CONTINUOUS RATIONAL FUNCTIONS AND WEAK NORMALISATION OF REAL ALGEBRAIC VARIETIES

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Abstract. We present the construction of the weak normalisation relative to the central locus of real algebraic varieties, as defined by Jean-Philippe Monnier, Ronan Quarez and the author. It uses the notion of continuous rational functions, which is also reviewed here.

1. Introduction

In real algebraic geometry, the choice of a good class of functions is an important matter. From polynomial functions to semialgebraic ones, passing through rational functions, regular functions (rational without poles) or Nash ones (real analytic and semialgebraic), a large class of different functions with a specific flavour is available. In this survey paper we focus on the class of continuous functions lying between rational function (quotient of two polynomials with real coefficients) and regular functions (quotient of two polynomials with real coefficients such that the denominator admits no real zero). A typical example is given by $f = \frac{x^3}{x^2+y^2}$ on $\mathbb{R}^2$. This rational function has a pole at $(0,0)$, and f admits a continuous extension on the whole of $\mathbb{R}^2$ by setting $f(0,0) = 0$.

Here are some arguments why such a function has a geometric interest. Consider the graph of f in $\mathbb{R}^3$: it coincides with the 2-dimensional sheet of the Cartan umbrella, given by $(x^2+y^2)z = x^3$, since as a graph it has only one point over $z = 0$, whereas the Cartan umbrella has a whole stick, the z-axis. Alternatively, consider its 1-level $\{f = 1\}$ in $\mathbb{R}^2$: it coincides with the 1-dimensional part of the irreducible cubic with an isolated point at the origin given by the equation $y^2 = x^2(x-1)$, since at $(0,0)$ the function f takes the value 0, not 1. Note that this 1-dimensional part is not an algebraic set, since its Zariski closure catches the isolated point $(0,0)$.

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In both cases, we see that such a function enables to eliminate the part of smaller dimension of the real algebraic sets involved. This phenomenon is quite general, since the zero sets of such functions, considered on smooth varieties, coincides with Zariski constructible sets closed with respect to Euclidean topology.

Another important fact about these functions is that they have a particular behaviour on singular real algebraic sets. As illustrated by Kollár [13], their restriction to an algebraic subset does not necessarily remain rational ! This will lead to make a difference between general continuous rational functions, and the subclass of those continuous rational functions that remain rational by restriction to subvarieties. These particular continuous rational functions are called regulous functions, after [9].

The first study of these rational functions admitting a continuous extension at their poles, was the work of Kucharz on the good way to approximate continuous maps between spheres [19]. The surprising behaviour of rational continuous functions on singular sets has been exhibited by Kollár ([13] or [16]), whereas their systematic study on smooth varieties have been performed in [9]. After that, Kucharz and Kurdyka discussed real algebraic vector bundles using this approach [20]. Notably, we know from [13] that the restriction of a rational continuous function remains rational, provided that the ambient space is smooth. We know also that such functions on smooth varieties are exactly those continuous functions that become regular after blowings-up [9]. Note that rational continuous functions appear also naturally in the algebraic identities which answer the 17th Hilbert Problem, namely Kreisel noticed that any non-negative polynomial is a sum of squares of rational continuous functions [17].

In this notes, we explain the construction of the weak normalisation of a real algebraic variety. The complex setting is settled for more than fifty years, first in the analytic case in [3] for studying the Chow variety of a complex analytic variety, and more generally for scheme in [2]. Recently, the concept of weak normalisation appears in the study of the singularities in the minimal model program by Kollár and Kovács [15], and also in [14]. If the complex setting is well known, the different attempts in real geometry were unsuccessful before the appearance of continuous rational functions [2, 23]. The two main difficulties one has to face in the real algebraic case are the complexity of the topology of the real points of algebraic varieties, notably with regards to centrality issues, and the intriguing behaviour of rational functions (as illustrated in [13]), whose understanding is somehow the key to construct the real weak normalisation.

2. Continuous rational functions on smooth varieties

Let n be an integer and let $X \subset \mathbb{R}^n$ be an irreducible, smooth, affine variety in the sense of [7].

Let $f \in \mathbb{R}(X)$ be a rational function on X. The domain of f, denoted by $\operatorname{dom}(f)$, is the biggest Zariski open subset of X on which f is regular, namely $f = \dfrac{p}{q}$ where p and q are polynomial functions on $\mathbb{R}^n$ such that q does not vanish on $\operatorname{dom}(f)$. The indeterminacy locus of f is defined to be the Zariski closed set $\operatorname{indet}(f) = X \setminus \operatorname{dom}(f)$.

2.1. **Regulous functions.**

We recall the definition of a regulous function, or more generally a k-regulous function, on a smooth affine real algebraic variety as given in [9]. We will consider X with its induced structure of a C^∞-variety.

Definition 2.1. Let n be an integer and let $X \subset \mathbb{R}^n$ be an irreducible smooth affine variety. Let $k \in \mathbb{N} \cup \{\infty\}$. We say that a function $f : X \to \mathbb{R}$ is k-regulous on X if f is of class C^k on X and f is a rational function on X, i.e. there exists a non-empty Zariski open subset $U \subseteq X$ such that $f|_U$ is regular (namely it is a rational function such that the denominator has no real zero).
A 0-regulous function on X is simply called a regulous function.

Let $k \in \mathbb{N} \cup \{\infty\}$. We denote by $\mathcal{R}^k(X)$ the ring of k-regulous functions on X. By Theorem 3.3 of [9] we know that $\mathcal{R}^\infty(X)$ coincides with the ring $\mathcal{O}(X)$ of regular functions on X.

Denote by $\mathcal{Z}(f)$ the zero set of the real function f.

Lemma 2.2. [9] *Let $X \subset \mathbb{R}^n$ be an irreducible smooth affine variety. Let $f \in \mathcal{R}^0(X)$ and suppose $f = \frac{p}{q}$ on $\operatorname{dom}(f)$ with p and q some real polynomials in n variables with no commun factor. Then $\mathcal{Z}(q) \subset \mathcal{Z}(p)$ and $\mathcal{Z}(f) \subset \mathcal{Z}(p)$, and the codimension of $\mathcal{Z}(q)$ in $\mathcal{Z}(p)$ is at least two.*

Note that for smooth curves, a regulous function is in particular regular. On smooth surfaces, it admits at most a finite number of poles.

Proposition 2.3. [10] *Let n be an integer and let $X \subset \mathbb{R}^n$ be an irreducible, smooth, affine variety of dimension 1. Then, for $k \in \mathbb{N}$, the ring $\mathcal{R}^k(X)$ of k-regulous functions on X coincides with the ring $\mathcal{R}^\infty(X)$ of regular functions on X.*

This property enables to prove that regulous functions are in fact arc-analytic in the sense of [22] : they remain analytic after composition with an analytic arc. In fact we have also an algebraic version of arc-analyticity.

Corollary 2.4. [10] *Let $f \in \mathcal{R}^0(\mathbb{R}^n)$. Then f is arc-algebraic.*

We resume now the main properties of the ring $\mathcal{R}^0(X)$, mainly taken from [9]. The zero sets of regulous functions are exactly the Zariski constructible subsets of X which are closed with respect to the Euclidean topology. The Zariski constructible subsets of X are those sets that we obtain from the algebraic subsets of X by taking difference (as in the example in the introduction of the cubic with one isolated point, when we remove the isolated point). As a ring, if $\mathcal{R}^0(X)$ is not Noetherian, however the classical Nullstellensatz is valid (whereas $\mathcal{R}^\infty(X)$ is Noetherian, but the Nullstellensatz has to be restricted to so-called real prime ideal to remain valid), and its Krull dimension is equal to the dimension of X.

All these properties show that, even if $\mathcal{R}^0(X)$ is not Noetherian, it is anyway a good ring to perform algebraic geometry.

To know more about regulous functions, a good reference is the survey paper [21].

2.2. **Locally bounded rational functions.** More generally, locally bounded rational functions have been investigated in [8] : a rational function $f \in \mathbb{R}(X)$ is called a locally bounded rational function on X if for every $x \in X$ there exists a euclidean neighbourhood V_x of x such that $f(V_x \cap \mathrm{dom}(f))$ is a bounded subset of $\mathbb{R}$. We denote by $\mathcal{R}_b(X)$ the ring of locally bounded rational functions on X.

For instance, consider $f \in \mathbb{R}(\mathbb{R}^2)$ given by,

$$f(x,y) = \frac{x^2}{x^2+y^2}.$$

Then $\mathrm{indet}(f) = \{(0,0)\}$ and f is bounded by 1 in any neighbourhood of the origin since $x^2 \leq x^2+y^2$ for any $(x,y) \neq (0,0)$. Moreover, for $y = ax$, we see that :

$$\lim_{x\to 0} \frac{x^2}{(1+a^2)x^2} = 1/(1+a^2),$$

which is bounded between 1 (corresponding to $a=0$) and 0 (corresponding to $a=\infty$). If we consider these limits as $(x,y) \to (0,0)$ along $y = ax$ as "values" of f, then we see that f takes all its values between 1 and 0 at $(0,0)$.

These functions are exactly those rational functions that becomes regular after resolution of singularities.

Theorem 2.5. *Let $X \subseteq \mathbb{R}^n$ be an irreducible, non-singular algebraic variety. If $f \in \mathbb{R}(X)$ becomes a regular function with values in $\mathbb{R}$ after a sequence of blowings-up along non-singular centres, then f is locally bounded.*

Theorem 2.5 was proved in [18], we extend it in [8] for a general real closed field. As a consequence :

Corollary 2.6. *Every birational proper morphism $\phi : \tilde{X} \to X$ between two affine, non-singular and irreducible real algebraic varieties, $\tilde{X}$ and X induces an isomorphism between $\mathcal{R}_b(X)$ and $\mathcal{R}_b(\tilde{X})$ given by $f \mapsto f \circ \phi$.*

3. Continuous rational functions on singular varieties

The behaviour of continuous rational functions on singular varieties is quite different that on smooth varieties, and in particular the classes of continuous rational functions and regulous ones are different. This is nicely illustrated by the following example [13] : consider the surface $S = \mathcal{Z}(y^3 - (1+z^2)x^3)$ in $\mathbb{R}^3$. The continuous function defined by $(x, y, z) \mapsto \sqrt[3]{1+z^2}$ is regular on S minus the z-axis, since it coincides with y/x, however its restriction to the z-axis is no longer rational. Regulous functions are those continuous rational functions that remain rational when restricted to subvarieties.

In these notes, we will concentrate on the larger class of continuous rational functions rather than the class of regulous functions. We will also restrict our attention to irreducible real algebraic sets in order to simplify the exposition.

Note that irreducible real algebraic varieties are in general not connected for the Euclidean topology, and it may even admit some parts of smaller dimension (like the stick in Cartan umbrella) or even isolated points (like in the cubic considered in the introduction). So in order to discuss some consideration about extension by continuity of a rational function on a real variety, we need to restrict our attention to the part of maximal dimension, called the central locus :

Definition 3.1. We define the central locus of an irreducible algebraic set $X \subset \mathbb{R}^n$, denoted by $\operatorname{Cent} X$, to be the Euclidean closure $\overline{X_{reg}}^{eucl}$ in X of the set X_{reg} of non-singular points of X. We say that X is central if X is equal to $\operatorname{Cent} X$.

Definition 3.2. Let $f : X \to \mathbb{R}$ be a continuous function. We say that f is a continuous rational function on X if there exists a Zariski-dense open subset $U \subset X$ such that $f_{|U}$ is regular. We denote by $\mathcal{R}(X)$ the ring of continuous rational functions on X.

Let $f : \operatorname{Cent} X \to \mathbb{R}$ be a continuous function. We say that f is a continuous rational function on $\operatorname{Cent} X$ if there exist a Zariski-dense open subset $U \subset X$ and a regular function g on U such that the restriction $f_{|U\cap \operatorname{Cent} X}$ is equal to the restriction $g_{|U\cap \operatorname{Cent} X}$. The ring of continuous rational functions on $\operatorname{Cent} X$ is denoted by $\mathcal{R}(\operatorname{Cent} X)$.

A map $Y \to X$ between real algebraic sets $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ is called rational continuous if its components are rational continuous functions on Y.

A map $\operatorname{Cent} Y \to \operatorname{Cent} X$ between the central loci of real algebraic sets $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ is called rational continuous if its components are rational continuous functions on $\operatorname{Cent} Y$.

It is important to consider the notion of continuous rational functions on $\operatorname{Cent} X$, because outside the central locus, the extension by continuity can be very wide.

Example 3.3. Consider again Cartan umbrella $X = \mathcal{Z}(z(x^2+y^2)-x^3)$ and the function $f = x^3/(x^2+y^2)$ extended by 0 along the z-axis. Then f is rational continuous on X and so on $\operatorname{Cent} X$, but f is also polynomial on $\operatorname{Cent} X$ since its coincides there with z.

Of course f and z are different on X. Even worst, one can extend f along the z-axis by any continuous functions of the variable z vanishing at 0 (like $\sin z$).

Remark that a polynomial function on a real algebraic set X is the restriction to X of a unique polynomial function on the complexification $X_{\mathbb{C}}$ of X, and uniqueness comes from the Zariski density of X in $X_{\mathbb{C}}$.

Note also that on a curve with isolated points, like the cubic curve $\mathcal{Z}(y^2-x^2(x-1))$, a function regular on the one-dimensional branches can be extended continuously by any real value at the isolated points. In particular, the natural ring morphism $\mathcal{R}(X) \to \mathbb{R}(X)$ which sends $f \in \mathcal{R}(X)$ to the class $(U, f_{|U})$ in $\mathbb{R}(X)$, where $U \subset X$ is a Zariski-dense open subset such that $f_{|U}$ is regular, is not injective in general. However, restricting our attention to the central locus, now the canonical map $\mathcal{R}(\operatorname{Cent} X) \to \mathbb{R}(X)$ is now injective since the domain of a rational function is dense in $\operatorname{Cent} X$ for Euclidean topology.

We can prove that the continuous rational functions on the central locus of X are exactly the functions on $\operatorname{Cent} X$ that become regular after a well chosen resolution of singularities of X.

Proposition 3.4. [11] *Let* $f : \operatorname{Cent} X \to \mathbb{R}$ *be a real function. Then* $f \in \mathcal{R}(\operatorname{Cent} X)$ *if and only if there exists a resolution of singularities* $\pi : Y \to X$ *such that* $f \circ \pi$ *is regular on* Y*.*

The proof relies on the fact that the image of Y under π is equal to the central locus of X. Note that the ring of continuous rational functions on a central algebraic set has some good algebraic properties. For instance, $\mathcal{R}(\operatorname{Cent} X)$ is integrally closed.

Theorem 3.5. [11] *Let* X *be an irreducible algebraic set such that* $\operatorname{Cent}(X) = X_{reg}$*. Then* $\mathcal{R}(\operatorname{Cent} X)$ *is integrally closed in* $\mathbb{R}(X)$*.*

A rational function which does not admit a continuous extension on a given algebraic set may admit different behaviors at a indeterminacy point. It can be unbounded like $1/x$ at the origin in $\mathbb{R}$, bounded with infinitely many limit points like $x^2/(x^2+y^2)$ at the origin in $\mathbb{R}^2$, or bounded with finitely many limit points like in the case of rational function satisfying an integral equation with continuous rational coefficients on the central locus.

Lemma 3.6. [11] *Let* X *be an irreducible algebraic set. Assume* $f \in \mathbb{R}(X)$ *satisfies an integral equation with coefficients in* $\mathcal{R}(\operatorname{Cent} X)$*. Then* f *admits finitely many different limits at its indeterminacy points that are central.*

In particular, such functions are locally bounded in the sense of the previous section, but now on a singular set.

4. Weak normalisation via continuous rational functions

Given an irreducible algebraic set X, the process of constructing the weak normalisation of X consists in adding to the ring of polynomial functions $\mathcal{P}(X)$ of X, all continuous rational functions which are integral over $\mathcal{P}(X)$. The results in this section come from [12].

4.1. **Adding one integral continuous rational function to $\mathcal{P}(X)$.** We investigate first the action of adding to $\mathcal{P}(X)$ one continuous rational function f which is integral over the ring of polynomial functions $\mathcal{P}(X)$ on X.

The inclusion $\mathcal{P}(X) \to \mathbb{R}(X)$ factorizes via the morphism $\phi : \mathcal{P}(X)[t] \to \mathbb{R}(X)$ defined by $t \mapsto f$, and it induces a morphism $\mathcal{P}(X)[t]/\operatorname{Ker}\phi \to \mathbb{R}(X)$. Since f is integral, $\mathcal{P}(X) \to \mathcal{P}(X)[t]/\operatorname{Ker}\phi$ is a finite ring homomorphism and $\mathcal{P}(X)[t]/\operatorname{Ker}\phi$ is the coordinate ring $\mathcal{P}(Y)$ of a certain algebraic set Y. The function f corresponds then to the new variable t.

Taking into account additionally the continuity of f leads to :

Proposition 4.1. *Let X be an irreducible algebraic set. Let f be a rational function on X that is integral on $\mathcal{P}(X)$ and assume that $f \in \mathcal{R}(\operatorname{Cent} X)$. Denote by Y the algebraic set such that $\mathcal{P}(Y) = \mathcal{P}(X)[f]$, by t the polynomial function in $\mathcal{P}(Y)$ that corresponds to f and by $\pi : Y \to X$ the associated finite birational morphism. Then the continuous rational function $f \circ \pi$ coincides with the polynomial function t on* $\operatorname{Cent} Y$.

So we see that adding such a function to the polynomial ring of X leads to a new variety on which the continuous rational function becomes a polynomial function. The idea to produce the weak normalisation is to saturate $\mathcal{P}(X)$ with such functions.

4.2. **Adding all integral continuous rational functions to $\mathcal{P}(X)$.** We define the weak normalisation of an irreducible algebraic set X by adding to $\mathcal{P}(X)$ all continuous rational functions f which are integral over the ring of polynomial functions $\mathcal{P}(X)$.

Definition 4.2. The weak normalization relative to the central locus X^{w_c} of X is the algebraic set whose ring of polynomial functions is the integral closure of $\mathcal{P}(X)$ in $\mathcal{R}(\operatorname{Cent} X)$.

The natural finite birational morphism $\pi^{w_c} : X^{w_c} \to X$ is called the weak normalisation (relative to the central locus) morphism. The algebraic set X is called centrally weakly-normal if $X = X^{w_c}$.

Remark 4.3. The fact that there exist such an algebraic set X^{w_c} corresponding to the integral closure A of $\mathcal{P}(X)$ in $\mathcal{R}(\operatorname{Cent} X)$ comes from a classical fact in algebra. Indeed, A is a submodule of the Noetherian $\mathcal{P}(X)$-module $\mathcal{P}(X')$ so that A is a finite $\mathcal{P}(X)$-module. As a consequence A is a finitely generated $\mathbb{R}$-algebra, so there exist such a variety X^{w_c}.

In particular the real algebraic set X^{w_c} is an intermediate algebraic set between X and X', and we have a sequence of finite birational morphisms $X' \to X^{w_c} \to X$. From the definition, we see that a variety X is centrally weakly-normal if and only if every function in $\mathcal{R}(\operatorname{Cent} X)$ integral over $\mathcal{P}(X)$ is the restriction to $\operatorname{Cent} X$ of a polynomial function on X.

Example 4.4.

(1) Let X be the cuspidal curve given by $y^2 = x^3$ in $\mathbb{R}^2$. The normalisation of X is obtained by adding the rational function $f = y/x$, which is integral since $f^2 = x$, to $\mathcal{P}(X)$. Note that f admits a continuous extension at the origin, so X' coincides with X^{w_c}.

(2) Let X be the nodal curve given by $y^2 = x^2(x+1)$ in $\mathbb{R}^2$. The normalisation of X is obtained by adding the rational function $f = y/x$, which is integral since $f^2 = x+1$, to $\mathcal{P}(X)$. Note that f does not admit a continuous extension at the origin (the possible limits of f at the origin are ± 1), and X^{w_c} coincides with X in this case.

(3) Let X be the curve given by $y^2 = x^4(1-x^2)$ in $\mathbb{R}^2$ (it looks like a symbol "infinity", a bit flat at the origin). The normalisation of X is obtained by adding the rational function $f = y/x^2$, which is integral since $f^2 = 1-x^2$, to $\mathcal{P}(X)$. Note that f does not admit a continuous extension at the origin. However $g = y/x$, which satisfies $g^2 = x^2(1-x^2)$, admits a continuous extension at the origin by setting $g(0,0) = 0$, and X^{w_c} is indeed the algebraic curve defined by $y^2 = x^2(1-x^2)$ in $\mathbb{R}^2$. So here $X \neq X^{w_c} \neq X'$.

We see with these examples that for curves, the weak normalisation smoothes the cusp and simplifies, but does not separates, the crossings.

Example 4.5. Consider Kollár surface $X = \mathcal{Z}(x^3 - y^3(1+z^2))$. The polynomial ring of the normalization is given by $\mathcal{P}(X') = \mathcal{P}(X)[x/y]$ and the rational function x/y can be extended to a continuous function on X. As a consequence X' coincides with X^{w_c}.

An important fact is that the new variety obtained is in bijection with the beginning variety, at least on the central loci.

Proposition 4.6. *The weak normalisation (relative to the central locus) morphism $\pi^{w_c} : X^{w_c} \to X$ is a bijection between the central loci and its inverse map is rational continuous on the center of X.*

This result leads to a universal property of the weak normalisation.

Theorem 4.7. *Let X be an algebraic set. Let Y be an algebraic set equipped with a finite birational morphism $\pi : Y \to X$.*

Then π induces a bijection from $\operatorname{Cent} Y$ *to* $\operatorname{Cent} X$ *if and only if $\pi^{w_c} : X^{w_c} \to X$ factorizes through π.*

The universal property enables to prove that some bijective maps are in fact isomorphisms.

Corollary 4.8. *Let X be an algebraic set. Suppose that X is centrally weakly-normal and that $\varphi : Y \to X$ is a finite birational polynomial morphism with Y an algebraic set. Then φ is a bijection on the central loci if and only if φ is an isomorphism.*

Remark 4.9. Note that a bijective finite birational polynomial morphism onto a central algebraic set is not necessarily an isomorphism while it is an isomorphism in the category of rational continuous maps. For instance, let X be the cuspidal curve given by $y^2 = x^3$ in $\mathbb{R}^2$, and X' be its normalization. The normalization morphism $\pi : X' \to X$ is birational, finite and bijective. It is even an homeomorphism with respect to the Zariski topology (the curves are irreducible, so the Zariski subsets are just points). However X is singular whereas X' is smooth.

4.3. **Other related works.** The construction of the weak normalisation relative to the central locus, using continuous rational functions, can also be performed using regulous functions. The variety obtained that way is called the seminormalisation relative to the central locus. It has similar properties than the weak normalisation, but this time the bijections appearing in the universal property are not only rational continuous, but biregulous. This construction is also described in [12].

Actually the seminormalisation of a complex algebraic variety can also be done using the notion of (complex) continuous rational functions, and this has been done by Bernard [4]. Bernard has also develop a complex bi-Lipschitz version [6], together with a real version taking into account complex points in the normalisation of X lying over real points of X [5].

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