接分布構造と多様体のトポロジーについて

— Existence and classification of the Cartan $(2,3,5)$-distribution —

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Abstract

The Cartan $(2,3,5)$-distribution is a tangent distribution of rank 2 on a 5-dimensional manifold satisfying certain generic conditions. The necessary and sufficient condition for a manifold to admit such a structure is established in this note. The condition obtained is purely topological. In addition, the classification of such structures, up to homotopy as formal Cartan distributions, is obtained.

1 Introduction

The existence and classification of geometric structures on smooth manifolds have long been central themes in differential topology. Among such structures, smooth distributions of tangent subspaces that satisfy specific non-integrability conditions provide a profound interplay between local geometry and global topology. A fundamental problem in this field is to determine whether a manifold admits a global distribution with a prescribed

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local behavior, and if so, to classify such distributions. While the local models of these distributions are rigidly governed by Lie theory and differential systems, their global existence poses a close relation to topological properties.

Among such structures, Cartan $(2, 3, 5)$-distributions on 5-dimensional manifolds take a unique position. It is defined as a tangent distribution of rank 2 whose successive Lie brackets generate a sequence of distributions with maximal growth vector $(2, 3, 5)$. Let $\mathscr{D} \subset TM$ be a distribution of rank 2 on a 5-dimensional manifold M. The distribution $\mathscr{D}$ is called a *Cartan* $(2, 3, 5)$*-distribution* if it satisfies the following conditions:

$$\text{rank } [\mathscr{D}, \mathscr{D}] = 3, \qquad \text{rank}[\mathscr{D}, [\mathscr{D}, \mathscr{D}]] = 5.$$

While it is related to control theory and various applied fields, it has long been a central subject of interest in the geometric study of differential equations. Since the seminal work of Cartan [C] on the equivalence problem of second-order differential systems, later modernized through the rigorous framework of exterior differential systems by Bryant et al. [BCG3], these distributions are widely known for their deep connections to the exceptional Lie group G_2 and their intrinsic local rigidity. Indeed their integral curves exhibit strict rigidity phenomena, as profoundly analyzed by Bryant and Hsu [BH]. Following the foundational framework of graded Lie algebras developed by Tanaka [T] and its systematic application to differential systems by Yamaguchi [Y], the local geometry of Cartan $(2, 3, 5)$-distributions is now deeply understood. Furthermore, as demonstrated by Nurowski [N], these distributions have attracted renewed interest due to their natural connection to conformal structures of signature $(3, 2)$. However, despite this rich geometric heritage (see [BCG3], [Mo], for example), their global existence and classification from a purely topological viewpoint have remained largely unexplored.

In contrast to such rigid properties, the results of this paper has flexibility flavors. One of the main goals of this paper is to establish the necessary and sufficient condition for a manifold to admit a Cartan $(2, 3, 5)$-

distribution. It should be emphasized that the condition imposed on the manifold is purely topological. In order to formulate this condition precisely, we introduce a formal topological counterpart to the geometric structure from the view point of the h-principle. An *almost Cartan structure* on a 5-dimensional manifold is defined as a sequence of tangent distributions $\mathscr{D} \subset \mathscr{E} \subset TM$ of rank 2 and 3 respectively, together with a certain triple $\{\omega_1,\ \omega_2;\ \omega_3\}$ of 2-forms.

Definition (almost Cartan structure)**.** An *almost Cartan structure* on a 5-dimensional manifold M is defined as a tuple $(\mathscr{D} \subset \mathscr{E}, \{\omega_1, \omega_2; \omega_3\})$, where $\mathscr{D} \subset \mathscr{E}$ is a sequence of tangent distributions of ranks 2 and 3 respectively on M, and ω_1, ω_2, ω_3 are 2-forms that satisfy the following conditions:

(1)-(i) ω_1 and ω_2 are degenerate on $\mathscr{D}$,

(1)-(ii) ω_3 is non-degenerate on $\mathscr{D}$,

(2) ω_1 and ω_2 are pointwise linearly independent on $\mathscr{E}$.

The relation between genuine Cartan $(2, 3, 5)$-distribution and almost Cartan structure is analogous to that between contact structure and almost contact structure. Then the necessary and sufficient condition is stated as follows.

Theorem A. *Let M be a possibly closed 5-dimensional manifold. Then M admits a Cartan $(2, 3, 5)$-distribution if and only if it admits an almost Cartan structure.*

By Theorem A, the existence problem reduces entirely to finding an almost Cartan structure. This is a standard problem in algebraic topology. We prove this theorem from the viewpoint of the h-principle. Consequently, any almost Cartan structure can be homotoped to a genuine Cartan (2,3,5)-distribution. The key to proving the h-principle is Gromov's convex integration method (see [Am],[CEM], [G], [S]).

In some specific cases, similar results were obtained by Dave and Haller [DH]. They dealt with the case where 5-dimensional manifolds are open and orientable, and the tangent distributions are also orientable. Since they applied Gromov's h-principle for open manifolds, then such assumptions were required. In that setting, they obtained the same result as Theorem A, and clearly describe the condition from the viewpoint of obstruction theory, using known topological invariants. Actually, their "formal" structure for the Cartan $(2,3,5)$-distribution turns out to be equivalent to that in this paper, almost Cartan structure. Furthermore, they studied necessary and sufficient condition for tangent distributions of rank 2 to be Cartan $(2,3,5)$-distributions on open orientable manifolds that satisfy the conditions above. The claims are described by the following topological terms. A manifold M is said to be *spin* if it admits a spin structure. In other words, M satisfies $w_1(M) = 0$, $w_2(M) = 0$, where $w_i(M)$ denotes Stiefel-Whitney class of TM. Let $\frac{1}{2}p_1(M) \in H^4(M;\mathbb{Z})$ denote the first fractional Pontryagin class of TM.

Theorem (Dave-Haller)**.** (1) *Let M be an open 5-dimensional manifold. The manifold M admits an orientable Cartan $(2,3,5)$-distribution if and only if M is spin.*
(2) *Let M be an open 5-dimensional spin manifold and $\mathscr{D}$ an orientable tangent distribution of rank 2 on M. Then there exists an orientable Cartan $(2,3,5)$-distribution $\tilde{\mathscr{D}}$ on M with $e(\tilde{\mathscr{D}}) = e(\mathscr{D})$, if and only if $e(\mathscr{D})^2 = \frac{1}{2}p_1(M)$, where $e(\mathscr{D}) \in H^2(M;\mathbb{Z})$ denotes the Euler class.*

In other words, the second statement characterizes the possible Euler classes that can be realized by the Cartan $(2,3,5)$-distributions on open 5-dimensional spin manifolds.

Even in the case where manifolds are closed, Dave and Haller [DH] also studied topological requirements for the almost generic rank-two distributions in dimension 5, that they regarded as a formal structure in their sense. However, Gromov's h-principle for open manifolds does not apply in this case. Combining their results with those obtained in this paper, we obtain the following theorem.

Theorem B. (1) *Let M be a closed connected 5-dimensional manifold. The manifold M admits an orientable Cartan $(2,3,5)$-distribution if and only if M is spin with vanishing Kervaire semicharacteristic $\kappa(M) = 0 \in \mathbb{Z}/2\mathbb{Z}$.*
(2) *Let M be a closed connected 5-dimensional spin manifold with vanishing Kervaire semicharacteristic $\kappa(M) = 0$, and $\mathscr{D}$ an orientable tangent distribution of rank 2 on M. Then there exists an orientable Cartan $(2,3,5)$-distribution $\tilde{\mathscr{D}}$ on M with $e(\tilde{\mathscr{D}}) = e(\mathscr{D})$, if and only if $e(\mathscr{D})^2 = \frac{1}{2}p_1(M)$, where $e(\mathscr{D}) \in H^2(M;\mathbb{Z})$ is the Euler class.*

In other words, the second statement characterizes the possible Euler classes that can be realized by the Cartan $(2,3,5)$-distributions on closed connected spin 5-manifolds with vanishing Kervaire semicharacteristic.

Examples. From Theorem B, we obtain some concrete examples of closed manifolds that either admit or do not admit Cartan $(2,3,5)$-distributions. It is well known that $S^3 \times S^2$ admits a Cartan $(2,3,5)$-distribution, the standard model (see [BCG3], [Mo], and so on). The 5-dimensional manifold $S^3 \times S^2$ is in fact spin, and satisfies $\kappa(S^3 \times S^2) = 0$. On the other hand, the 5-dimensional sphere S^5 and the manifold $\mathbb{C}P^2 \times S^1$ do not admit the Cartan $(2,3,5)$-distribution. S^5 is spin but its Kervaire semicharacteristic is nonzero. $\mathbb{C}P^2 \times S^1$ has vanishing Kervaire semicharacteristic but is not spin. Thus neither of them admits a Cartan $(2,3,5)$-distribution. In addition to them other examples are obtained. For example, direct product of a Riemannian surface with any closed orientable 3-manifold, directproduct of any spin 4-manifold with S^1, such non-trivial fiber bundle, and certain connected sum of $S^2 \times S^3$ admit the structure. On the other hand, certain 5-dimensional lense space, and certain connected sum of $S^2 \times S^3$ do not.

Another main goal of this paper is a classification of Cartan $(2,3,5)$-distributions. While Theorem A establishes their topological condition for existence, their geometric flexibility is further demonstrated by the one-parametric version of the h-principle. Because our proof relies on Gromov's convex integration method, it naturally extends to the one-parametric version of the h-principle. It implis that homotopy as formal structures yield homotopy as genuine distributions. Consequently, we obtain the following classification result:

Theorem C. *Let $\mathscr{D}_1$, $\mathscr{D}_2 \subset TM$ be Cartan $(2,3,5)$-distributions on a 5-dimensional manifold M. If they are homotopic through almost Cartan structures, then they are homotopic through the genuine Cartan $(2,3,5)$-distributions.*

Concerning the methods to show the existence of geometric structures, there are several important works. The existence of a contact structure on a closed orientable 3-manifold is proved by Martinet [Ma] by a constructive method. On the other hands, methods from the viewpoints of the h-principles are applied to some geometric structures. For higher-dimensional manifolds, it is proved by Borman, Eliashberg, and Murphy [BEM] from the viewpoint of the h-principle that, if a manifold admits an almost contact structure, it admits a genuine contact structure. The existence of Engel structure is proved by Casals, Pérez, del Pino, and Presas [CPPP] in a similar way. One of the works that has most strongly influenced this works is McDuff's h-principle [Mc] for even-contact structures. Gromov's convex integration method is applied to show the h-principle.

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