On a construction of stable maps from 3-manifolds into surfaces

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Stable maps from 3-manifolds into 2-manifolds have been studied as a generalization of Morse functions on smooth 3-manifolds. For example, for a stable map f from the closed orientable smooth 3-manifold S^3 into $\mathbb{R}^2$, characterizations of links L have been obtained for which there exists such a stable map $f : S^3 \to \mathbb{R}^2$ satisfying certain conditions on singularities and $L \subset S_0(f)$ ([3], [1]). In this paper, we introduce an explicit construction of stable maps $f : S^3 \to \mathbb{R}^2$ that satisfy similar conditions on singularities and, in addition, satisfy $S_0(f) = L$.

1 Introduction

We first define a stable map. As a preparation for this, we introduce the following notion. Let X and Y be smooth manifolds, and let f and g be smooth maps from X to Y. We say that f and g are **left-right equivalent** if there exist diffeomorphisms $\Phi : X \to X$ and $\phi : Y \to Y$ such that the following diagram commutes. Consider the space of all smooth maps from a closed orientable smooth 3-manifold M to $\mathbb{R}^2$ endowed with the Whitney topology. A smooth map $f : M \to \mathbb{R}^2$ is called **stable** if there exists a neighborhood U_f of f in this space such that every smooth map $g \in U_f$ is left-right equivalent to f.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \Phi}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \phi} \\ X & \xrightarrow[g]{} & Y \end{array}$$

We introduce several definitions and facts. Let $f : M \to \mathbb{R}^2$ be a smooth map. A point of M at which the rank of the differential of f is deficient called a **singular point**, and the set of all such points is called the **singular set**. In this paper, we denote the singular set by $S(f)$.

Fact 1.1 ([6]) *Let M be a smooth 3-manifold. A smooth maop $f : M \to \mathbb{R}^2$ is a stable map if and only if fis locally given in one of the following:*

1. $(u, x, y) \mapsto (u, x)$
2. $(u, x, y) \mapsto (u, x^2 + y^2)$
3. $(u, x, y) \mapsto (u, x^2 - y^2)$
4. $(u, x, y) \mapsto (u, y^2 + ux - x^3)$ *and f globaly satisfies*

5. $f^{-1}(f(p)) \cap S(f) = \{p\}$ *for a cusp point* p

6. *the restriction of* f *to* $S(f) - C(f)$ *is an immersion with only normal crossings.*

Levin [6] gave a characterization of stable maps from closed orientable smooth 3-manifolds to $\mathbb{R}^2$. Denote the points on M corresponding locally to $2, 3$, and 4 by definite fold points, indefinite fold points and cusp points, respectively. And we denote by $S_0(f)$, $S_1(f)$, and $C(f)$ the sets of definite fold singularities, indefinite fold singularities, and cusp points, respectively. Then we have $S(f) = S_0(f) \cup S_1(f) \cup C(f)$, where $S(f)$ is the singular set of f. Moreover, Levine gave the following result concerning cusp points.

Proposition 1.2 ([6]) *For a closed smooth* 3*-manifold, the cusp points of any such stable map can be eliminated by smooth homotopy.*

Thus, in the rest of this paper, we will assume that all stable maps from 3-manifolds into $\mathbb{R}^2$ have no cusp points.

By studying the singularities of f described above, information about the domain manifold of a stable map can be extracted, and several results along these lines have been established. Saeki [8] obtained the following result. Here we omit the definition of a graph manifold; see [8] for details.

Theorem 1.3 ([8],Theorem 3.1) *Let* M *be a closed orientable smooth* 3*-manifold. Then the following three are equivalent:*

1. *There exists a stable map* $f : M \to \mathbb{R}^2$ *with* $f|_{S(f)}: S(f) \to \mathbb{R}^2$ *a smooth embedding.*

2. *There exists a simple stable map* $f : M \to N$ *for some* 2*-manifld* N.

3. M *is graph manifold.*

Remark 1.4 *the image* $f(S_1(f))$ *is (possibly non-simple) closed curves in* $\mathbb{R}^2$.

The above result treats the case where $f(S_1(f))$ has no self-intersections in $\mathbb{R}^2$. A natural question arising from this is what happens when $f(S_1(f))$ has self-intersections in $\mathbb{R}^2$. To address this, we first consider the preimage under f of the self-intersection points of $f(S_1(f))$ in $\mathbb{R}^2$.

Definition 1.5 ([5]) *For a stable map* f *from a* 3*-manifold* M *into the plane, singular fibers of* f *are defined to be a connected component of the preimage of a self-intersection point of* $f(S_1(f))$ *that contains at least two points of* $S_1(f)$. *The singular fibers over crossings in* $f(S_1(f))$ *are of two types,* type II^2 *and* type II^3, *as shown in Fig. 1. The sets of singular fibers of* type II^2 *and* type II^3 *of* f *are denoted by* $\mathrm{II}^2(f)$ *and* $\mathrm{II}^3(f)$, *respectively.*

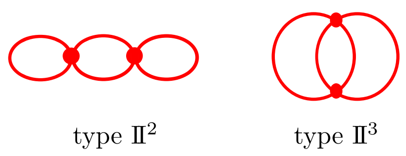

Figure 1: The types of the singular fibers.

Ishikawa and Koda [3] studied stable maps $f : S^3 \to \mathbb{R}^2$ satisfying $|\mathrm{I\!I}^2(f)| = 1$ and $|\mathrm{I\!I}^3(f)| = 0$, and obtained the following result concerning a link L contained in $S_0(f)$. Here $| * |$ denote the cardinarity of $\mathrm{I\!I}^2(f)$ and $\mathrm{I\!I}^3(f)$.

Theorem 1.6 ([3]) *Let L be a link in S^3. Then there exists a stable map $f : S^3 \to \mathbb{R}^2$ such that $C(f) = \emptyset$, $L \subset S_0(f)$, $|\mathrm{I\!I}^2(f)| = 1$, and $|\mathrm{I\!I}^3(f)| = 0$ if and only if $E(L)$ is diffeomorphic to a 3-manifold obtained by Dehn filling the exterior of one of the six links $L_1, \ldots, L_6$ in S^3 along some (possibly none) of its boundary tori. The links $L_1, \ldots, L_6$ are illustrated below.*

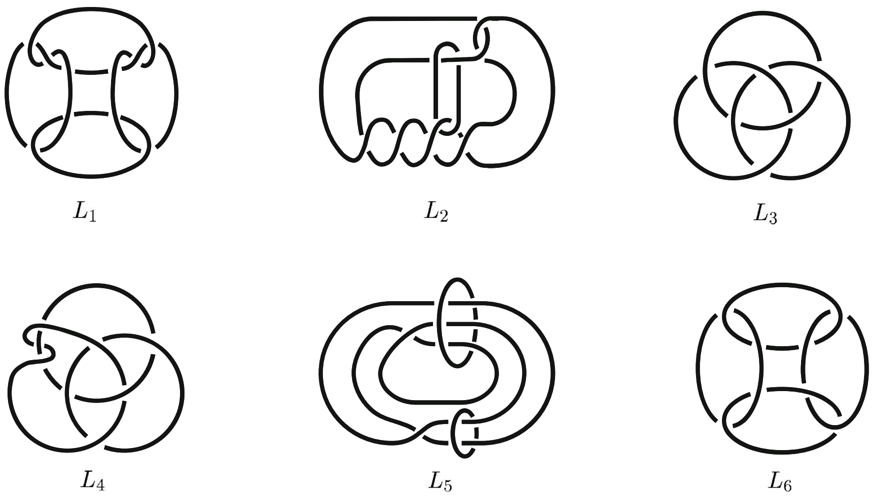

Figure 2: [3], Fig.2

We omit the details about the above theorem. See [3]. Furthermore, Furutani and Koda [1] obtained the following result.

Theorem 1.7 ([1]) *Let L be a link in S^3. Then there exists a stable map $f : S^3 \to \mathbb{R}^2$ such that $C(f) = \emptyset$, $L \subset S_0(f)$, $|\mathrm{I\!I}^2(f)| = 0$, and $|\mathrm{I\!I}^3(f)| = 1$ if and only if $E(L)$ is diffeomorphic to a 3-manifold obtained by Dehn filling the exterior of one of the four links L'_1, L'_2, L'_3, L'_4 in S^3 along some (possibly none) of its boundary tori.*

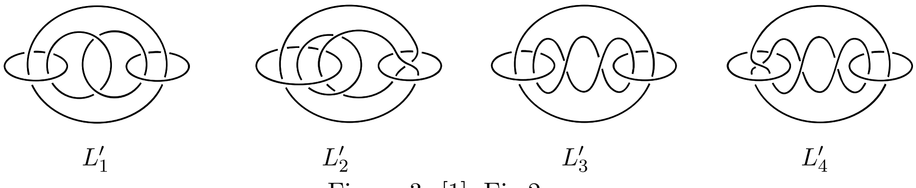

Figure 3: [1], Fig.2

Although these results were obtained by constructions using 4-manifolds, we were able to give a more explicit and visually understandable construction of stable maps for certain restricted classes of links by using only information up to the level of 3-manifolds.

Theorem 1.8 ([2]) *Let L be a two-bridge link of 2-components in S^3. Then there exists a stable map $f : S^3 \to \mathbb{R}^2$ such that $C(f) = \emptyset$, $S_0(f) = L$ and $\mathrm{I\!I}^2(f) = \emptyset$.*

2 The result

Before presenting the main result, we prepare one preliminary fact.

The n-string braids form a group B_n with respect to the above product. The standard generating element σ_i is shown in Fig. 4. See [7], for a *braid of n-strings*.

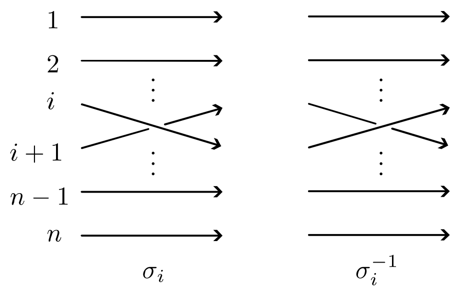

Figure 4: The generator σ_i and σ_i^{-1}.

In the result of [2], the restriction on the link can be removed.

Theorem 2.1 ([2]) *Let L be a link in S^3 and $b \in B_n$ is given as a word $\sigma_{z_1}^{m_1}\sigma_{z_2}^{m_2}\cdots\sigma_{z_l}^{m_l}$ ($n \geq 2$) such that the closure of $\hat{b}$ is L, where $1 \leq z_i \leq n-1$, $z_{i-1} \neq z_i$ and m_i are integers for $i \in \{1, 2, \cdots, l\}$. Then, there exists a stable map f from S^3 into the plane $\mathbb{R}^2$ such that $C(f) = \emptyset$, $S_0(f) = L$, $|\mathrm{II}^2(f)| = 2(l - X)$ and $\mathrm{II}^3(f) = \emptyset$. Here X denotes the number of σ_1 in $\sigma_{z_1}, \sigma_{z_2}, \cdots, \sigma_{z_l}$.*

Furthermore, by performing Dehn surgery along the link in the source manifold of the stable map obtained from the above result, we obtain the following.

Corollary 2.2 ([2]) *Let M be a closed orientable 3-manifold. Suppose that M is obtained by an integral Dehn surgery on a link L in S^3 such that L is represented as the closure of a pure braid $b = \sigma_{z_1}^{m_1}\sigma_{z_2}^{m_2}\cdots\sigma_{z_i}^{m_i}\cdots\sigma_{z_l}^{m_l}$. Then, there is a stable map f_0 from M into the 2-sphere S^2 satisfying $C(f_0) = S_0(f_0) = \mathrm{II}^3(f_0) = \emptyset$ and $|\mathrm{II}^2(f_0)| = 2(l - X)$. Here, X denotes the number of σ_1 in $\sigma_{z_1}, \sigma_{z_2}, \cdots, \sigma_{z_l}$.*

3 A construction of a stable map $S^3 \to \mathbb{R}^2$

This section outlines the general flow and the key idea of a construction. See [4] for further details.Using Levine's characterization of stable maps from 3-manifolds to 2-manifolds, we construct the stable map as claimed in the statement. In particular, we prove the assertion by constructing a smooth map that satisfies four local conditions and global conditions (see Fcat 1.1). While the details of a construction are omitted, the smooth map is obtained by decomposing S^3 and the disk into several components, constructing a smooth map for each part, and then gluing them together.

The key idea of a construction lies in constructing a smooth map from a cylinder to a rectangular region. To this end, we construct these smooth maps by considering a 1-parameter family of Morse functions based on a Morse function on a disk. First, we introduce the base Morse function as shown in Fig. 5. By taking the product with an interval for both the source and target, we construct a smooth map from a cylinder to a rectangular region in Fig. 6. Furthermore, by introducing three types of base Morse functions as shown Fig. 7, we obtain the following smooth map as a natural extension as shown Fig. 8.

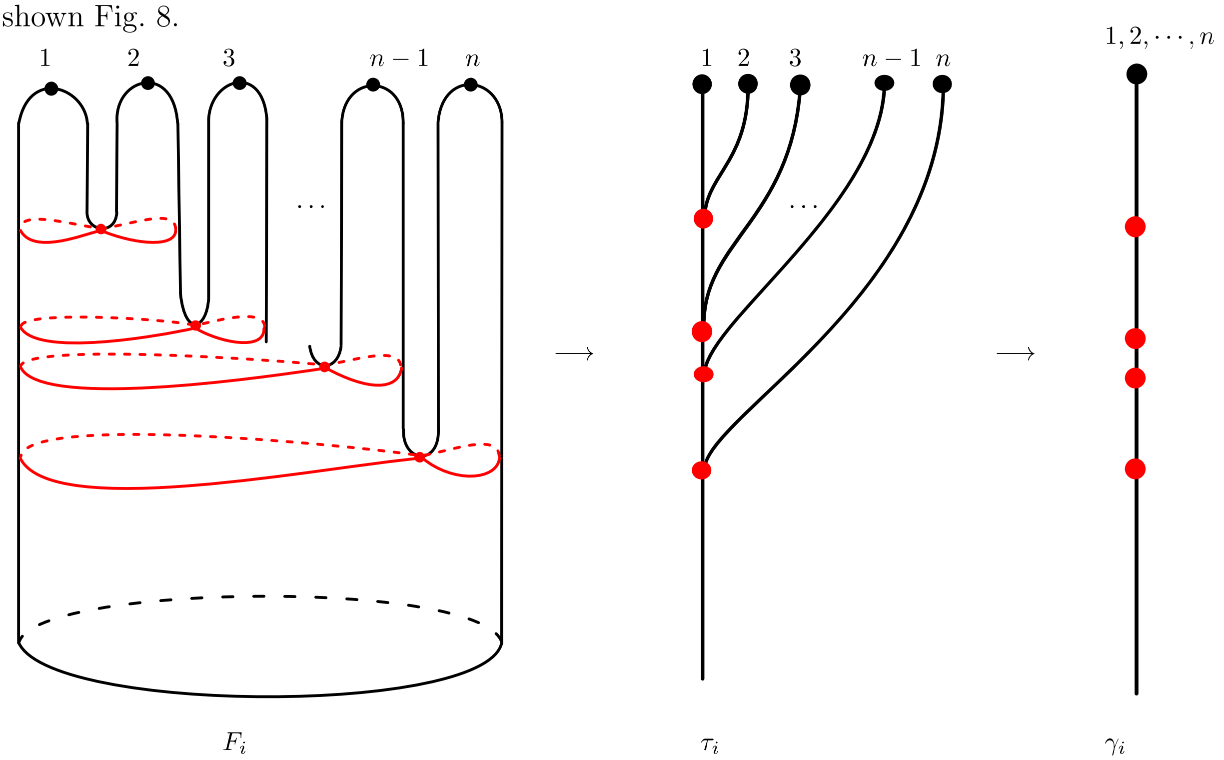

Figure 5: A smooth map $\psi_i : F_i \to \gamma_i$ for $1 \leq i \leq l$. The numbers in Fig. 5 correspond to the indices of the strands in the braid. See Figure 4.

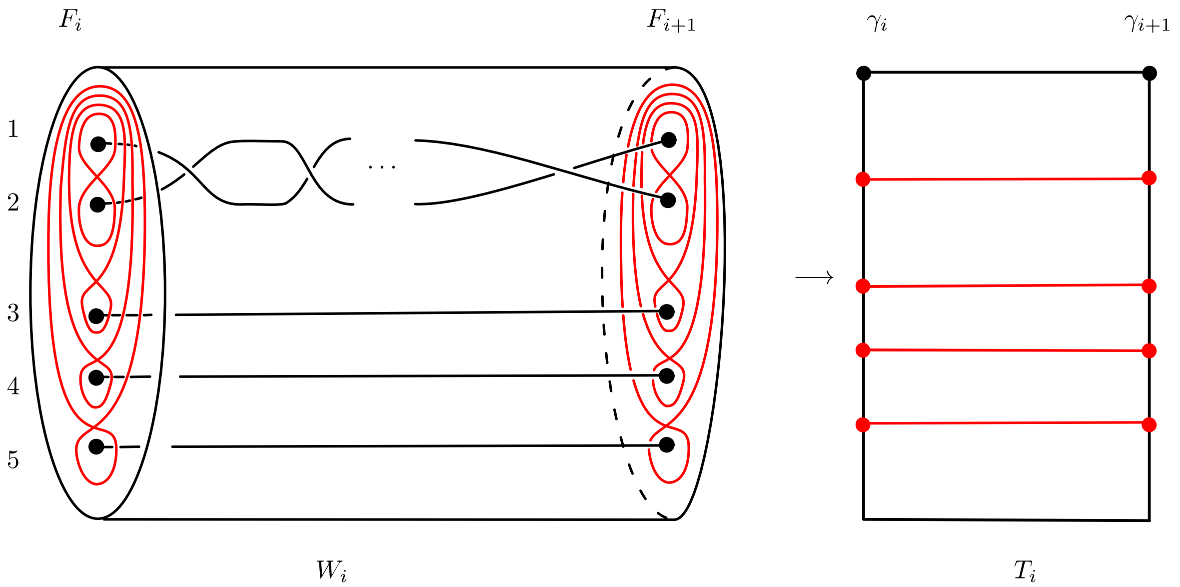

Figure 6: A smooth map $\Psi_i : W_i \to T_i$, when $z_i = 1$ and $n = 5$.

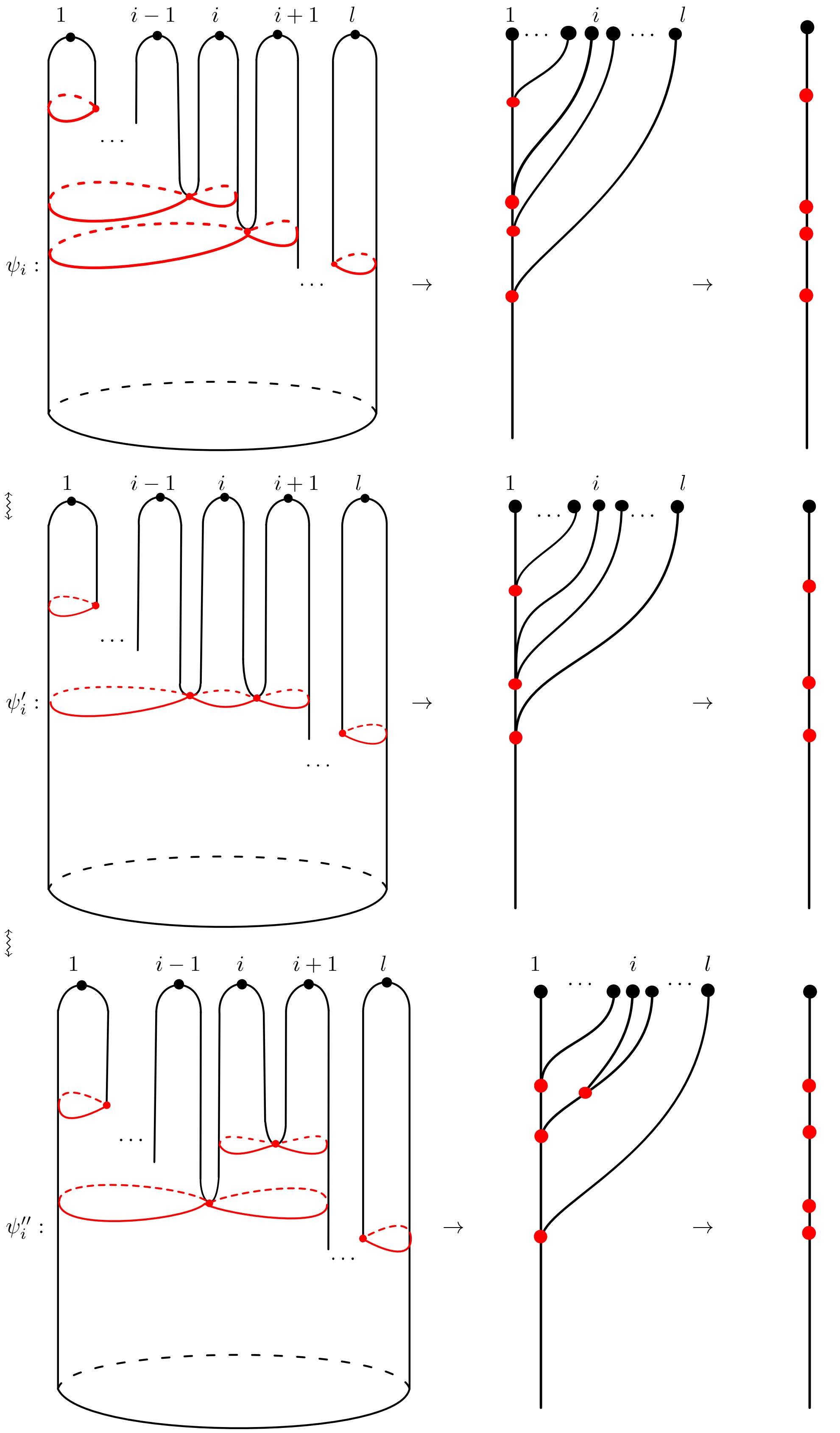

Figure 7: A deformation from ψ_i to ψ_{i+1}, when $z_i \geq 2$.

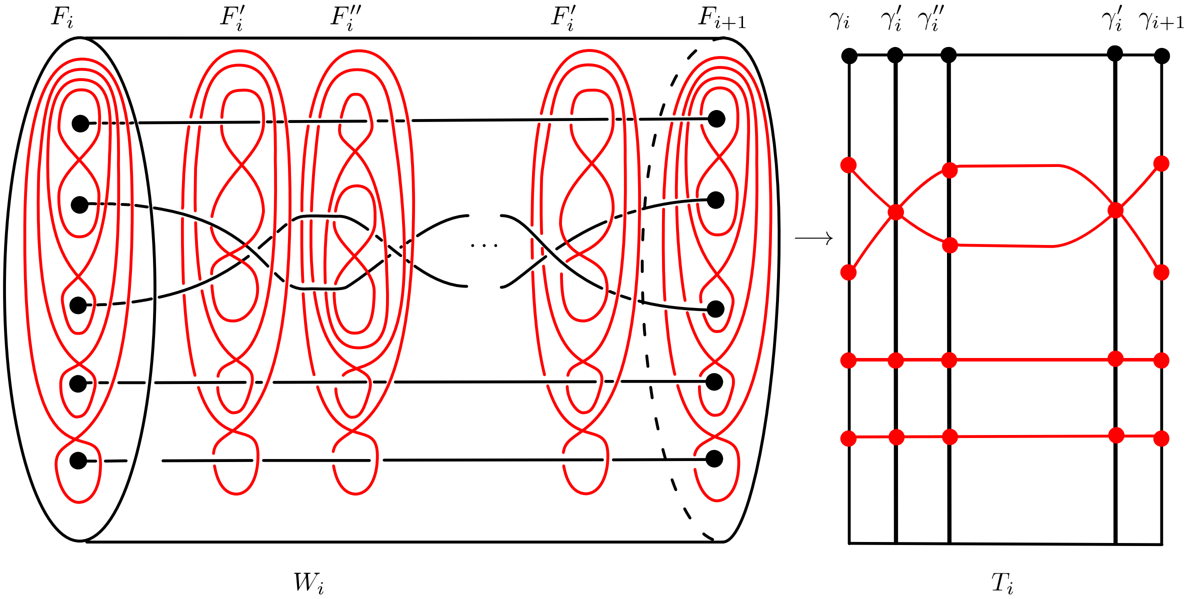

Figure 8: When $z_i = 2$ and $n = 5$, a smooth map $\Psi_i : W_i \to T_i$.

We glue the two patterns of smooth maps from a cylinder to a rectangular region, prepared thus far, so as to be compatible with the braid presentation of L. We obtain the stable map in Theorem 2.1.

Example 3.1 *We see an example of stable maps for the* 10_{128} *link. A stable map* $f : S^3 \to \mathbb{R}^2$ *with the* 10_{128} *link as* $S_0(f)$ *obtained by the construction given in this section is presented in Fig. 9. However, while the image of* $S_1(f)$ *represents the whole object on* $\mathbb{R}^2$*, the singular fibers of* f *depicts only a part of it on* S^3*. This* f *has no fiber of type* II^3*. Some of the fibers containing an indefinite fold point are also shown in the figure.*

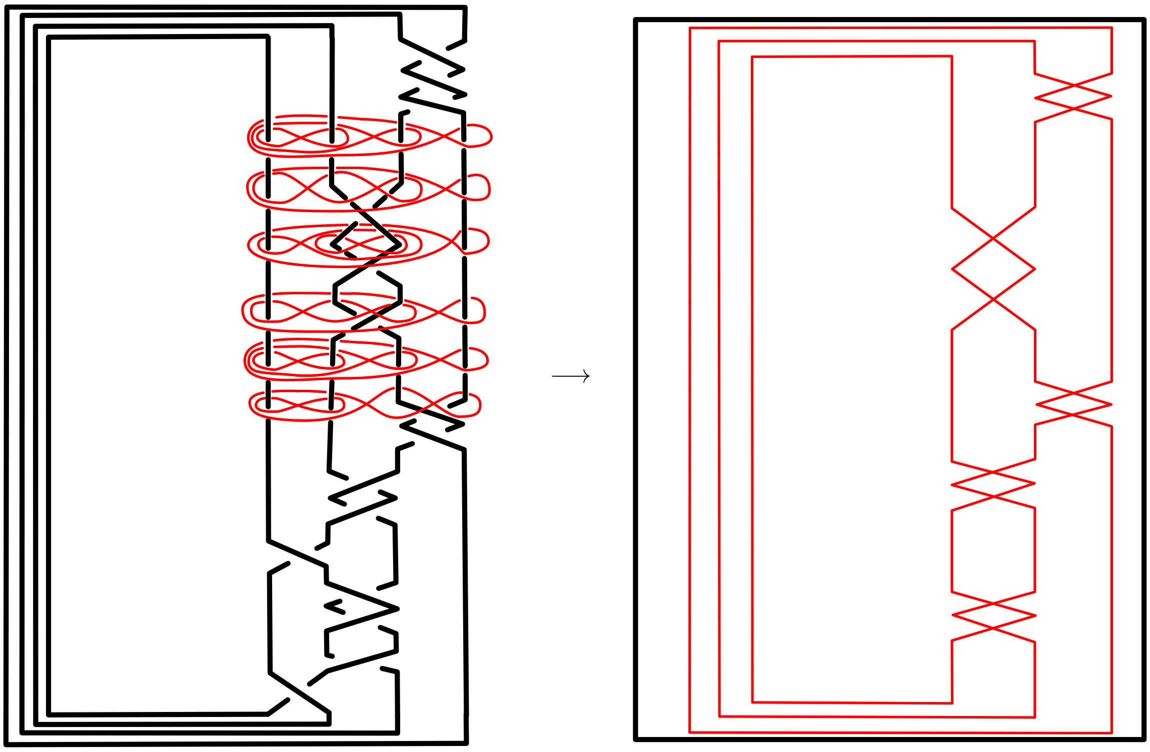

Figure 9: A stable map $S^3 \to \mathbb{R}^2$ for the 10_{128} link,.

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