On Singularities of Ruled Surfaces with Finite Multiplicity

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Abstract

A pseudo-cylindrical ruled surface is a ruled surface with finite multiplicity, which includes all analytic ruled surfaces except cylinders. In this paper, we extend the class of non-cylindrical ruled surfaces to the broader class of pseudo-cylindrical ruled surfaces. For a non-cylindrical ruled surface, singularities such as the cross caps, cuspidal edges, and swallowtails may appear, and there always exists a striction curve closely related to these singularities. On the other hand, pseudo-cylindrical ruled surfaces may exhibit singularities that do not appear in the non-cylindrical case, such as cuspidal beaks and the Scherbak surfaces, and in some cases the striction curve may not exist. We describe the relationship between the behavior of the striction curve and the singularities, and review fundamental properties of singularities, including frontals, wave fronts, non-degenerate singular points, and singular points of the n-th kind. The results presented here are based on [4].

1 Introduction

The study of ruled surfaces in $\mathbb{R}^3$ is a classical subject in differential geometry and has been given in [1]. Ruled surfaces have singularities in general. The properties of singularities appearing on non-cylindrical ruled surfaces have singularities appearing on non-cylindrical ruled surfaces have been well studied (see [2, 5, 6, 7]). In addition, there exist ruled surfaces that are neither a non-cylindrical ruled surface nor a cylinder.

Pseudo-cylindrical ruled surfaces were introduced by the author in [3] as a natural extension of non-cylindrical ruled surfaces, by allowing the director curve to have finite multiplicity. Such surfaces include all analytic ruled surfaces other than cylinders. However, the singularity structure of pseudo-cylindrical ruled surfaces has not been systematically investigated.

The present work fills this gap by providing the first comprehensive classification of singularities on pseudo-cylindrical ruled surfaces and by clarifying their relationship with the behavior of the striction curve.

Except for cones, non-cylindrical developable surfaces are tangent developables. Cuspidal edges, swallowtails, and cuspidal cross caps appear on non-cylindrical developable surfaces as singular points. Moreover, cuspidal beaks and Scherbak surfaces are also singularities that do not appear on non-cylindrical developable surfaces but do appear on other tangent developables.

In this paper, we study pseudo-cylindrical developable surfaces, since they include tangent developables. A ruled surface is said to be *k-th pseudo-cylindrical* if director curve ξ satisfies

$$d\xi(x)/dx = \tilde{\xi}(x)x^k, \ \tilde{\xi}(0) \neq 0.$$

If $k \geq 1$, then the ruled surface is neither a cylinder nor a non-cylindrical ruled surface. To study pseudo-cylindrical developable surfaces, we consider the behavior of the striction curves on these surfaces, and give the fundamental properties of singularities such as frontals, wave fronts, non-degenerate singular points, and singular points of the n-th kind. Moreover we consider the relationship between the striction curves and the rulings.

2 Basic notion of ruled surfaces

We review the basic notions and properties of ruled surfaces and developable surfaces. Let $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^3, 0)$ and $\xi : (\mathbb{R}, 0) \to \mathbb{R}^3\backslash\{\mathbf{0}\}$ be C^∞-maps. Then we define a *ruled surface* $F : (\mathbb{R}, 0) \times \mathbb{R} \to \mathbb{R}^3$ by

$$F(x, t) = \gamma(x) + t\xi(x).$$

We call the map γ a *base curve* and the map ξ a *director curve*. If we fix a parameter $x_0 \in \mathbb{R}$, the line $F(x_0, t)$ is called a *ruling* on x_0. We call a ruled surface with vanishing Gaussian curvature on the regular part a *developable surface*. If the direction of the director curve ξ is constant, we call F a *cylinder*.

We set $\bar{\xi}(x) = \xi(x)/||\xi(x)||$ and $\bar{F}(x, t) = \gamma(x) + t\bar{\xi}(x)$. Then the image $\operatorname{Im} F$ is equal to the image $\operatorname{Im} \bar{F}$. Therefore, we may assume $|\xi(x)| = 1$. Then it is known that a ruled surface F is a developable surface if and only if

$$\det(\gamma'(x), \xi(x), \xi'(x)) = 0, \tag{2.1}$$

for any $x \in (\mathbb{R}, 0)$ where $(\ )' = d/dx$ (cf., [7]). We say that F is *non-cylindrical* if it holds that $\xi'(x) \neq \mathbf{0}$ for any $x \in (\mathbb{R}, 0)$. The curve $s(x) = F(x, t(x))$ is called a *striction curve* if it satisfies that

$$\langle s'(x), \xi'(x) \rangle = 0 \tag{2.2}$$

for any $x \in (\mathbb{R}, 0)$. If there exists a striction curve and the striction curve is constant, we call F a *cone*. We assume that F is non-cylindrical. The striction curve is given by

$$s(x) = \gamma(x) - \frac{\langle \gamma'(x), \xi'(x) \rangle}{\langle \xi'(x), \xi'(x) \rangle}\xi(x).$$

It is known that a singular point of the non-cylindrical ruled surface is located on the striction curve. Moreover the set of singular points of a non-cylindrical developable surface coincides with its striction curve [6].

A map $f : (\mathbb{R}^2, 0) \to \mathbb{R}^3$ is a *frontal* if there exists a map $\nu : (\mathbb{R}^2, 0) \to S^2$ such that for any $p \in ((\mathbb{R}^2; (u, v)), 0)$, it holds that $\langle \nu(p), f_u(p) \rangle = \langle \nu(p), f_v(p) \rangle = 0$, where $\langle , \rangle$ is an inner product. We call ν a *unit normal vector field* of f. Let f be a frontal. We set a map $L = (f, \nu) : (\mathbb{R}^2, p_0) \to \mathbb{R}^3 \times S^2$. Then f is said to be a *wave front* if L is an immersion. It is known (cf., [7, 5]) the following fact.

Fact 2.1. If a non-cylindrical ruled surface has singularities, then it is a frontal if and only if it is developable. a non-cylindrical developable surface F is a wave front if and only if

$$\psi(x) = \det\big(\xi(x), \xi'(x), \xi''(x)\big) \neq 0. \tag{2.3}$$

For a coordinate system (u, v) on $(\mathbb{R}^2, 0)$, we define a function $\lambda : (\mathbb{R}^2, 0) \to \mathbb{R}$ by $\lambda = \det(f_u, f_v, \nu)$ and call it the *signed area density of* f. A singular point $p_0 \in (\mathbb{R}^2, 0)$ is called a *non-degenerate singular point* if $d\lambda(p_0) \neq 0$. In this case, there exists a smooth parametrization $\gamma(t) : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ of $S(f)$. Moreover, there exists a vector field η on $(\mathbb{R}^2, 0)$ such that $\langle \eta(p_0) \rangle_{\mathbb{R}} = \operatorname{Ker} df_{p_0}$ for any $p_0 \in S(f)$, where $S(f)$ is the set of singular points of f. We call η a *null vector field.* Now a singular point $p_0 \in (\mathbb{R}^2, 0)$ is called a *singular point of the n-th kind* if

$$\eta\lambda(p_0) = \eta\eta\lambda(p_0) \cdots = \eta^{(n-1)}\lambda(p_0) = 0,\ \eta^{(n)}\lambda(p_0) \neq 0,$$

Here, $\eta\lambda : (\mathbb{R}, 0) \to \mathbb{R}$ means the directional derivative of λ by the vector field $\eta \in \mathfrak{X}(U)$ is an extended vector field of η to $(\mathbb{R}^2, 0)$. Similarly, $\eta^{(j)}\lambda$ denotes the j-th directional derivative of λ along η. Let F be a non-cylindrical developable surface. Then by (2.1), there exist functions $\alpha, \beta : (\mathbb{R}, 0) \to \mathbb{R}$ such that $\gamma'(x) = \alpha(x)\xi(x) + \beta(x)\xi'(x)$. The striction curve of F is written as $s(x) = \gamma(x) - \beta(x)\xi(x)$ and we see that $\lambda(x, t) = ||\xi'(x)||(t + \beta(x))$. Since we have a null vector field $\eta = \partial_x - \alpha(x)\partial_t$ of F, we obtain the following lemma.

Lemma 2.2. *Assume that the above notation. Then a singular point $(x, -\beta(x))$ is always non-degenerate. A singular point $(x, -\beta(x))$ is of the n-th kind if and only if*

$$\beta' - \alpha = \beta'' - \alpha' = \cdots = \beta^{(n-1)} - \alpha^{(n-2)} = 0, \quad \beta^{(n)} - \alpha^{(n-1)} \neq 0.$$

Proof. Since $\lambda_t(x, -\beta(x))||\xi'(x)|| \neq 0$, $d\lambda(x, -\beta(x)) \neq 0$. Next, we have

$$\eta^{(n)}\lambda(x, t) = ||\xi(x)||^{(n)}(t + \beta(x)) + \sum_{i=0}^{n-1} {}_nC_i ||\xi(x)||^{(i)}(x)\{\beta^{(n-i)}(x) - \alpha^{(n-i-1)}(x)\}.$$

Since $||\xi(x)|| \neq 0$ and $t = -\beta(x)$, we have

- $\eta\lambda(x, \beta(x)) \neq 0$ if and only if $\beta'(x) - \alpha(x) \neq 0$,
- $\eta\lambda(x, \beta(x)) = 0$, $\eta\eta\lambda(x, \beta(x)) \neq 0$ if and only if $\beta'(x) - \alpha(x) = 0$, $\beta''(x) - \alpha'(x) \neq 0$.

By an induction on n, we have this assertion. □

By Fact 2.1. and Lemma 2.2., fundamental singular points appear on developable surfaces. For example, cuspidal edges is a singular point of wave front, non-degenerate, and the first kind. swallowtails is a singular point of wave front, non-degenerate, and the second kind. See Figure 2.1.

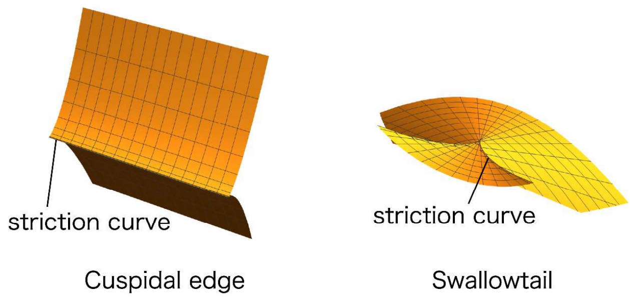

Figure 2.1: Examples of cuspidal edge (left) and swallowtail (right)

3 Pseudo-cylindrical ruled surfaces

Non-cylindrical ruled surfaces and cylinders are classical and well studied. However these ruled surfaces have the property that the derivative of the director curve $\xi'(x)$ is identically either equal to 0 or never 0. On the other hand, if we have $\xi'(x) = \tilde{\xi}(x)x^k$, $\tilde{\xi}(0) \neq 0$ $(k \geq 1)$, then the ruled surface is neither a cylinder nor a non-cylindrical ruled surface.

Let $g : (\mathbb{R}, 0) \to \mathbb{R}^3$ be a C^∞-map. We say that g is *of finite multiplicity* if there exist an integer $m \in \mathbb{Z}_{\geq 0}$ and a C^∞-map $\tilde{g} : (\mathbb{R}, 0) \to \mathbb{R}^3$ with the following property:

$$g(x) = \tilde{g}(x)x^m, \tilde{g}(0) \neq \mathbf{0}.$$

We define a pseudo-cylindrical ruled surface as follows.

Definition 3.1 ([3, Section 4.1]). Let $\xi'(x)$ be of finite multiplicity. Then the ruled surface $F(x,t) = \gamma(x) + t\xi(x)$ is called *k-th pseudo-cylindrical* if

$$\xi'(x) = \tilde{\xi}(x)x^k, \tilde{\xi}(0) \neq \mathbf{0}$$

holds for $k \in \mathbb{Z}_{\geq 0}$.

If the ruled surface F is analytic (that is, γ and ξ are analytic), then F is either cylinder or k-th pseudo-cylindrical ruled surface. Hereafter, we assume that γ and ξ are analytic. We can take a unit vector field $\xi_d : (\mathbb{R}, 0) \to S^2$ such that

$$\xi_d(x) = \frac{\tilde{\xi}(x)}{||\tilde{\xi}(x)||}.$$

Since $||\xi(x)|| = 1$, two vectors ξ, ξ_d are perpendicular, we give an orthonormal frame $\{\xi(x), \xi_d(x), \xi(x) \times \xi_d(x)\}$ along F. We have the following Frenet-Serret type formula

$$\begin{pmatrix} \xi'(x) \\ \xi_d'(x) \\ (\xi(x) \times \xi_d(x))' \end{pmatrix} = \begin{pmatrix} 0 & \delta(x) & 0 \\ -\delta(x) & 0 & \rho(x) \\ 0 & -\rho(x) & 0 \end{pmatrix} \begin{pmatrix} \xi(x) \\ \xi_d(x) \\ \xi(x) \times \xi_d(x) \end{pmatrix},$$

where

$$\delta(x) = \langle \xi'(x), \xi_d(x) \rangle = ||\tilde{\xi}(x)|| x^k, \tag{3.1}$$

and

$$\rho(x) = \langle \xi_d'(x), \xi(x) \times \xi_d(x) \rangle = \det(\xi(x), \xi_d(x), \xi_d'(x)). \tag{3.2}$$

We set

$$\gamma'(x) = p(x)\xi(x) + q(x)\xi_d(x) + r(x)(\xi(x) \times \xi_d(x)), \tag{3.3}$$

where

$$p(x) = \langle \gamma'(x), \xi(x) \rangle,\ q(x) = \langle \gamma'(x), \xi_d(x) \rangle,\ r(x) = \langle \gamma'(x), (\xi(x) \times \xi_d(x)) \rangle.$$

Since γ and ξ are analytic, we can write

$$p(x) = \begin{cases} 0, \\ \tilde{p}(x)x^P,\ \tilde{p}(0) \neq 0, \end{cases} \quad q(x) = \begin{cases} 0, \\ \tilde{q}(x)x^Q,\ \tilde{q}(0) \neq 0, \end{cases} \quad r(x) = \begin{cases} 0, \\ \tilde{r}(x)x^R,\ \tilde{r}(0) \neq 0, \end{cases}$$

where $\tilde{p}, \tilde{q}, \tilde{r} : (\mathbb{R}, 0) \to \mathbb{R}^3$ and integers $P, Q, R \in \mathbb{Z}_{\geq 0}$. If $p(x) = 0$ (resp., $q(x) = 0$, $r(x) = 0$), then we regard its order of vanishing as $P = \infty$ (resp., $Q = \infty$, $R = \infty$).

Proposition 3.2. *Let $F(x,t) = \gamma(x) + t\xi(x)$ be k-th pseudo-cylindrical and analytic. The striction curve $s(x)$ is well-defined at $x = 0$ if and only if $Q \geq k$.*

Proof. We have that

$$\begin{aligned} \langle s'(x), \xi'(x) \rangle &= \langle \gamma'(x) + t'(x)\xi(x) + t(x)\xi'(x), \xi'(x) \rangle \\ &= (q(x) + t(x)\delta(x))\delta(x). \end{aligned}$$

If $\langle s'(x), \xi_d(x) \rangle = 0$, then

$$t(x) = -\frac{q(x)\delta(x)}{\delta(x)\delta(x)} = -\frac{\tilde{q}(x)}{||\tilde{\xi}(x)||} x^{Q-k}. \tag{3.4}$$

The striction curve is given by

$$s(x) = \gamma(x) - \frac{\tilde{q}(x)}{||\tilde{\xi}(x)||} x^{Q-k} \xi(x). \tag{3.5}$$

Therefore, the striction curve $s(x)$ is well-defined at $x = 0$ if and only if $Q \geq k$. □

Example 3.3. Define map-germs $f_1, f_2 : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ by

$$f_1(x,t) = \left(x, -\frac{x^3}{3}, -\frac{x^4}{4} \right) + t\left(1, x^2, x^3\right),$$

and

$$f_2(x,t) = \int_0^x g(u)\,du + tg(x), \qquad g(x) = \frac{(1, x^4, x^5)}{\sqrt{1 + x^8 + x^9}}.$$

Then the ruled surfaces f_1 and f_2 give examples of developable surfaces in the case $k = 1$. Figure 3.1 shows the developable surface given by f_2, where the striction curve can be defined at $x = 0$. In fact, the striction curve and the ruling $x = 0$ appear as singular loci, and a singularity called a *cuspidal beaks* appears at the origin.

On the other hand, when $Q < k$, the striction curve behaves asymptotically toward the ruling $x = 0$. Figure 3.2 shows an example of the developable surface f_2 satisfying $Q < k$. In this case, the striction curve approaches the ruling $x = 0$ asymptotically.

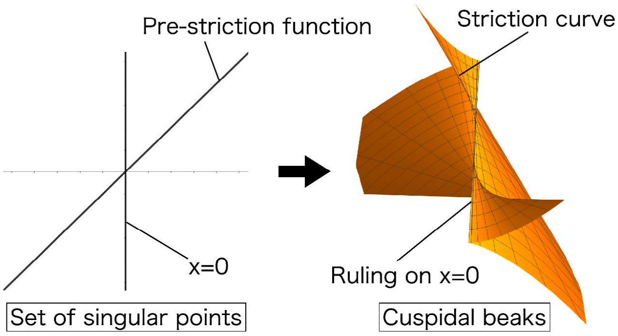

Figure 3.1: An example of a pseudo-cylindrical ruled surface (cuspidal beaks)

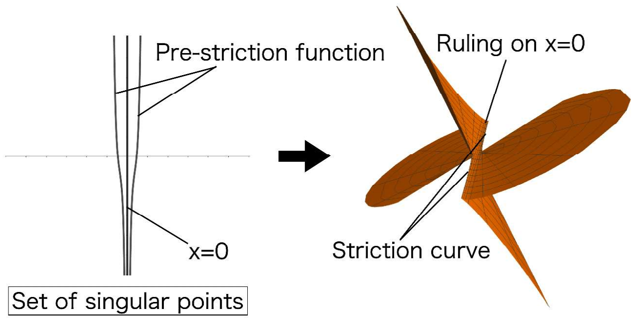

Figure 3.2: Example where no striction curve exists at $x = 0$ $(Q < k)$

We define $m = \min\{Q, R, k\}$ and

$$A(x,t) = -\frac{q(x) + t\delta(x)}{x^m}, \; B(x) = \frac{r(x)}{x^m}.$$

As an extension of Fact 2.1, we obtain the following theorem.

Theorem 3.4 (Frontals and wave fronts)**.** *Let F be k-th pseudo-cylindrical and analytic.*

(1) *F is a frontal if and only if one of the following holds:*

(a) *F is developable, i.e., $r(x) = 0$;*

(b) *if* $r(x) = \tilde{r}(x)x^R$ *with* $\tilde{r}(0) \neq 0$, *then either* $k > Q$ *or* $k \geq R$.

(2) *Assume that* F *is a frontal. Then* F *is a wave front if and only if one of the following holds:*

(a) *if* $r(x) = 0$, *then* $\rho(x) \neq 0$;

(b) *if* $r(x) = \tilde{r}(x)x^R$ *with* $\tilde{r}(0) \neq 0$, *then*

$$\rho(x)\big(A(x,t)^2 + B(x)^2\big) + A(x,t)B'(x) - A_x(x,t)B(x) \neq 0.$$

If there exist a striction curve $s(x)$ on ruling of $x = 0$, then we define a function $\sigma(x) = \delta(x)||s(x)||$. As an extension of Lemma 2.2, we obtain the following theorem.

Theorem 3.5 (Classification of singular points)**.** *Let* F *be* k*-th pseudo-cylindrical, analytic, and frontal.*

(1) F *has a non-degenerate singular point if and only if*

$$\min\{Q, R, k+1\} = 1.$$

(2) *A singular point of* F *is of the* n*-th kind if and only if one of the following holds:*

(a) *if* $k > Q$, *or* $r(x) = \tilde{r}(x)x^R$ *with* $\tilde{r}(0) \neq 0$, *then* $\min\{Q, R\} = n$;

(b) *if* $Q \geq k$, *and* $r(x) \equiv 0$, *then*

$$\sigma(x) = \sigma'(x) = \cdots = \sigma^{(n-2)}(x) = 0, \ \sigma^{(n-1)}(x) \neq 0.$$

Example 3.6. Define map-germs $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ by

$$f(x,t) = \left(\frac{x^{n-k}}{n-k}, \frac{x^{n+1}}{n+1}, \frac{x^{n+2}}{n+2}\right) + t(1, x^{k+1}, x^{k+2}),$$

and

$$g(x,t) = \left(\frac{x^{n-k}}{n-k}, \frac{x^{n+1}}{n+1}, \frac{x^{n+3}}{n+3}\right) + t(1, x^{k+1}, x^{k+3}),$$

where $n, k \in \mathbb{Z}_{\geq 0}$ and $n > k$.

Then f and g are k-th order pseudo-cylindrical ruled surfaces, and both are developable surfaces. The map f has a singular point at the origin and defines a wave front ($\rho(x) \neq 0$) with a singularity of type n (see Figure 3.3). On the other hand, g also has a singular point at the origin and defines a frontal which is not a wave front ($\rho(x) = 0$, $\rho'(x) \neq 0$), having a singularity of type n (see Figure 3.4).

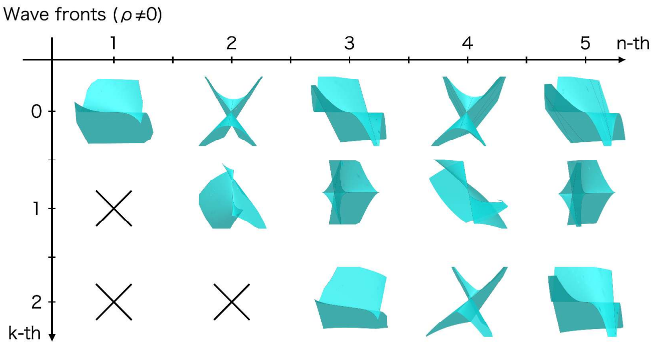

Figure 3.3: Classification table of singularities (wave fronts)

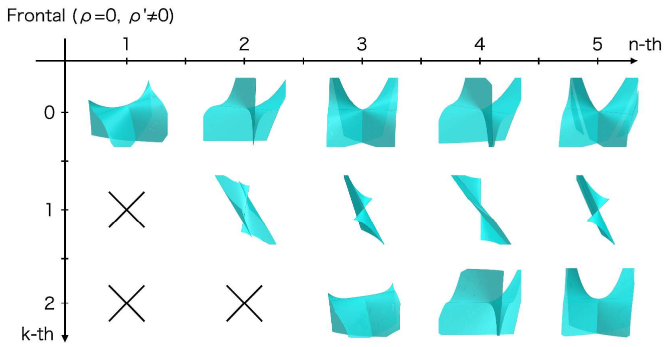

Figure 3.4: Classification table of singularities (frontals)

4 Relationship between the striction curve and the ruling of $x = 0$

We consider the relationship between striction curve and ruling of $x = 0$.

We introduce the *contact exponent for a space curve with the tangent line.*

Definition 4.1. Let $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^3, 0)$ be a C^∞-curve with multiplicity $m(\gamma)$. Let L be a line through the origin. Assume γ and L share the same tangent direction at 0. Then the contact exponent of γ with L at 0 is defined by

$$\zeta_0(\gamma, L) = \frac{(\gamma, L)_0}{m(\gamma)},$$

where $(\gamma, L)_0$ is a multiplicity of $w(x)$ such that $\gamma(x) = x^m L + w(x)$, $\langle w(x), L \rangle = 0$.

We have the following theorem.

Theorem 4.2. *Let F be a k-th pseudo-cylindrical developable surface having a striction curve $s(x)$. Assume that a singular point 0 is of the n-th kind. Then*

$$\zeta_0(s(x), R(x)) = \frac{n+1}{n-k} = \frac{(k+1)+(l+1)}{l+1},$$

where, $R(x) = \xi(0)x$ is the ruling on $x = 0$, and the integer l is the multiplicity of $s(x)$.

From this theorem, we see that as n (respectively, l) increases, ζ_0 decreases monotonically and approaches 1. Conversely, as k increases, ζ_0 increases monotonically.

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