On the Number of Fold Components of Image Simple Fold Maps

Rustam Sadykov
Department of Mathematics, Kansas State University

Osamu Saeki
Institute of Mathematics for Industry, Kyushu University

Abstract. In this article, we briefly survey the study of the number of fold components of image simple fold maps into surfaces. In particular, we clarify under what conditions this number modulo two is a homotopy invariant. We also consider the algebraic number of fold components for image simple fold maps in the case where the target is the plane $\mathbf{R}^2$.

1. Introduction

This is a survey article about the parity of the number of fold components for image simple fold maps of closed manifolds into surfaces. Details can be found in [2, 5].

Throughout the paper, we work in the C^∞ category: manifolds and maps between them are smooth of class C^∞ unless otherwise stated.

Let M^n and N^p denote manifolds of dimensions n and p, respectively, with $n \geq p \geq 1$. We consider a smooth map $f\colon\ M^n \to N^p$.

Definition 1.1. A point $q \in M^n$ is a *singular point* if the rank of the differential $df_q\colon\ T_qM^n \to T_{f(q)}N^p$ is strictly smaller than p. We denote by $S(f)$ the set of singular points of f. We say that f is *image simple* if the restriction $f|_{S(f)}\colon\ S(f) \to N^p$ is a topological embedding.

Definition 1.2. A point $q \in S(f)$ is a *fold singularity* (or, simply, a *fold*) if, around q, f can be represented by the map

$$(x_1, x_2, \ldots, x_n) \mapsto (x_1, x_2, \ldots, x_{p-1}, \pm x_p^2 \pm x_{p+1}^2 \pm \cdots \pm x_n^2)$$

with respect to certain local coordinates around q and $f(q)$. The number of negative signs in the last coordinate function is called the *index* with respect to the direction specified by the last coordinate in the target.

The point q is a *definite fold singularity* (or, simply, a *definite fold*) if the signs appearing above are all the same, i.e. if the index is equal to 0 or $n-p+1$. Otherwise, q is called an *indefinite fold singularity* (or, simply, *indefinite fold*). We say that f is a *fold map* if $S(f)$ consists only of fold singularities.

It is easy to check that if f is a fold map, then $S(f)$ is a closed $(p-1)$-dimensional submanifold of M^n and the restriction $f|_{S(f)}\colon S(f) \to N^p$ is a (codimension one) immersion. In particular, such an f is image simple if and only if $f|_{S(f)}$ is a smooth embedding.

In the following, we mainly consider the case $p = 2$.

Definition 1.3. A singular point $q \in S(f)$ of a smooth map $f\colon M^n \to N^2$ that has the normal form

$$(t_1, t_2, x_1, x_2, \dots, x_{n-2}) \mapsto (t_1, t_2^3 + t_1 t_2 \pm x_1^2 \pm x_2^2 \pm \cdots \pm x_{n-2}^2)$$

is called a *cusp singularity* (or, simply, a *cusp*).

Smooth maps $f\colon M^n \to N^2$ with only folds and cusps as their singularities are often said to be *generic*. It has been known that the set of generic maps is open and dense in the space $C^\infty(M^n, N^2)$ of smooth maps of M^n into N^2 endowed with the Whitney C^∞ topology [1, 11, 12].

Example 1.4. Let $f_1\colon S^3 \to S^2$ be the Hopf fibration. Its homotopy class $[f_1]$ is a generator of $\pi_3(S^2) \cong \mathbf{Z}$. By the birth move [10, Figure 5], the map f_1 is homotopic to a map $g_1\colon S^3 \to S^2$ such that $S(g_1) \cong S^1$ consists of two cusps, an arc of definite folds and another arc of indefinite folds. In its turn, by Levine's elimination technique [3], the map g_1 is homotopic to a fold map $h_1\colon S^3 \to S^2$ with one circle of indefinite folds and one circle of definite folds (see Fig. 1, where the red curves represent the image of definite folds and the black curves represent the image of indefinite folds).

Now, for a positive integer $n \geq 1$, we consider the "connected sum" $h_n\colon S^3 \to S^2$ of n copies of $h_1\colon S^3 \to S^2$. More precisely, we consider $t_m \circ h_1$, where $t_m\colon S^2 \to S^2$ is a certain rotation through angle $m\pi/n$ with respect to a fixed axis for $m = 0, 1, \dots, n-1$, and take their "connected sum" (for this "connected sum" operation, refer to [6, Lemma 5.4]). See Fig. 2. Note that h_n represents $n \in \mathbf{Z} \cong \pi_3(S^2)$. Note also that h_n is an image simple fold map.

Motivated by the above example, Masamichi Takase posed the following question (see [8]).

Question 1.5 (Takase, 2019). Let $f\colon S^3 \to S^2$ be an arbitrary image simple fold map representing $n \in \mathbf{Z} \cong \pi_3(S^2)$. Let $\sharp S(f)$ denote the number of components of $S(f)$. Then, does the congruence $\sharp S(f) \equiv n + 1 \pmod 2$ always hold?

Note that for the image simple fold map h_n constructed above the congruence in the Takase question holds.

Then, the second author of the present paper showed that the answer to the above question is negative in general by constructing a counter example (see [9]).

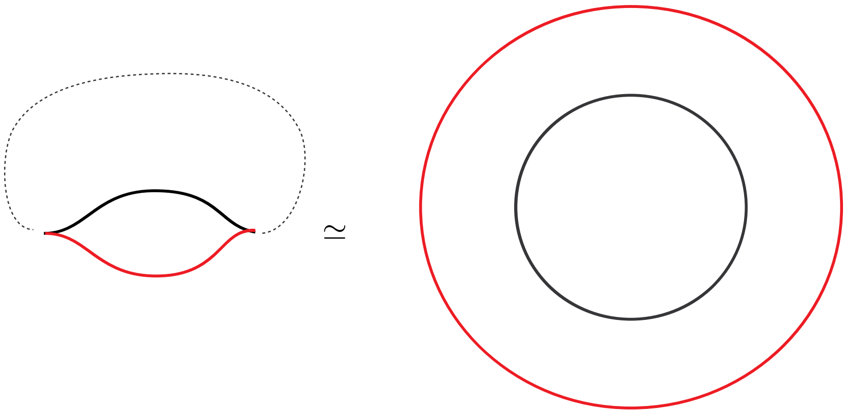

Figure 1. The Hopf fibration f_1 is homotopic to $g_1\colon S^3 \to S^2$ such that $g_1(S(g_1)) \subset S^2$ is as depicted in the left-hand figure. The two cusps of g_1 can be eliminated along the dotted curve using Levine's technique so that we obtain an image simple fold map h_1 such that $h_1(S(h_1))$ is as depicted in the right-hand figure. The maps f_1, g_1 and h_1 are homotopic to each other and represent a generator of $\pi_3(S^2) \cong \mathbf{Z}$.

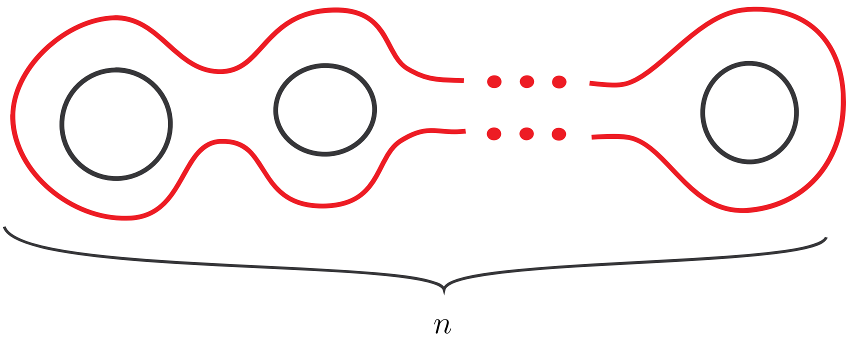

Figure 2. The figure represents the image of the singular point set of h_n in S^2.

The existence of such a counter example leads to the following more general problem.

Problem 1.6. Let $f\colon M^n \to S$ be an image simple fold map of a closed n-dimensional manifold with $n \geq 2$ into a surface S. Under what conditions is $\sharp S(f) \pmod 2$ a homotopy invariant of f?

2. Results

In this section we essentially answer Problem 1.6. Theorem 2.1 gives a positive answer under the assumptions that the dimension of the source manifold is even, and the target surface is orientable. The next two theorems show that if either of these assumptions is dropped, then the parity of the number of fold components need not be a homotopy invariant.

The following criterion has been proved in [2].

Theorem 2.1 (Kahmeyer and Sadykov, 2024)**.** *The parity of the number of fold components,* $\sharp S(f) \pmod 2$*, is a homotopy invariant of image simple fold maps* $f\colon M^n \to S$ *if* $n = \dim M^n \geq 2$ *is even and* S *is orientable.*

In §3, we give a proof of Theorem 2.1 different from the original one in [2].

In the case where $S = \mathbf{R}^2$ the statement of Theorem 2.1 can be strengthened by considering the algebraic number of fold components rather than the parity of the number of fold components. This has been proved by Levine [4]. In §4 we give another proof of this fact, see Theorem 4.2.

When n is odd, the situation changes drastically. The following negative result has been proved in [5].

Theorem 2.2 (Sadykov and Saeki, 2025)**.** *Suppose* $n \geq 3$ *is odd. For an arbitrary image simple fold map* $f\colon M^n \to S$ *of a closed* n*-dimensional manifold* M^n *into a surface* S*, there exists an image simple fold map* $g\colon M^n \to S$ *homotopic to* f *such that* $\sharp S(g) = \sharp S(f) + 1$*.*

For maps into non-orientable surfaces, again we have a negative result [5].

Theorem 2.3 (Sadykov and Saeki, 2025)**.** *Let* $n \geq 2$ *be an arbitrary even integer. Then, there exist a closed* n*-dimensional non-orientable manifold* M *and two image simple fold maps* $f, g\colon M^n \to \mathcal{M}$ *into the open Möbius band* $\mathcal{M}$ *such that* f *is homotopic to* g *but* $\sharp S(f) \not\equiv \sharp S(g) \pmod 2$*.*

In §5 we go over the constructions that prove Theorems 2.2 and 2.3.

3. Homotopy invariance of $\sharp S(f) \pmod 2$

In this section we prove Theorem 2.1. In the following, χ denotes the Euler characteristic.

We may assume that S is connected. Let $f\colon M^n \to S$ be an image simple fold map, where $n \geq 2$ is even and S is an orientable surface. First, we suppose that S is a closed surface. Note that $f(S(f))$ is a finite disjoint union of simple closed curves in S. Let $S \setminus f(S(f)) = \sqcup R_i$ be the decomposition into connected components. For each i, take a point $y_i \in R_i$ and set χ_i to be the Euler characteristic $\chi(f^{-1}(y_i))$ of the regular fiber $f^{-1}(y_i)$ with the convention that $\chi_i = 0$ if $f^{-1}(y_i) = \emptyset$. Note that the value χ_i does not depend on the choice of y_i by

Ehresmann's fibration theorem applied to the submersion f restricted to $f^{-1}(R_i)$.

Since S is connected, the regular fibers of f are all cobordant. Hence the parity of χ_i does not depend on i. Another way to see this is to take an embedded arc γ in S which intersects $f(S(f))$ transversely at exactly one point in its interior, and which connects y_i and y_j for some i and j. Then, $f|_{f^{-1}(\gamma)}\colon f^{-1}(\gamma) \to \gamma$ can be identified with a Morse function with exactly one critical point on a cobordism between $f^{-1}(y_i)$ and $f^{-1}(y_j)$. Since the dimensions of the regular fibers are even, the difference $\chi_i - \chi_j$ is equal to ± 2.

We set $r_f = 0 \in \mathbf{Z}/4\mathbf{Z}$ if χ_i are all even, and set $r_f = 1 \in \mathbf{Z}/4\mathbf{Z}$ otherwise. Let $\widetilde{R}_0$ (or $\widetilde{R}_1$) be the union of those R_i's for which $\chi_i \equiv r_f \pmod 4$ (resp. $\chi_i \equiv r_f + 2 \pmod 4$). Since the Euler characteristics χ_i of fibers over adjacent regions differ by 2 modulo 4, we have $S = \overline{\widetilde{R}_0} \cup \overline{\widetilde{R}_1}$ and $\overline{\widetilde{R}_0} \cap \overline{\widetilde{R}_1} = f(S(f))$, where $\overline{\widetilde{R}_0}$ and $\overline{\widetilde{R}_1}$ denote the closures in S of $\widetilde{R}_0$ and $\widetilde{R}_1$, respectively. Since $f(S(f))$ is a finite disjoint union of embedded circles, we have $\chi(\widetilde{R}_1) = \chi(S) - \chi(\widetilde{R}_0)$ and

$$\begin{aligned}\chi(M^n) &= \sum \chi_i \chi(R_i) \\ &\equiv r_f \chi(\widetilde{R}_0) + (r_f + 2)\chi(\widetilde{R}_1) \pmod 4 \\ &\equiv 2\chi(\widetilde{R}_0) + (r_f + 2)\chi(S) \pmod 4.\end{aligned}$$

Furthermore, since $\overline{\widetilde{R}_0}$ is a compact orientable surface with boundary $f(S(f))$, we have $2\chi(\widetilde{R}_0) \equiv 2\sharp S(f) \pmod 4$. Consequently,

$$\sharp S(f) \equiv (\chi(M^n) - (r_f + 2)\chi(S))/2 \pmod 2.$$

Since r_f is a homotopy invariant of f, this proves Theorem 2.1 in the case where S is closed.

Remark 3.1. The above proof shows that $\chi(M^n)$ is always even, since $\chi(S)$ is even. This can also be deduced from a result due to Whitney [12] and Thom [11] stating that the parity of $\chi(M^n)$ coincides with that of the number of cusp points, or from an Euler characteristic formula modulo 2 obtained in [7]. The homotopy invariance of r_f can also be proved by using arguments in [7], for example.

When S is not closed, let $H\colon M^n \times [0,1] \to S$ be a homotopy between two image simple fold maps f and $g\colon M^n \to S$. Since $H(M^n \times [0,1])$ is a compact subset of S, we may assume that H is a map into a closed orientable surface containing $H(M^n \times [0,1])$. Then, the above argument applies. This completes the proof of Theorem 2.1.

Remark 3.2. The above proof shows that if $r_f = 0$, then $\sharp S(f) \equiv \chi(M^n)/2 \pmod 2$. For example, this congruence holds if f is a map into an open surface S. See also §4 for a refined result in the case where $S = \mathbf{R}^2$.

Remark 3.3. The following comment is due to Kenta Hayano. When M^n is of even dimension and S is orientable, the parity of $\sharp S(f)$ seems to be an invariant of something much weaker than the homotopy class of f.

In fact, according to our new proof, it is an invariant of the parity of the Euler characteristic of a regular fiber. For example, when M^4 is orientable of dimension 4, a regular fiber is also orientable and its Euler characteristic is always even. This means that the parity of $\sharp S(f)$ depends only on $\chi(M^4)$ and $\chi(S)$. A sharper argument may be possible.

4. Integral lift of $\sharp S(f)$ (mod 2)

In this section, we consider a refined version of Theorem 2.1: i.e., the homotopy invariance of a certain integral lift of $\sharp S(f)$ (mod 2).

The following problem has been posed by Tatsuro Shimizu.

Problem 4.1. When $\sharp S(f)$ (mod 2) is a homotopy invariant, can we define a "sign" $\in \{\pm 1\}$ for each component of $S(f)$ in such a way that the number of connected components of $S(f)$ counted with signs is an integer homotopy invariant of f?

The answer is, in fact, given by Levine [4] for the case where the target surface S is the plane $\mathbf{R}^2$. Here, we give another formulation and its proof in line with the discussion in §3.

Let $f\colon M^n \to \mathbf{R}^2$ be an image simple fold map of a closed manifold M^n of even dimension $n \geq 2$ to the plane. Let γ be a component of $S(f)$. The complement $\mathbf{R}^2 \smallsetminus f(\gamma)$ is a disjoint union of an open 2-disk and an unbounded region. Let x_0 be a point in the open 2-disk near $f(\gamma)$ and x_∞ be a point in the unbounded region near $f(\gamma)$. We say that γ (or $f(\gamma)$) is *positive* if $\chi(f^{-1}(x_0)) > \chi(f^{-1}(x_\infty))$. Otherwise, we say that γ (or $f(\gamma)$) is *negative*.

The following theorem is essentially due to Levine [4]. Here, we give another proof based on an argument similar to that in §3.

Theorem 4.2. *Let $f\colon M^n \to \mathbf{R}^2$ be an image simple fold map of a closed manifold M of even dimension $n \geq 2$ to $\mathbf{R}^2$. Then, the Euler characteristic $\chi(M^n)$ is even, and the algebraic number of fold components counted with signs is equal to*

$$p(f) - n(f) = \chi(M^n)/2,$$

where $p(f)$ and $n(f)$ are the numbers of positive and negative components of $S(f)$, respectively. In particular, the algebraic number of fold components is a homotopy invariant of f: it depends only on $\chi(M^n)$.

Proof. First, let us define the following notion. For any component R of $\mathbf{R}^2 \smallsetminus f(S(f))$, the *depth* $d(R)$ of the region R is the algebraic number of times a generic ray from a point in R crosses $\Delta = f(S(f))$, where crossings with positive components of Δ are counted with positive signs, while crossings with negative components of Δ are counted with negative signs. Note that this is well-defined due to the vanishing of $H_1(\mathbf{R}^2; \mathbf{Z})$.

The complement $\mathbf{R}^2 \smallsetminus \Delta$ of $\Delta = f(S(f))$ is a disjoint union $R_\infty \sqcup R_1 \sqcup \cdots \sqcup R_k$ of regions, where R_∞ is the unbounded region. For $y_i \in R_i$, set $F_i = f^{-1}(y_i)$. In particular, the fiber F_∞ is empty. As we have seen in §3, the Euler characteristics of the fibers over two adjacent regions differ by ± 2. Therefore $d_i = \chi(F_i)/2$ is the depth of the region R_i.

For each component S_j of Δ, choose a small closed annular neighborhood U_j of S_j, and choose a large closed 2-disk $B \subset \mathbb{R}^2$ containing all U_j in its interior. Put $U = \cup U_j$. Then, we have

$$M^n = f^{-1}(B \smallsetminus \operatorname{Int} U) \cup f^{-1}(U).$$

Since $f^{-1}(U)$ and $f^{-1}(\partial U)$ are disjoint unions of locally trivial bundles over circles, their Euler characteristics are zero. Thus, we have $\chi(M^n) = \chi(f^{-1}(B \smallsetminus \operatorname{Int} U))$.

Let Q_i denote the component of $B \smallsetminus \operatorname{Int} U$ in the region R_i, $1 \le i \le k$. Then Q_i is diffeomorphic to $\overline{R_i}$, and we have

$$\chi(M^n) = \sum \chi(f^{-1}(Q_i)) = \sum \chi(F_i)\chi(Q_i) = \sum 2d_i \chi(\overline{R_i}).$$

It remains to compute this sum. Note that the sum on the right hand side only depends on the signed components of Δ, which determine the regions R_i and their depths. For this reason, we will denote the sum $\sum 2d_i \chi(\overline{R_i})$ by χ_Δ. We emphasize that the sum χ_Δ is well-defined for every finite family of simple closed curves Δ in $\mathbf{R}^2$ with signed components.

Choose an innermost component C of Δ, and let D be the open 2-disk bounded by C. Let R be the region adjacent to C on the other side. For $R' = R \cup D$, we have $\chi(\overline{R'}) = \chi(\overline{R}) + 1$. If C is positive, then $d(D) = d(R) + 1$, and therefore

$$2d(D)\chi(\overline{D}) + 2d(R)\chi(\overline{R}) = 2(d(R)+1) + 2d(R)\chi(\overline{R}) = 2 + 2d(R)\chi(\overline{R'}).$$

Thus, for $\Delta' = \Delta \smallsetminus C$, we have $\chi_\Delta = 2 + \chi_{\Delta'}$. Similarly, in the case of a negative innermost component C, deleting C from Δ results in increasing the sum by 2.

Iterating this procedure, we essentially remove all components of Δ and reduce the sum to zero. Hence, if $p(f)$ and $n(f)$ denote the numbers of positive and negative fold components of f, respectively, then we have $\chi(M^n) = \sum 2d_i \chi(\overline{R_i}) = 2p(f) - 2n(f)$. This completes the proof. □

Remark 4.3. Levine's proof of Theorem 4.2 given in [4] is based on generic projections $\pi : \mathbf{R}^2 \to \mathbf{R}$ such that the composition $\pi \circ f \colon M^n \to \mathbf{R}$ is a Morse function.

We do not know the answer to Problem 4.1 when S is not $\mathbf{R}^2$.

5. Changing $\sharp S(f)$ (mod 2)

Let us prove Theorem 2.2. Let $f\colon M^n \to S$ be an image simple fold map of a closed n-dimensional manifold M^n into a surface S, where $n \geq 3$ is odd.

Let Δ be a small 2-disk contained in $f(M^n) \smallsetminus f(S(f))$. We modify the image simple fold map f by homotopy with support on a connected component of $f^{-1}(\Delta)$, following the procedure as described in Fig. 3. These figures depict the images of singular point sets of generic maps inside Δ. The integers attached to some curves indicate the corresponding indices of the relevant folds with respect to the directions indicated by arrows. The lower left figure depicts the singular value set of f inside Δ, i.e. the empty set. Then, we create three wrinkles, using the birth move three times (see [10, Fig. 4]). Next, by three flip moves, we turn the wrinkles into three "bow ties" made up of folds of indices $n-1$, n and $n+1$ (see [10, Fig. 4]). Note that reversing the chosen direction interchanges the indices $n-1$ and $n+1$. Then, we merge the three "bow ties" as in the upper right figure by the cusp elimination technique due to Levine [3]. This is possible, since we are modifying the maps only on a connected component of $f^{-1}(\Delta)$ so that the paths between the images of the cusps that are merged can be lifted to curves in M^n connecting the relevant cusps. As the three fold curves all have indices $n \geq 2$, we can then perform the type III crossing as in the lower right figure (see [10, Fig. 6]). Finally, we perform three unflips. This is possible, since the involved folds are of indices n and $n+1$ with $2 \leq n \leq 2n-1$ (see [10, Fig. 4]). This completes the proof of Theorem 2.2.

Remark 5.1. There is also another proof of Theorem 2.2 using higher-dimensional Dehn twists. For details, see [5].

Let us now prove Theorem 2.3.

Let w and $z\colon S^1 \to \mathbf{R}$ be two Morse functions such that (refer to Fig. 4)

- w has exactly two critical points a_1 and a_2 of indices 0 and 1, respectively, with $w(a_1) = -1$ and $w(a_2) = 1$,
- z has exactly four critical points b_1, b_2, b_3 and b_4 of indices $0, 0, 1$ and 1, respectively, with $z(b_1) = -1$, $z(b_2) = -1/2$, $z(b_3) = 1/2$ and $z(b_4) = 1$,
- there exists an orientation reversing involution $\sigma\colon S^1 \to S^1$ such that $w \circ \sigma = -w$, $\sigma(a_1) = a_2$ and $\sigma(a_2) = a_1$,
- there exists an orientation reversing involution $\tau\colon S^1 \to S^1$ such that $z \circ \tau = -z$, $\tau(b_1) = b_4$, $\tau(b_2) = b_3$, $\tau(b_3) = b_2$ and $\tau(b_4) = b_1$.

Then, w induces a fold map

$$\mathrm{id}_{[0,1]} \times w\colon\ [0,1] \times S^1 \to [0,1] \times \mathbf{R},$$

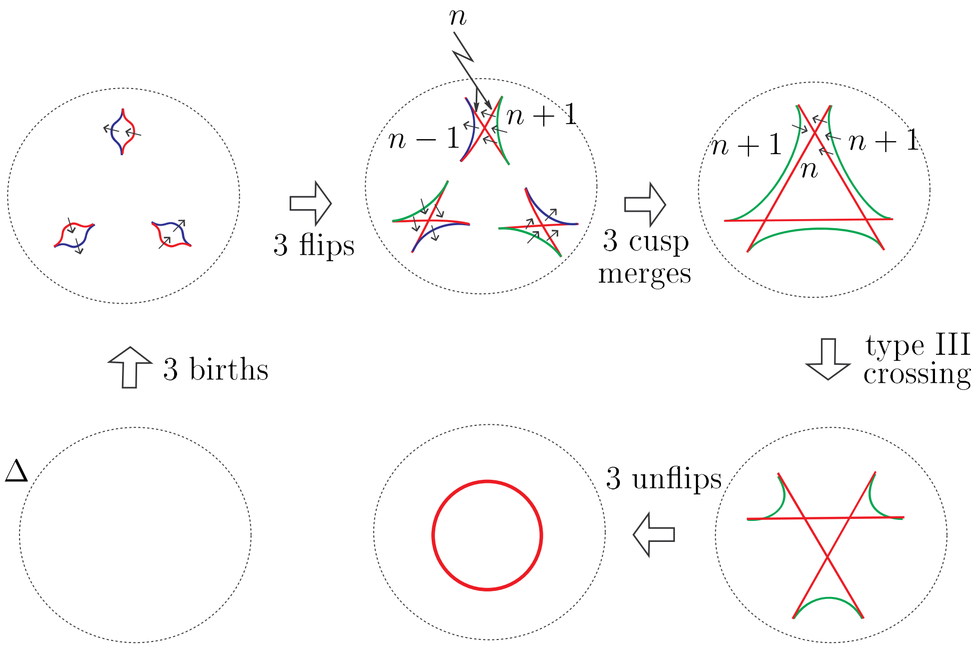

FIGURE 3. Sequence of moves

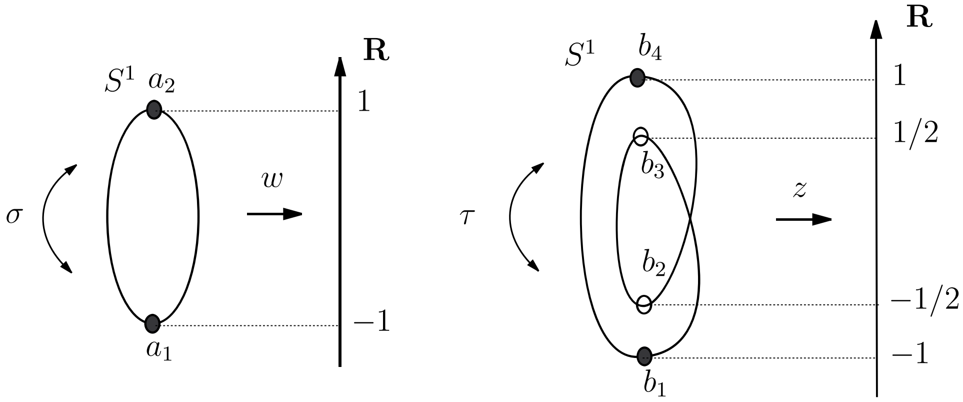

FIGURE 4. Two Morse functions w and z on S^1 and the involutions σ and τ of S^1

which further induces a fold map

$$\widetilde{w} = \mathrm{id}_{S^1} \widetilde{\times} w \colon\ S^1 \widetilde{\times} S^1 \to S^1 \widetilde{\times} \mathbf{R},$$

where $\mathrm{id}_{[0,1]}$ is the identity map of the interval $[0,1]$, $S^1 \widetilde{\times} S^1$ is obtained from $[0,1] \times S^1$ by identifying $(1,x)$ with $(0,\sigma(x))$ for each $x \in S^1$, and $S^1 \widetilde{\times} \mathbf{R}$ is obtained from $[0,1] \times \mathbf{R}$ by identifying $(0,t)$ with $(1,-t)$ for each $t \in \mathbf{R}$. Similarly, we get a fold map

$$\widetilde{z} = \mathrm{id}_{S^1} \widetilde{\times} z \colon\ S^1 \widetilde{\times} S^1 \to S^1 \widetilde{\times} \mathbf{R}.$$

We see that $\widetilde{w}$ and $\widetilde{z}$ are image simple fold maps from the Klein bottle $S^1 \widetilde{\times} S^1$ to the open Möbius band $\mathcal{M}$. Furthermore, we have $\sharp S(\widetilde{w}) = 1$, while $\sharp S(\widetilde{z}) = 2$. On the other hand, since w and z are null-homotopic, the maps $\widetilde{w}$ and $\widetilde{z}$ are homotopic. Hence, the parity of the number of connected components of singular point set is not a homotopy invariant for image simple fold maps in general. This completes the proof for the case $n = 2$.

For $n \geq 4$ even, a similar construction works (for details, see [5]). This completes the proof of Theorem 2.3.

6. Further problems

We conclude with some related problems.

Problem 6.1. The following problem has been posed by Kentaro Saji. Let $f\colon M^n \to S$ be an image simple fold map of a closed n-dimensional manifold M^n into a surface S with $n \geq 2$. When $n = \dim M^n \geq 3$ is odd, is it possible, under some circumstances, to decrease the number of fold components of $\sharp S(f)$ by one?

Problem 6.2. The following problem has been posed by Toshizumi Fukui. What can be said for other dimension pairs (n, p)? Are there such dimension pairs for which a similar problem can be interesting?

Acknowledgements

The authors would like to thank Liam Kahmeyer for discussions of the topic. The second author would like to thank the Department of Mathematics, Kansas State University, for the hospitality which enabled the authors to conduct fruitful joint research. This work has been partially supported by JSPS KAKENHI Grant Numbers JP22K18267, JP23H05437. This work was also supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University. The authors would like to thank the organizers and the participants of the workshop, especially Kenta Hanayo, Tatsuro Shimizu, Kentaro Saji, and Toshizumi Fukui, for the valuable discussions and questions about the topic.

References

[1] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Graduate Texts in Mathematics, Vol. 14, Springer-Verlag, New York-Heidelberg, 1973.

[2] L. Kahmeyer and R. Sadykov, *A homotopy invariant of image simple fold maps to oriented surfaces*, J. Singul. **27** (2024), 193–219.

[3] H. I. Levine, *Elimination of cusps*, Topology **3**, Suppl. 2 (1965), 263–296.

[4] H. I. Levine, *Mappings of manifolds into the plane*, Amer. J. Math. **88** (1966), 357–365.

[5] R. Sadykov and O. Saeki, *On the number of components of folds of image simple fold maps*, to appear in Ensaios Matemáticos, preprint is available at arXiv:2509.03878 [math.GT].

[6] O. Saeki, *Topology of special generic maps of manifolds into Euclidean spaces*, Topology Appl. **49** (1993), 265–293.
[7] O. Saeki, *Studying the topology of Morin singularities from a global viewpoint*, Math. Proc. Camb. Phil. Soc. **117** (1995), 223–235.
[8] O. Saeki, *Elimination of definite fold. II*, Kyushu J. Math. **73** (2019), 239–250.
[9] O. Saeki, *A signature invariant for stable maps of* 3*-manifolds into surfaces*, Rev. Roumaine Math. Pures Appl. **64** (2019), 541–563.
[10] O. Saeki, *Simplifying generic smooth maps to the* 2*–sphere and to the plane*, arXiv:2407.10145 [math.GT].
[11] R. Thom, *Les singularités des applications différentiables*, Ann. Inst. Fourier, Grenoble **6** (1955–1956), 43–87.
[12] H. Whitney, *On singularities of mappings of euclidean spaces. I. Mappings of the plane into the plane*, Ann. of Math. (2) **62** (1955), 374–410.

R. Sadykov: Department of Mathematics, Kansas State University, Manhattan, Kansas
Email address: `sadykov@ksu.edu`

O. Saeki: Institute of Mathematics for Industry, Kyushu University, Motooka 744, Nishi-ku, Fukuoka 819-0395, Japan
Email address: `saeki@imi.kyushu-u.ac.jp`