT], Thm. 4.6). *Let $(X, \mathscr{O}_X)$ be a reduced complex space and let $\mathscr{I} \subset \mathscr{O}_X$ denote the coherent ideal sheaf which defines a thin subspace Z.*

Then for any compact subset $K \subset X$, there exists an open neighbourhood U of K such that the following holds. Let $\Pi : Y \longrightarrow X|U$ be the normalised blowup of $X|U$ with centre $Z|U$. Then the number of the set $\{E_i\}$ of irreducible components of the exceptional divisor $\Pi^{-1}(Z|U)$ is finite and we have

$$\overline{\nu}_{\mathscr{I}_\xi}(f) = \min_i \frac{v_i(\tilde{f})}{v_i(\mathscr{I})} \qquad (f \in \Gamma(Z \cap U, \mathscr{O}_X)).$$

Here $v_i(f)$ denotes the generic vanishing order of $f \circ \Pi$ along E_i (defined above) and

$$v_i(\mathscr{I}) = \inf\{v_i(f)|\ f \in \Gamma(Z \cap U, \mathscr{I}_\xi)\}.$$

We need only reduced one point case $Z = \{\xi\}$ to explain LCI_1 below.

<u>*D. Order of product.*</u> Recall our plan:

$$\underset{\overline{\nu}(f) \leq \nu(f) + b}{\underline{\underline{(\mathrm{LCI}_0)}}} + (\mathrm{LCI}_1) \Longrightarrow (\mathrm{LCI}_2) \Longrightarrow \underset{\mu_T(f) \leq a \cdot \nu(f) + b}{\underline{\underline{(\mathrm{LCI}_3)}}}$$

So, let us explain (LCI_1) for product. Let $(A, \mathfrak{m})$ be a Noetherian local ring. The following is obvious by $\mathfrak{m}^p\mathfrak{m}^q \subset \mathfrak{m}^{p+q}$.

$$(\mathrm{I}_1) \qquad \forall f, g \in A,\ \nu(f) + \nu(g) \leq \nu(fg) \quad \text{(a condition for a valuation)}.$$

Given a Noetherian local ring $(A, \mathfrak{m})$ we can define its extension $(\hat{A}, \hat{\mathfrak{m}})$ using the distance on A defined as

$$d(f, g) = \exp(-\sup\{k|\ f - g \in \mathfrak{m}^k\}).$$

Here we need Krull's intersection theorem to prove the implication $d(f, g) = 0 \Longleftrightarrow f = g$. We call $(\hat{A}, \hat{\mathfrak{m}})$ *completion.* The algebraic version of Theorem 2.2 + 2.1 is the following Rees' *valuation theorem.*

Theorem 2.3 ([Re4], Thm. 1.8). *Let $(R, \mathfrak{m})$ be a Noetherian local ring such that $\overline{A}$ has no nilpotent element. Let $I \subset R$ be an $\mathfrak{m}$-primary ideal. Then there exist integer valued valuations $v_1, v_2, \ldots, v_p$ on R and natural numbers $e_1, e_2, \ldots, e_p$ and $t(I)$ such that*

(1) $v_I(f) = \min\{v_1(f)/e_1, v_2(f)/e_2, \ldots, v_p(f)/e_p\}$
(2) $v_i(f) = \infty$ *implies that f belongs a minimal prime ideal.*
(3) $\overline{v}_I(f) - v_I(f) \leq t(I)$ *for any $f \in R$.*

The conclusion (3) is Theorem 2.1. Rees pointed out that the following theorem settles the old problem of characterising local rings with integral completion.

Theorem 2.4 ([Iz2], Rees [Re5]). *The following conditions are equivalent for a Noetherian local ring $(A, \mathfrak{m})$.*

(1) *The completion $\hat{A}$ is an integral domain.*
(2) *We have the following inequality.*

$$(\mathrm{LCI}_1) \qquad \exists a > 0,\ \exists b \geq 0,\ \forall f, \forall g \in A:\ \nu(fg) \leq a \cdot \nu(f) + \nu(g) + b.$$

(3) *We have the following for the valuations v_i in Theorem 2.2, 2.3 above.*

$$(\mathrm{LCI}'_1) \qquad \exists a_{ij} > 0,\ \forall f \in \mathscr{O}_{X,\xi}:\ v_i(f) \leq a_{ij} \cdot v_j(f)\ (1 \leq i, j \leq k).$$

We can easily show that (LCI_1) and (LCI'_1) are equivalent using Theorem 2.3. If A is a regular local ring (e.g. convergent power series ring), we have $\nu(fg) = \nu(f) + \nu(g)$ always and we may put $a = 1$ and $b = 0$. If A is not an integral domain, (LCI_1) does not hold. We may say that

The smaller $a \geq 1$ and $b \geq 0$ are, the simpler A is.

The present author proved only the case of analytic local algebras $\mathscr{O}_{X,\xi}$. (Note that, in this case, A is an integral domain if and only if its completion $\hat{A}$ is so, by a theorem of Nagata). The 2-dimensional case of (LCI'_1) is proven by taking a resolution of singularity of the blowup with the reduced centre $\{\xi\}$ and applying the theorem of du Val: the intersection matrix of the exceptional fibres $\{E_i\}$ is negative definite, which was the most difficult point to find out. The higher dimensional case is proved using a Bertini type theorem of Flenner [F] for local rings, which enables to slice down the dimension and to apply induction.

Rees generalised to Noetherian local rings in a similar way. He has shown also that we may put (the coefficient of $\nu(g)$) $= 1$ in (LCI_1). Anyway, Rees' valuation theorem, Theorem 2.3, or its analytic valiant Theorem 2.2, is essential for the proof of (LCI_1), (LCI'_1).

E. Order of pullback. Next, let us introduce (I_2) and (LCI_2): inequalities for orders of pullbacks. Let (X_ξ, A) and (Y_η, B) be irreducible analytic singularities, that is $A := \mathscr{O}_{X,\boldsymbol{\xi}}$ and $B := \mathscr{O}_{Y,\boldsymbol{\eta}}$ are integral domains. If $\Phi : Y_{\boldsymbol{\eta}} \longrightarrow X_{\boldsymbol{\xi}}$ is a morphism. It induces a homomorphism by pullback: $\varphi : A \longrightarrow B$, $\varphi(f) := f \circ \Phi$. Substantially, this is defined by the substitutions of coordinates centred at ξ by power series without constant terms in the coordinates centred at η. Hence it is a local homomorphism of local $\mathbb{C}$-algebras: $\varphi(\mathfrak{m}_A) \subset \mathfrak{m}_B$. Then we have

$$(\mathrm{I}_2) \qquad \nu(f) \leq \nu(\varphi(f)).$$

Since φ is local, φ is continuous with respect to the maximal ideal adic metric defined in §2, D and induces $\hat{\varphi} : \hat{A} \longrightarrow \hat{B}$ between completions canonically. In this situation, the important invariants are the following defined by Gabrièlov:

(1) $r_1 := \inf\{\mathrm{topdim}\, \Phi(U) |\ U$ is a neighbourhood of $\xi\}/2$ (*topological rank* = topological dimension of the image divided by that of $\mathbb{C}$).
(2) $r_2 := \dim \hat{A}/\operatorname{Ker} \hat{\varphi}$ (*formal rank*, the Krull dimension of "the formal closure of the image" (imagine this as a phantom) of Φ).
(3) $r_3 := \dim A/\operatorname{Ker} \varphi$ (*generic rank, geometric rank*, the Krull dimension of the analytic closure of the image of Φ).

We know that $r_1 \leq r_2 \leq r_3$. The inequality $r_1 \leq r_2$ is not trivial, cf. [Iz4], (1.5). One of the important results is the following.

Theorem 2.5 (Gabrièlov [G2]). *$r_1 = r_2$ implies $r_1 = r_2 = r_3$. .*

Although these ranks are found in differential analysis, we can generalise to the algebraic results.

Theorem 2.6 ([Iz2, Iz7], [Ro2]). *Let $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ be Noetherian integral local k-algebras and $A \longrightarrow B$ an injective local homomorphism. Then $r_1 = r_3$ if and only if*

$$(\mathrm{LCI}_2) \qquad \exists a \geq 1,\ \exists b \geq 0,\ \forall f \in A: \ \nu_{\mathfrak{n}}(\varphi(f)) \leq a \cdot \nu_{\mathfrak{m}}(f) + b.$$

This is proved as $(\mathrm{LCI}_0) +\ (\mathrm{LCI}_1) \Longrightarrow (\mathrm{LCI}_2)$. Essential point is generalisation of rank r_3 defined above in addition to (LCI_1).

(1) Tougeron [T] (p.178, Lem. 1.3) was very early to prove the case when A is a convergent (formal) power series in an elementary way.
(2) The general characteristic 0 case is proved in [Iz7] using (LCI_1), ideas of Tougeron above, Eakin-Harris' result [EH] and generalisation of r_3 by Kähler differential [SS].
(3) The positive characteristic case is done by Rond [Ro1] using (LCI_1) and further generalisation of r_3. Rond noted that this generalisation of generic rank is due to Spivakovsky (unpublished).

When we study mappings, the case when the target space is smooth is generally easier because singularity of the source space can be avoided by Hironaka's resolution [H1].

<u>*F. Geometric order.*</u> Now we come back to the geometric order. This is defined only in the analytic case. For $T \subset X$ accumulating at ξ, geometric order is defined as follows.

$$\mu_T(f) := \sup \{p |\ \exists \alpha > 0,\ \exists \text{ a neighbourhood } U \text{ of } \xi \ \text{ s.t. } |f(x)| \leq \alpha |x - \xi|^p\ (x \in T \cap U)\}.$$

In §2 A, we have stated the obvious inequality:

$$(\mathrm{I}_3) \qquad \forall f \in \mathscr{O}_{X,\xi}: \ \nu_{\mathfrak{m}}(f) \leq \mu_T(f).$$

Here $\mathfrak{m}$ denotes the maximal ideal defining the subspace $\{\xi\}$.

Theorem 2.7 ([Iz2]). *Let X be a real or complex analytic space. If $S \subset X$ contains an open subanalytic set adherent to ξ, we have*

$$(\mathrm{LCI}_3) \qquad \exists a \geq 1,\ \exists b \geq 0,\ \forall f \in \mathscr{O}_{X,\xi}: \ \mu_T(f) \leq a \cdot \nu(f) + b.$$

In real case, we can prove this applying (LCI_2) to Hironaka's *rectilinearisation map* [H2] for subanalytic sets, because that map satisfies Gabrièlov's condition $r_1 = r_3$. The complex case follows from this real one.

3. **Other Proofs, Variants and Analogies of the Basic Inequalities**

The following is a list of papers which seems to be closely related to the basic inequalities above. The present author has not read them closely and there may be some omissions.

<u>*A. Generalisation.*</u> Rees [Re5] was first to point out the importance of (LCI_1) and (LCI'_1) in ring theory. There appeared many different proofs, variants and generalisations. Hübl–Swanson [HS] (cf. Swanson's remark [Sw], p.18); Beddani [B]; Moghadam [M]. Rond–Spivakovsky [RS] generalised to the case of Abhyankar valuations with archimedian value semigroup. Boucksom–Favre–Jonsson [BFJ] not only considered the orders on the irreducible

components of the exceptional divisors but observed the behaviour of order on the whole "dual complex" formed by the intersections of the irreducible components.

Beginning from Tomari-Watanabe [TW], many applications of (LCI_1) and (LCI'_1) appeared in the field of commutative ring theory.

Returning to complex spaces, local uniformity results of (LCI_1) and (LCI_2) are obtained by Wang [W] and Adamus–Bierstone–Milman [ABM].

B. Analogy with the Theorem of Gabrièlov. There are analogical theorems between theories of convergence (or increase) and vanishing order in analytic category, which is studied in [Iz5]. Many results are obtained by this analogy in that paper.

The following analogy concerning homomorphisms of local analytic algebras $\varphi : A \longrightarrow B$ is most important.

Gabrièlov's rank condition assures that a certain property of an image of $\hat{\varphi}$ implies the same property of some preimage as follows.

(1) Theorem of Gabrièlov (affirmative answer to Grothendieck's problem):

$$r_1 = r_2 \Longrightarrow \hat{\varphi}(\hat{A}) \cap B = \varphi(A).$$

(2) Theorem of Iz. and Rond, (LCI_2):

$$r_1 = r_2 \Longrightarrow \exists a \geq 1, \exists b \geq 0, \forall p \in \mathbb{N} : \hat{\varphi}(\hat{A}) \cap \mathfrak{m}_B^{ap+b} = \varphi(\mathfrak{m}_A^p).$$

The proof of Gabrièlov's theorem is very difficult and there appeared many proofs and comments. Hüble [Hu] obtained a simple algebraic proof of a weaker version where the assumption $r_1 = r_2$ is replaced by $r_1 = r_3$. He pointed out that Gabrièlov's theorem may be peculiar to analytic algebras. The thick paper da Silva, Curmi, Rond [BCR] seems to give a summary.

4. **Local Zero Estimate**

In this section, we treat the zero estimate theory which is imported from analytic number theory. By zero estimate, the present author means the function $\theta_\Phi(k)$ defined below.

Let $(R, \mathfrak{m})$ be a local K-algebra over a field K of characteristic 0. Take a finite subset $\Phi := (\Phi_1, \ldots, \Phi_m) \subset R$ and put

$$K[\Phi]^{\leq k} := \{f \in K[\Phi] | \ \deg f \leq k\}.$$

In particular,

$$K[1, \Phi]^{\leq 0} = K[\Phi]^{\leq 0} = K[1]^{\leq k} = K[\emptyset]^{\leq k} = K \quad (k = 0, 1, ...).$$

Definition 2 (the growth function of vanishing order and its degree)**.**

$$\theta_\Phi(k) := \sup\left\{\nu_\mathfrak{m}(f) | \ f \in K[\Phi]^{\leq k} \setminus \{0\}\right\},$$
$$\alpha(\Phi) = \alpha(\Phi_1, \ldots, \Phi_m) := \limsup_{k \to \infty} \log_k \theta_\Phi(k).$$

Namely, $\alpha(\Phi)$ is the upper degree of $\theta_\Phi(k)$ in k.

Example 2 ([Iz6])**.** If $\Phi := (x, x \exp x) \subset \mathscr{O}_{\mathbb{C}}$, the vector subspace $\mathbb{C}[\Phi]^{\leq k}$ is spanned by

$$1;\ x, x\exp x;\ x^2, x^2\exp x, x^2 \exp 2x;\ \cdots;\ x^k, x^k \exp x, \dots, x^k \exp kx.$$

These form a basis of the space of solutions of

$$\left\{ \left(\frac{d}{dx}\right)\left(\frac{d}{dx}-1\right)\left(\frac{d}{dx}-2\right)\cdots\left(\frac{d}{dx}-k\right)\right\}^{k+1} f = 0$$

The solution space has a linear bijection with the space $\mathbb{C}^{(k+1)^2}$ through the initial conditions $(f(0), f''(0), \cdots, f^{((k+1)^2-1)}(0))$. Excepting the identically 0 solution, the maximal vanishing order is attained by the solution with the initial condition $(0, 0, \dots, 1)$. Then we have $\theta(k) = (k+1)^2 - 1$ and $\alpha(\Phi) = 2$.

The following is an analogue of Liouville numbers, namely they are constructed in the similar way to the transcendental numbers with big gap sequences.

Example 3 ([Iz6], Ueda function)**.** If we put

$$\Phi := \left(x,\ \sum_{i=1}^{\infty} \frac{1}{(2i!)!} x^{2i!} \right) \subset \mathscr{O}_{\mathbb{C}}$$

we have $\alpha(\Phi) = 2$. The sequence $\{2i! |\ i \in \mathbb{N}\}$ of powers has big gaps, which is similar to Liouville numbers. The denominator is added only for the sake of convergence on whole the complex plane. With these denominators, Φ becomes an entire function.

Theorem 4.1 ([Iz8], (2.2) lower estimate)**.** *Let $(R, \mathfrak{m})$ be a d-dimensional local K-algebra with* $\operatorname{char} K = 0$*. If $\Phi \subset R$ is a finite subset and $e := \operatorname{trdeg}_K K(\Phi)$, there exists $c > 0$ such that $\theta_\Phi(k) \geq c \cdot k^{e/d}$, $\alpha(\Phi) \geq e/d$.*

This is easily obtained by comparison of Samuel function $\varphi(t) := \dim_K R/\mathfrak{m}^t$ of R and Hilbert function $\psi(t) := \dim_K [\Phi]^{\leq t}$ of $K(\Phi)$.

Theorem 4.2 ([Iz8], Thm. 1.2, upper estimate)**.** *Let K be a field of characteristic 0 and $(R, \mathfrak{m})$ a local K-algebra which is an integral domain. Suppose that there are two non-empty finite subsets $\Phi := (\Phi_1, \dots, \Phi_m) \subset \mathfrak{m}$ and $\Psi := (\Psi_1, \dots, \Psi_n) \subset \mathfrak{m}$. Let $L := K(\Phi)$ be the field of quotients of the integral domain $K[\Phi] \subset R$ and let $L(\Psi)$ be the field generated by Ψ over L. Suppose that all the elements of Ψ are algebraic over L. Then we have $\theta_\Psi(k) \leq \theta_\Phi(ks)$ $(k \in \mathbb{N})$ for some $s > 0$. If $\Phi \subset \Psi$ further, we have $\alpha(\Psi) = \alpha(\Phi)$.*

Although the author used this theorem later, the given proof is **wrong**. A corrected proof is proposed in §6.

Corollary 4.3 ([Iz8], Thm. 2.3, algebraicity criterion)**.** *Let K be a field with* $\operatorname{char} K = 0$ *and $(R, \mathfrak{m})$ a local K-domain of $d := \dim R \geq 1$. Let $\Phi \subset R$ be a finite system of generators of an $\mathfrak{m}$-primary ideal of R. Then the following are equivalent.*

(1) $\operatorname{trdeg}_K K(\Phi) = d$.
(2) $\exists a > 0, \exists b \geq 0 : \theta_\Phi(k) \leq ak + b$.
(3) $\alpha(\Phi) = 1$.

An important development is brought by Rond [Ro2]. Let $M = K[[x]]^s/N$ be a finitely generated $K[[x]]$-module. Here $K[[x]]$ is the formal power series ring with coefficients in K, a field of *any characteristic*. Furthermore, he has given a necessary and sufficient condition for N to be generated by a submodule whose components are algebraic over $K[x]$. It is expressed by two kinds of zero estimate inequalities. Rond noted that such an algebraicity problem is a long standing problem since Samuel and M. Artin.

As we have seen above, gaps of power series are related to non-algebraicity. Hironaka posed a problem on the division theorem introduced by Grauert, Hironaka and Galligo: *characterise the residue of the division of an algebraic power series by another.* Rond shows that the gaps of the residues are tame in that case, a partial answer to this problem.

5. **Local Complexity of an Affine Embedding of a Smooth Manifold**

Bos–Calvi [BC1, BC2] have found a kind of *singularity* for a plane algebraic curve $X \subset \mathbb{C}^2$ in their research of interpolation theory. They defined the Taylor polynomial of degree $k \in \mathbb{N}$ of analytic functions defined in a neighbourhood of $\xi \in X$ embedded in an affine space. They use "parametrisation" which means a local embedding of an open subset of an affine space into an open subset of X and "polynomial" is one in the affine coordinates of the ambient affine space of X. They called a point $\xi \in X$ k-*Taylorian* when this Taylor polynomial of degree k is well-defined, namely, when the Taylor polynomial at $\xi \in X$ is independent of the parametrisation of the curve. We define the notion of Taylorian in the case of a higher dimensional embedded manifold similarly.

Theorem 5.1 (Bos-Calvi, [BC2], Thm. 4.10; [Iz11], Thm. 11.1)**.** *An analytic curve $X \subset \mathbb{C}^m$ is k-Taylorian at all points excepting a countable set.*

This was first proved by Bos and Calvi in the case of a plane algebraic curve. They used the tangents of order k of a curve at a point, which are defined as the pushforwards of tangents of the parameter space of order k using the idea of de Boor–Ron [BR1, BR2] (see below). Bos-Calvi treat an algebraic curve as a whole. In such a case we must choose one branch at a self-intersection point and avoid other singularities. They have proved that non-k-Taylorian points is finite in the case of algebraic plane curve.

Unfortunately, in the case of a higher dimensional analytic manifold $X \subset \mathbb{C}^n$, generic independence of Taylor polynomial from parametrisation fails (cf. [Iz11], Example 10.9). Let us consider the general case $m = \dim X \geq 2$ a little geometrically, as a short supplementary explanation of the later sections of the complicated paper [Iz11].

We suppose that $O = (0, \ldots, 0) \in X$ for the sake of simplicity. Let U be an open neighbourhood of $O \subset \mathbb{C}^m$ and let $\Phi = (\Phi_1, \ldots, \Phi_n) : U \longrightarrow X,\ O \mapsto O$ be a parametrisation of X around O. Let $t = (t_1, \ldots, t_m)$ denote the fixed affine coordinates defined on U centred at O. Let $f_O\!\downarrow$ denote the *least part*, namely the non-zero homogenous part of the lowest degree of the convergent power series expansion of $f(t) \in \mathbb{C}\{t\}$. We put $0\!\downarrow = 0$ specially. Consider the space $\mathbb{C}[\Phi]^{\leq k}$ (introduced in §4) of polynomial functions of degrees at most k in $\Phi = (\Phi_1, \ldots, \Phi_n)$. Then de Boor–Ron [BR1, BR2] have found that

$$\mathbb{C}[\Phi]^{\leq k}_O\!\downarrow := \text{the linear span of } \left\{ f_O\!\downarrow : f(t) \in \mathbb{C}[\Phi]^{\leq k} \right\}$$

can be seen as a dual vector space of $\mathbb{C}[\Phi]^{\le k}$. If we express the independent variables of $f_O{\downarrow}$ by $\tau = (\tau_1, \dots, \tau_m)$ $(\tau_i = (t_i)_O{\downarrow})$, the vector space $\mathbb{C}[\Phi]_O^{\le k}{\downarrow}$ is generated by homogeneous polynomials in τ. It is convenient to define a sesqui-linear form between the elements of $\mathbb{C}[\Phi]^{\le k}$ and $\mathbb{C}[\Phi]_O^{\le k}{\downarrow}$ by $\langle x^\nu | \tau^\mu \rangle = \nu! \delta_{\nu,\mu}$ ([BR2], §2; [Iz11], §5). Here we have put $\delta_{\nu,\mu} = 1$ $(\nu = \mu)$, $\delta_{\nu,\mu} = 0$ $(\nu \neq \mu)$. This implies the identifications $\tau_1 = \partial/\partial t_1$, $\tau_1^2 \tau_2 = \partial^3/\partial t_1^2 \partial t_2$, etc. Then the elements of $\mathbb{C}[\Phi]_O^{\le k}{\downarrow}$ can be considered as higher order tangents of $\mathbb{C}^n$ at O. Identifying $\mathbb{C}[x]_O^{\le k}{\downarrow}$ with $\mathbb{C}[\Phi]_O^{\le k}{\downarrow}$ by $x_i = \Phi_i(t)$, $\mathbb{C}[x]_O^{\le k}{\downarrow}$ can be seen as the space of pushforwards of high order tangents of $\mathbb{C}^m$ to those of $\mathbb{C}^n$. Their tangency to X can be easily verified. Thus we can consider $\mathbb{C}[\Phi]_O^{\le k}{\downarrow}$ as a space of high order tangents of X at O.

In the case of a curve, the condition for ξ to be k-Taylorian is that the set of powers of τ appearing in $\mathbb{C}[\Phi]_\xi^{\le k}{\downarrow}$ has no gap ([BC2], Thm. 3.4). For a general parametrisation

$$\Phi : \mathbb{C}^m \supset U \longrightarrow X \subset \mathbb{C}^n,$$

this gap-free condition can be paraphrased to *D-invariance of* $\mathbb{C}[\Phi]_\xi^{\le k}{\downarrow}$, namely it is closed with respect to the derivations by τ_i $(1 \le i \le m)$. Let $U^k \subset \mathbb{C}^m$ denote the set of points where the k-jets of elements of Φ form a vector bundle in a sufficiently small neighbourhood. The intersection $U^{bdl} := \bigcap U^k$ is dense in U as a complement of a countable union of closed thin analytic subsets (the set of first category in the sense of Baire).

Lemma 5.2 ([Iz11], Thm. 4.5)**.** *The set* $\mathbb{C}[\Phi]_\xi^{\le k}{\downarrow}$ *is D-invariant for any* $t \in U^{bdl}$.

This is a consequence of *formal theory of differential equations*, which was suggested to the author by Professor Tohru Morimoto.

D-invariance implies that all the derivatives of an element p of $\mathbb{C}[\Phi]_\xi^{\le k}{\downarrow}$ with the maximal order are all contained in it. The number of such derivatives (including p) is $\binom{m+\theta_\Phi(k)}{m}$. These are linearly independent and included in $\mathbb{C}[\Phi]_\xi^{\le k}{\downarrow}$, whose dimension is the Hilbert function of $\mathbb{C}(\Phi)$. It is known to coincides with a polynomial of degree $\dim \overline{X}_\xi$ in k for large k. Here, $\overline{X}_\xi$ is the algebraic closure of X_ξ, the minimal algebraic set which includes the germ X_ξ. Then we can deduce the following zero estimate at a general point, an *improved* form of [Iz11], Theorem 12.6.

Theorem 5.3. *Let X be an m-dimensional regular complex submanifold of an open subset $\Omega \subset \mathbb{C}^n$ $(1 \le m \le n)$. Then, for any local parametrisation Φ at ξ, we have*

$$\binom{m + \theta_\Phi(k)}{m} \le \dim_{\mathbb{C}} \mathbb{C}[\Phi]_\xi^{\le k}{\downarrow}$$

at ξ where $\mathbb{C}[\Phi]_\xi^{\le k}{\downarrow}$ is D-invariant.

If $\xi \in U^{bdl}$, we have $\alpha(\Phi) \le \dim \overline{X}_\xi / m$.

We may consider that $\theta_\Phi(k)$ and α_Φ are measures of complexity of the embedding Φ and that the theorem asserts that Φ is not very wild at general point.

As far as the present author knows, this kind of tameness in non-algebraic case is given only in [G3]: zero estimate is given for Noetherian functions along a trajectory of a Noetherian vector field. A very sharp estimate for polynomial functions along a trajectory of a polynomial vector field is given in [G4]. Of course, these are estimates on one dimensional manifolds but there is no points of exclusion as ours.

6. Correction of a Former Paper on Local Zero Estimate (Thm. 4.2 above; [Iz8], Thm. 1.2)

There is a stupid mistake in the proof of [Iz8], Theorem 1.2, which is one of a starting point of the paper and used in [Iz11]. The conclusion seems to remain valid by the following correction.

Corrected Proof. Since θ_Φ is monotone with respect to Φ, excluding algebraically dependent elements, we may assume that the elements of Φ are algebraically independent over K. Let L be the field of quotients of $K[\Phi]$. Since $\operatorname{char} K = 0$, Ψ is contained in a simple extension of L, that is, there exists $\tau \in L(\Psi)$ (the field generated by Ψ over L) such that τ is algebraic over L and $L(\Psi) = L(\tau)$. Let

$$P(t) := t^p + a_1 t^{p-1} + \cdots + a_p = 0 \quad (a_i \in L)$$

be the minimal polynomial of τ over L. Replacing τ by a suitable multiple by an element of $K[\Phi]$, we may assume from the first that τ is integral over $K[\Phi]$, that is $a_i \in K[\Phi]$. Since $\tau^{p-1}, \tau^{p-2}, \ldots, \tau, 1$ form a vector basis of $L(\Psi) = L(\tau)$ over L, we can express Ψ_i as $\Psi_i = Q_i(\tau)$, with

$$Q_i(t) := (b_{i,1} t^{p-1} + b_{i,2} t^{p-2} + \cdots + b_{i,p-1})/c \quad (b_{i,1}, b_{i,2}, \ldots, b_{i,p-1}, c \in K[\Phi]).$$

For any $k \in \mathbb{N}$, let us take $f \in K[\Psi]^{\leq k} \setminus K[\Psi]^{\leq k-1}$, if any. Then we have $f = F(\Psi)$ for some $F \in K[x]$ of degree just k. Let $\tau_1, \tau_2, \ldots, \tau_p$ denote all the conjugates of $\tau = \tau_1$ over L and put

$$f_j := F(Q_1(\tau_j), \ldots, Q_n(\tau_j)) \quad (j = 1, \ldots, p).$$

The conjugates $\tau_1, \tau_2, \ldots, \tau_p$ are distinct by the assumption that $\operatorname{char} K = 0$. Since $c\Psi_i$ are integral over $K[\Phi]$, $c^k f_j$ is also so. Let us put

$$M(t) := \prod_{j=1}^{p} (t - c^k f_j) = t^p + d_1 t^{p-1} + \cdots + d_p \quad (d_i \in K[\Phi]).$$

Let $\nu := \nu_{\mathfrak{m}}$ denote the valuation associated with $\mathfrak{m}$. The equation

$$d_p = -(c^k f)^p - d_1 (c^k f)^{p-1} - \cdots - d_{p-1}(c^k f)$$

implies that

$$\nu(d_p) \geq (c^k f)^p \geq \nu(c^k f) \geq k \cdot \nu(c) + \nu(f). \tag{1}$$

Let us put

$$r := \max\left\{1, \max_i \deg_\Phi a_i,\ \max_{i,j} \deg_\Phi b_{i,j},\ \deg_\Phi c\right\}.$$

These are independent of k and of f and it holds that $(c^k f_1)\cdots(c^k f_p) = (-1)^p d_p$. If F_i denotes the homogeneous part of degree i of F, we have

$$\nu(d_p) = \nu\left((c^k f_1)\cdots(c^k f_p)\right) = \nu\left(\prod_{j=1}^{p}\sum_{i=0}^{k} c^{k-i}F_i(cp_1(\tau_j),\dots,cp_n(\tau_j))\right) \tag{2}$$

$$= \nu\left(\prod_{j=1}^{p}\sum_{i=0}^{k} c^{k-i}F_i\left(b_{1,1}\tau_j^{p-1}+b_{1,2}\tau_j^{p-2}+\cdots+b_{1,p-1},\dots,b_{n,1}\tau_j^{p-1}+b_{n,2}\tau_j^{p-2}+\cdots+b_{n,p-1}\right)\right).$$

Let P denote the product in the last expression of (2).

- ◇ Then P is a symmetric polynomial in $\tau_1,\dots,\tau_p$ of degree in $[0, k(p-1)p]$ with coefficients in $K[\Phi]$. Since $(-1)^i a_i$ is the elementary symmetric polynomial of degree i in $\tau_1,\dots,\tau_p$, we can replace P by polynomials of degree at most $k(p-1)p$ in $a_1, a_2, \dots, a_p$ with coefficients in $K[b_{1,1}, b_{1,2},\dots,b_{n,p-1}, c]$.
- ◇ The degree of the coefficients ($\in K[b_{1,1}, b_{1,2},\dots,b_{n,p-1}, c]$) of the monomials in $a_1,\dots,a_n$ (obtained by replacement above) in the expansion of P are majorised by kp.

By these observation, we see that the degree of P in $a_{i,j}, b_{i,j}, c_i$ is majorised by $k(p-1)p+kp = kp^2$. This proves that the degree of P is majorised by kp^2r as an element of $K[\Phi]$. This proves

$$\nu(f) \le \nu(d_p) = \nu(P) \le \theta_\Phi(kp^2r)$$

by (1). We have only to put $s = p^2r$.

Suppose further that $\Phi \subset \Psi$ holds. Then $\theta_\Phi(k) \le \theta_\Psi(k)$, $\alpha(\Phi) \le \alpha(\Psi)$ are obvious. Since

$$\alpha(\Psi) = \limsup_{k\to\infty} \log_k(\theta_\Psi(k)) \le \limsup_{k\to\infty} \log_k(\theta_\Phi(kp^2r))$$
$$= \limsup_{k\to\infty}\left[\log_k(kp^2)\cdot\log_{kp^2}\theta_\Phi(kp^2r)\right] = \limsup_{k\to\infty}\left[\log_k k\cdot\log_k\theta_\Phi(k)\right] = \alpha(\Phi),$$

we have $\alpha(\Psi) = \alpha(\Phi)$ which completes the proof.

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