Hyperbolic generalized framed surfaces in hyperbolic 3-space

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1 Introduction

We investigate differential geometric properties of surfaces with singularities in hyperbolic 3-space and have defined the hyperbolic framed surface in [10]. Many surfaces are hyperbolic framed base surfaces and the theory of hyperbolic framed surfaces can be applied to study their properties. However, there are some surfaces that may not be hyperbolic framed surfaces. A typical example is the one-parameter family of hyperbolic framed curves. A curve in hyperbolic 3-space with curvature 1 and torsion 0 is known as a *horocycle*. A one-parameter family of horocycles is called a *horocyclic surface* (cf. [4]), which may not conform to the definition of a hyperbolic framed base surface. These examples, along with others, motivate the development of a more general notion of singular surfaces in hyperbolic 3-space.

In Euclidean space, the framed surface possesses a smooth moving frame at its singularities (cf. [2]). The relation between one-parameter families of framed curves and framed surfaces was discussed in [3]. To address a broader class of singular surfaces, authors in [8] introduced generalized framed surfaces. In hyperbolic 3-space, we give the concept of hyperbolic generalized framed surfaces, extending hyperbolic framed surfaces and one-parameter families of hyperbolic framed curves. The relations between these surfaces are discussed in Section 3. Additionally, we provide necessary and sufficient conditions for a smooth surface to be a hyperbolic generalized framed base surface (Theorem 3.9). In order to study mixed-type surfaces with singularities in Lorentz-Minkowski 3-space, authors in [5] introduced the lightcone framed surface equipped with a lightcone frame. The lightcone framed surfaces are closely related to generalized framed surfaces in Euclidean 3-space, see Theorem 3.5 in [5]. We further discuss the connections between hyperbolic generalized framed surfaces, generalized framed surfaces of Euclidean 3-space and lightcone framed surfaces of Lorentz-Minkowski 3-space in Section 4. To illustrate the theory, two examples are presented in Section 5.

All maps and manifolds considered here are of class C^∞ unless otherwise stated.

The content of this paper is based on joint research with Donghe Pei and Masatomo Takahashi (cf. [6]).

2 Preliminaries

Let $\mathbb{R}^n_1$ denote *Lorentz-Minkowski n-space* with the pseudo inner product $\langle \boldsymbol{x}_1, \boldsymbol{x}_2\rangle$ and the pseudo vector product $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_{n-1}$ where $\boldsymbol{x}_i \in \mathbb{R}^n_1$ $(i = 1, 2, \ldots, n-1)$. A vector $\boldsymbol{x} \in \mathbb{R}^n_1 \setminus \{\boldsymbol{0}\}$ is *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x}\rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x}\rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x}\rangle < 0$, respectively. The norm of vector $\boldsymbol{x}$ is given by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x}\rangle|}$. We define the *hyperbolic 3-space* as $H^3 = \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = -1\}$

and the *de Sitter 3-space* as $S^3_1 = \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}$. The submanifold Δ_5 in [1] is defined as $\Delta_5 = \{(\boldsymbol{\nu}_1, \boldsymbol{\nu}_2) \in S^3_1 \times S^3_1 \mid \langle \boldsymbol{\nu}_1, \boldsymbol{\nu}_2 \rangle = 0\}$.

It is well known that there are two connected branches in hyperbolic 3-space. Here we only consider curves or surfaces on the branch $\{\boldsymbol{x} = (x_1, x_2, x_3, x_4) \in H^3 \mid x_1 > 0\}$ and still denote it by H^3. In this paper, we denote U a simply connected open subset in $\mathbb{R}^2$ and I an interval in $\mathbb{R}$.

2.1 Hyperbolic framed surfaces

Definition 2.1 ([10])**.** We define $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to H^3 \times \Delta_5$ as a *hyperbolic framed surface* if $\langle \boldsymbol{x}_u, \boldsymbol{n} \rangle(u,v) = \langle \boldsymbol{x}_v, \boldsymbol{n} \rangle(u,v) = \langle \boldsymbol{x}, \boldsymbol{n} \rangle(u,v) = \langle \boldsymbol{x}, \boldsymbol{s} \rangle(u,v) = 0$ for all $(u,v) \in U$, where $\boldsymbol{x}_u(u,v) = (\partial \boldsymbol{x}/\partial u)(u,v)$, $\boldsymbol{x}_v(u,v) = (\partial \boldsymbol{x}/\partial v)(u,v)$. $\boldsymbol{x} : U \to H^3$ is a *hyperbolic framed base surface* if there exists $(\boldsymbol{n}, \boldsymbol{s}) : U \to \Delta_5$ such that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to H^3 \times \Delta_5$ is a hyperbolic framed surface.

Let $\boldsymbol{t}(u,v) = (\boldsymbol{x} \wedge \boldsymbol{n} \wedge \boldsymbol{s})(u,v)$. Then $\{\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}, \boldsymbol{t}\}$ is a moving frame along $\boldsymbol{x}$. Thus, we get

$$\begin{pmatrix} \boldsymbol{x}_u(u,v) \\ \boldsymbol{n}_u(u,v) \\ \boldsymbol{s}_u(u,v) \\ \boldsymbol{t}_u(u,v) \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_1(u,v) & b_1(u,v) \\ 0 & 0 & e_1(u,v) & f_1(u,v) \\ a_1(u,v) & -e_1(u,v) & 0 & g_1(u,v) \\ b_1(u,v) & -f_1(u,v) & -g_1(u,v) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(u,v) \\ \boldsymbol{n}(u,v) \\ \boldsymbol{s}(u,v) \\ \boldsymbol{t}(u,v) \end{pmatrix},$$

$$\begin{pmatrix} \boldsymbol{x}_v(u,v) \\ \boldsymbol{n}_v(u,v) \\ \boldsymbol{s}_v(u,v) \\ \boldsymbol{t}_v(u,v) \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_2(u,v) & b_2(u,v) \\ 0 & 0 & e_2(u,v) & f_2(u,v) \\ a_2(u,v) & -e_2(u,v) & 0 & g_2(u,v) \\ b_2(u,v) & -f_2(u,v) & -g_2(u,v) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(u,v) \\ \boldsymbol{n}(u,v) \\ \boldsymbol{s}(u,v) \\ \boldsymbol{t}(u,v) \end{pmatrix},$$

where $a_i, b_i, e_i, f_i, g_i : U \to \mathbb{R}$, $i = 1, 2$ are smooth functions. They are called the *basic invariants* of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$.

2.2 One-parameter families of hyperbolic framed curves

Definition 2.2. We call $(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : U \to H^3 \times \Delta_5$ a *one-parameter family of hyperbolic framed curves with respect to u* (respectively, *with respect to v*) if $\langle \boldsymbol{\gamma}, \boldsymbol{\nu}_i \rangle(u,v) = \langle \boldsymbol{\gamma}_u, \boldsymbol{\nu}_i \rangle(u,v) = 0$ (respectively, $\langle \boldsymbol{\gamma}, \boldsymbol{\nu}_i \rangle(u,v) = \langle \boldsymbol{\gamma}_v, \boldsymbol{\nu}_i \rangle(u,v) = 0$) for all $(u,v) \in U$, $i = 1, 2$, where $\boldsymbol{\gamma}_u(u,v) = (\partial \boldsymbol{\gamma}/\partial u)(u,v)$, $\boldsymbol{\gamma}_v(u,v) = (\partial \boldsymbol{\gamma}/\partial v)(u,v)$. $\boldsymbol{\gamma} : U \to H^3$ is called a *one-parameter family of hyperbolic framed base curves with respect to u* (respectively, *with respect to v*) if there exists $(\boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : U \to \Delta_5$ such that $(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : U \to H^3 \times \Delta_5$ is a one-parameter family of hyperbolic framed curves with respect to u (respectively, with respect to v).

Define $\boldsymbol{\mu}(u,v) = (\boldsymbol{\gamma} \wedge \boldsymbol{\nu}_1 \wedge \boldsymbol{\nu}_2)(u,v)$, then $\{\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \boldsymbol{\mu}\}$ is a moving frame along $\boldsymbol{\gamma}$.

Let $(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$ be a one-parameter family of hyperbolic framed curves with respect to u. We have the following formulas

$$\begin{pmatrix} \boldsymbol{\gamma}_u(u,v) \\ \boldsymbol{\nu}_{1u}(u,v) \\ \boldsymbol{\nu}_{2u}(u,v) \\ \boldsymbol{\mu}_u(u,v) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & m(u,v) \\ 0 & 0 & n(u,v) & a(u,v) \\ 0 & -n(u,v) & 0 & b(u,v) \\ m(u,v) & -a(u,v) & -b(u,v) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma}(u,v) \\ \boldsymbol{\nu}_1(u,v) \\ \boldsymbol{\nu}_2(u,v) \\ \boldsymbol{\mu}(u,v) \end{pmatrix},$$

$$\begin{pmatrix} \boldsymbol{\gamma}_v(u,v) \\ \boldsymbol{\nu}_{1v}(u,v) \\ \boldsymbol{\nu}_{2v}(u,v) \\ \boldsymbol{\mu}_v(u,v) \end{pmatrix} = \begin{pmatrix} 0 & P(u,v) & Q(u,v) & M(u,v) \\ P(u,v) & 0 & N(u,v) & A(u,v) \\ Q(u,v) & -N(u,v) & 0 & B(u,v) \\ M(u,v) & -A(u,v) & -B(u,v) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma}(u,v) \\ \boldsymbol{\nu}_1(u,v) \\ \boldsymbol{\nu}_2(u,v) \\ \boldsymbol{\mu}(u,v) \end{pmatrix},$$

where $m, n, a, b, M, N, A, B, P, Q : U \to \mathbb{R}$ are smooth functions. They are called the *curvature* of the one-parameter family of hyperbolic framed curves with respect to u.

3 Hyperbolic generalized framed surfaces

Definition 3.1. We say that $(\boldsymbol{x}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : U \to H^3 \times \Delta_5$ is a *hyperbolic generalized framed surface* if $\langle \boldsymbol{x}, \boldsymbol{\nu}_1 \rangle(u,v) = \langle \boldsymbol{x}, \boldsymbol{\nu}_2 \rangle(u,v) = 0$ and there exist smooth functions $\alpha, \beta : U \to \mathbb{R}$ such that $(\boldsymbol{x} \wedge \boldsymbol{x}_u \wedge \boldsymbol{x}_v)(u,v) = \alpha(u,v)\boldsymbol{\nu}_1(u,v) + \beta(u,v)\boldsymbol{\nu}_2(u,v)$ for all $(u,v) \in U$. We define $\boldsymbol{x} : U \to H^3$ as a *hyperbolic generalized framed base surface* if there exists $(\boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : U \to \Delta_5$ such that $(\boldsymbol{x}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : U \to H^3 \times \Delta_5$ is a hyperbolic generalized framed surface.

Let $\boldsymbol{\nu}_3(u,v) = (\boldsymbol{x} \wedge \boldsymbol{\nu}_1 \wedge \boldsymbol{\nu}_2)(u,v)$. Then $\{\boldsymbol{x}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \boldsymbol{\nu}_3\}$ is a moving frame along $\boldsymbol{x}$. Thus, we get

$$\begin{pmatrix} \boldsymbol{x}_u(u,v) \\ \boldsymbol{\nu}_{1u}(u,v) \\ \boldsymbol{\nu}_{2u}(u,v) \\ \boldsymbol{\nu}_{3u}(u,v) \end{pmatrix} = \begin{pmatrix} 0 & a_1(u,v) & b_1(u,v) & c_1(u,v) \\ a_1(u,v) & 0 & e_1(u,v) & f_1(u,v) \\ b_1(u,v) & -e_1(u,v) & 0 & g_1(u,v) \\ c_1(u,v) & -f_1(u,v) & -g_1(u,v) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(u,v) \\ \boldsymbol{\nu}_1(u,v) \\ \boldsymbol{\nu}_2(u,v) \\ \boldsymbol{\nu}_3(u,v) \end{pmatrix},$$

$$\begin{pmatrix} \boldsymbol{x}_v(u,v) \\ \boldsymbol{\nu}_{1v}(u,v) \\ \boldsymbol{\nu}_{2v}(u,v) \\ \boldsymbol{\nu}_{3v}(u,v) \end{pmatrix} = \begin{pmatrix} 0 & a_2(u,v) & b_2(u,v) & c_2(u,v) \\ a_2(u,v) & 0 & e_2(u,v) & f_2(u,v) \\ b_2(u,v) & -e_2(u,v) & 0 & g_2(u,v) \\ c_2(u,v) & -f_2(u,v) & -g_2(u,v) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(u,v) \\ \boldsymbol{\nu}_1(u,v) \\ \boldsymbol{\nu}_2(u,v) \\ \boldsymbol{\nu}_3(u,v) \end{pmatrix},$$

where $a_i, b_i, c_i, e_i, f_i, g_i : U \to \mathbb{R}$, $i = 1, 2$ are smooth functions with $a_1 b_2 - a_2 b_1 = 0$. They are called the *basic invariants* of $(\boldsymbol{x}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$. Denote

$$\boldsymbol{a}(u,v) = \big(a_1(u,v)\ a_2(u,v)\big)^T, \boldsymbol{b}(u,v) = \big(b_1(u,v)\ b_2(u,v)\big)^T, \boldsymbol{c}(u,v) = \big(c_1(u,v)\ c_2(u,v)\big)^T,$$

where $\boldsymbol{a}^T$ denotes the transpose of $\boldsymbol{a}$. Thus we can get

$$\alpha(u,v) = \det\big(\boldsymbol{b}(u,v)\ \boldsymbol{c}(u,v)\big),\ \beta(u,v) = \det\big(\boldsymbol{c}(u,v)\ \boldsymbol{a}(u,v)\big).$$

Since $(\boldsymbol{x}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$ is smooth, $\boldsymbol{x}_{uv} = \boldsymbol{x}_{vu}$, $\boldsymbol{\nu}_{1uv} = \boldsymbol{\nu}_{1vu}$ and $\boldsymbol{\nu}_{2uv} = \boldsymbol{\nu}_{2vu}$ hold. It follows that we have the integrability conditions:

$$\begin{cases} a_{1v} - b_1 e_2 - c_1 f_2 = a_{2u} - b_2 e_1 - c_2 f_1, \\ b_{1v} + a_1 e_2 - c_1 g_2 = b_{2u} + a_2 e_1 - c_2 g_1, \\ c_{1v} + a_1 f_2 + b_1 g_2 = c_{2u} + a_2 f_1 + b_2 g_1, \end{cases} \tag{1}$$

$$\begin{cases} e_{1v} - f_1 g_2 = e_{2u} - f_2 g_1, \\ f_{1v} + e_1 g_2 + a_1 c_2 = f_{2u} + e_2 g_1 + a_2 c_1, \\ g_{1v} - e_1 f_2 + b_1 c_2 = g_{2u} - e_2 f_1 + b_2 c_1. \end{cases} \tag{2}$$

Then we give fundamental theorems for hyperbolic generalized framed surfaces using their basic invariants.

Definition 3.2. Let $(\boldsymbol{x}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2), (\overline{\boldsymbol{x}}, \overline{\boldsymbol{\nu}}_1, \overline{\boldsymbol{\nu}}_2) : U \to H^3 \times \Delta_5$ be two hyperbolic generalized framed surfaces. We say that $(\boldsymbol{x}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$ and $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{\nu}}_1, \overline{\boldsymbol{\nu}}_2)$ are *congruent* as hyperbolic generalized framed surfaces if there exists $\boldsymbol{A} \in SO(1,3)$ such that $\overline{\boldsymbol{x}}(u,v) = \boldsymbol{A}(\boldsymbol{x}(u,v))$, $\overline{\boldsymbol{\nu}}_1(u,v) = \boldsymbol{A}(\boldsymbol{\nu}_1(u,v))$, $\overline{\boldsymbol{\nu}}_2(u,v) = \boldsymbol{A}(\boldsymbol{\nu}_2(u,v))$ for all $(u,v) \in U$. Here,

$$SO(1,3) = \left\{ \boldsymbol{A} \in \boldsymbol{M}_4(\mathbb{R}) \mid \boldsymbol{A}^T \boldsymbol{G} \boldsymbol{A} = \boldsymbol{G},\ \det(\boldsymbol{A}) = 1,\ \boldsymbol{G} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

Theorem 3.3 (Existence theorem for hyperbolic generalized framed surfaces)**.** *Let $a_i, b_i, c_i, e_i, f_i, g_i : U \to \mathbb{R}$, $i = 1, 2$ be smooth functions with $a_1 b_2 - a_2 b_1 = 0$ and the integrability conditions* (1) *and* (2)*. Then there exists a hyperbolic generalized framed surface $(\boldsymbol{x}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : U \to H^3 \times \Delta_5$ whose associated basic invariants are $a_i, b_i, c_i, e_i, f_i, g_i : U \to \mathbb{R}$, $i = 1, 2$.*

Theorem 3.4 (Uniqueness theorem for hyperbolic generalized framed surfaces)**.** *Let $(\boldsymbol{x}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$, $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{\nu}}_1, \overline{\boldsymbol{\nu}}_2) : U \to H^3 \times \Delta_5$ be two hyperbolic generalized framed surfaces with the same basic invariants $a_i, b_i, c_i, e_i, f_i, g_i : U \to \mathbb{R}$, $i = 1, 2$. Then $(\boldsymbol{x}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$ and $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{\nu}}_1, \overline{\boldsymbol{\nu}}_2)$ are congruent as hyperbolic generalized framed surfaces.*

3.1 Relations between hyperbolic generalized framed surfaces, hyperbolic framed surfaces, one-parameter families of hyperbolic framed curves and smooth surfaces.

Proposition 3.5. (1) *Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to H^3 \times \Delta_5$ be a hyperbolic framed surface with basic invariants a_i, b_i, e_i, f_i, g_i, $i = 1, 2$. Then $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a hyperbolic generalized framed surface with $\alpha(u,v) = (a_1 b_2 - a_2 b_1)(u,v)$, $\beta(u,v) = 0$.*

(2) *Let $(\boldsymbol{x}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : U \to H^3 \times \Delta_5$ be a hyperbolic generalized framed surface with basic invariants $a_i, b_i, c_i, e_i, f_i, g_i$, $i = 1, 2$.*

(i) *If $a_1(u,v) = a_2(u,v) = 0$ for all $(u,v) \in U$, then $(\boldsymbol{x}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$ is a hyperbolic framed surface.*

(ii) *If $b_1(u,v) = b_2(u,v) = 0$ for all $(u,v) \in U$, then $(\boldsymbol{x}, \boldsymbol{\nu}_2, \boldsymbol{\nu}_1)$ is a hyperbolic framed surface.*

(iii) *If there exists a smooth function $\theta : U \to \mathbb{R}$ such that $(a_1 \cos\theta - b_1 \sin\theta)(u,v) = (a_2 \cos\theta - b_2 \sin\theta)(u,v) = 0$ holds for all $(u,v) \in U$, then $\boldsymbol{x}$ is a hyperbolic framed base surface.*

Remark 3.6. Regular surfaces in hyperbolic 3-space are hyperbolic framed base surfaces, and therefore, regular surfaces are also hyperbolic generalized framed base surfaces.

Proposition 3.7. (1) *Let $(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : U \to H^3 \times \Delta_5$ be a one-parameter family of hyperbolic framed curves with respect to u. Then $(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$ is a hyperbolic generalized framed surface with $\alpha(u,v) = -m(u,v)Q(u,v)$, $\beta(u,v) = m(u,v)P(u,v)$.*

(2) *Let $(\boldsymbol{x}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : U \to H^3 \times \Delta_5$ be a hyperbolic generalized framed surface with basic invariants $a_i, b_i, c_i, e_i, f_i, g_i$, $i = 1, 2$. If $a_1(u,v) = b_1(u,v) = 0$ for all $(u,v) \in U$, then $(\boldsymbol{x}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$ and $(\boldsymbol{x}, \boldsymbol{\nu}_2, \boldsymbol{\nu}_1)$ are one-parameter families of hyperbolic framed curves with respect to u.*

(3) *Let $(\boldsymbol{x}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : U \to H^3 \times \Delta_5$ be a hyperbolic generalized framed surface with basic invariants $a_i, b_i, c_i, e_i, f_i, g_i$, $i = 1, 2$. Assume there exists a smooth function $\theta : U \to \mathbb{R}$ such that $(a_1 \cos\theta - b_1 \sin\theta)(u,v) = 0$ for all $(u,v) \in U$.*

(i) *If* $c_1(u,v)=0$ *for all* $(u,v)\in U$, *then* $\boldsymbol{x}$ *is a one-parameter family of hyperbolic framed base curves with respect to* u.

(ii) *If there exists a point* $(u_0,v_0)\in U$ *such that* $c_1(u_0,v_0)\neq 0$, *then* $\boldsymbol{x}$ *is a one-parameter family of hyperbolic framed base curves with respect to* u *at least locally around* (u_0,v_0).

Remark 3.8. Here we have only considered one-parameter family of hyperbolic framed curves with respect to u. Similar conclusions hold for one-parameter family of hyperbolic framed curves with respect to v.

Theorem 3.9. *Let* $\boldsymbol{x}:U\to H^3$ *be a smooth surface. Then* $\boldsymbol{x}$ *is a hyperbolic generalized framed base surface if and only if there exists a smooth map* $\boldsymbol{\nu}:U\to S^3_1$ *such that* $\langle\boldsymbol{x},\boldsymbol{\nu}\rangle(u,v)=\langle\boldsymbol{x}\wedge\boldsymbol{x}_u\wedge\boldsymbol{x}_v,\boldsymbol{\nu}\rangle(u,v)=0$ *for all* $(u,v)\in U$.

3.2 Singularities of hyperbolic generalized framed base surfaces

We say a point $\boldsymbol{p}$ is a cross cap singular point (respectively, $S_1^{\pm}$ singular point) of $\boldsymbol{x}$ if $\boldsymbol{x}$ is $\mathcal{A}$-equivalent to $(u,v)\mapsto(u,v^2,uv)$ (respectively, $(u,v)\mapsto(u,v^2,v(u^2\pm v^2))$) at $\boldsymbol{p}$. We use the following Lemma to obtain the conditions for cross cap and $S_1^{\pm}$ singular points.

Lemma 3.10 ([7, 9])**.** *Let* $\boldsymbol{x}:(U,\boldsymbol{p})\to(H^3,\boldsymbol{x}(\boldsymbol{p}))$ *be a map germ and* $\boldsymbol{p}=(u_0,v_0)$ *a corank one singular point of* $\boldsymbol{x}$. *Then,*

(1) $\boldsymbol{x}$ *at* $\boldsymbol{p}$ *is* $\mathcal{A}$*-equivalent to cross cap if and only if* $d\varphi(u_0,v_0)\neq\boldsymbol{0}$.

(2) $\boldsymbol{x}$ *at* $\boldsymbol{p}$ *is* $\mathcal{A}$*-equivalent to* S_1^+ *singular point if and only if* $d\varphi(u_0,v_0)=\boldsymbol{0}$, $\det\operatorname{Hess}\varphi(u_0,v_0)<0$, $\xi\boldsymbol{x}(\boldsymbol{p})$ *and* $\eta\eta\boldsymbol{x}(\boldsymbol{p})$ *are linearly independent.*

(3) $\boldsymbol{x}$ *at* $\boldsymbol{p}$ *is* $\mathcal{A}$*-equivalent to* S_1^- *singular point if and only if* $d\varphi(u_0,v_0)=\boldsymbol{0}$ *and* $\det\operatorname{Hess}\varphi(u_0,v_0)>0$.

Here, $\varphi(u,v)=\det(\boldsymbol{x},\xi\boldsymbol{x},\eta\boldsymbol{x},\eta\eta\boldsymbol{x})(u_0,v_0)$, ξ *is a vector field transverse to the null vector field* η *and* $\det\operatorname{Hess}\varphi(u,v)=(\varphi_{uu}\varphi_{vv}-\varphi_{uv}^2)(u,v)$.

Proposition 3.11. *Let* $(\boldsymbol{x},\boldsymbol{\nu}_1,\boldsymbol{\nu}_2):U\to H^3\times\Delta_5$ *be a hyperbolic generalized framed surface with basic invariants* a_i,b_i,c_i,e_i,f_i,g_i, $i=1,2$. *If* $\boldsymbol{p}=(u_0,v_0)$ *is a singular point of* $\boldsymbol{x}$, *that is,* $\alpha(u_0,v_0)=\det\big(\boldsymbol{b}(u_0,v_0)\ \boldsymbol{c}(u_0,v_0)\big)=0$ *and* $\beta(u_0,v_0)=\det\big(\boldsymbol{c}(u_0,v_0)\ \boldsymbol{a}(u_0,v_0)\big)=0$. *Then,*

(1) $\boldsymbol{p}$ *is a cross cap singular point of* $\boldsymbol{x}$ *if and only if* $(a_1,a_2,b_1,b_2)(u_0,v_0)=(0,0,0,0)$, $(c_1,c_2)(u_0,v_0)\neq(0,0)$ *and* $\big(\det\big(\boldsymbol{b}_u\ \boldsymbol{c}\big)\det\big(\boldsymbol{a}_v\ \boldsymbol{c}\big)-\det\big(\boldsymbol{b}_v\ \boldsymbol{c}\big)\det\big(\boldsymbol{a}_u\ \boldsymbol{c}\big)\big)(u_0,v_0)\neq 0$.

(2) $\boldsymbol{p}$ *is a* S_1^+ *singular point of* $\boldsymbol{x}$ *if and only if* $(a_1,a_2,b_1,b_2)(u_0,v_0)=(0,0,0,0)$, $(c_1,c_2)(u_0,v_0)\neq(0,0)$, $\big(\det\big(\boldsymbol{b}_u\ \boldsymbol{c}\big)\det\big(\boldsymbol{a}_v\ \boldsymbol{c}\big)-\det\big(\boldsymbol{b}_v\ \boldsymbol{c}\big)\det\big(\boldsymbol{a}_u\ \boldsymbol{c}\big)\big)(u_0,v_0)=0$, $\det\operatorname{Hess}\varphi(u_0,v_0)<0$ *and* $\big(-c_1\det\big(\boldsymbol{a}_v\ \boldsymbol{c}\big)+c_2\det\big(\boldsymbol{a}_u\ \boldsymbol{c}\big),c_2\det\big(\boldsymbol{b}_u\ \boldsymbol{c}\big)-c_1\det\big(\boldsymbol{b}_v\ \boldsymbol{c}\big)\big)(u_0,v_0)\neq(0,0)$.

(3) $\boldsymbol{p}$ *is a* S_1^- *singular point of* $\boldsymbol{x}$ *if and only if* $(a_1,a_2,b_1,b_2)(u_0,v_0)=(0,0,0,0)$, $(c_1,c_2)(u_0,v_0)\neq(0,0)$, $\big(\det\big(\boldsymbol{b}_u\ \boldsymbol{c}\big)\det\big(\boldsymbol{a}_v\ \boldsymbol{c}\big)-\det\big(\boldsymbol{b}_v\ \boldsymbol{c}\big)\det\big(\boldsymbol{a}_u\ \boldsymbol{c}\big)\big)(u_0,v_0)=0$ *and* $\det\operatorname{Hess}\varphi(u_0,v_0)>0$.

4 Relations between hyperbolic generalized framed surfaces in H^3, generalized framed surfaces in $\mathbb{R}^3$ and lightcone framed surfaces in $\mathbb{R}^3_1$

4.1 Relations between hyperbolic generalized framed surfaces and generalized framed surfaces

Let $\mathbb{R}^3$ denote Euclidean 3-space with the inner product $\boldsymbol{x}\cdot\boldsymbol{y}$ and the vector product $\boldsymbol{x}\times\boldsymbol{y}$, where $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^3$. Denote $\Delta=\{(\boldsymbol{x},\boldsymbol{y})\in S^2\times S^2 \mid \boldsymbol{x}\cdot\boldsymbol{y}=0\}$, where $S^2=\{\boldsymbol{x}\in\mathbb{R}^3 \mid \boldsymbol{x}\cdot\boldsymbol{x}=1\}$.

Definition 4.1 ([8]). We call $(\overline{\boldsymbol{x}},\overline{\boldsymbol{\nu}}_1,\overline{\boldsymbol{\nu}}_2):U\to\mathbb{R}^3\times\Delta$ a *generalized framed surface* if there exist smooth functions $\overline{\alpha},\overline{\beta}:U\to\mathbb{R}$ such that $(\overline{\boldsymbol{x}}_u\times\overline{\boldsymbol{x}}_v)(u,v)=\overline{\alpha}(u,v)\overline{\boldsymbol{\nu}}_1(u,v)+\overline{\beta}(u,v)\overline{\boldsymbol{\nu}}_2(u,v)$ for all $(u,v)\in U$. $\overline{\boldsymbol{x}}:U\to\mathbb{R}^3$ is a *generalized framed base surface* if there exists $(\overline{\boldsymbol{\nu}}_1,\overline{\boldsymbol{\nu}}_2):U\to\Delta$ such that $(\overline{\boldsymbol{x}},\overline{\boldsymbol{\nu}}_1,\overline{\boldsymbol{\nu}}_2)$ is a generalized framed surface.

Let $\overline{\boldsymbol{\nu}}_3(u,v)=(\overline{\boldsymbol{\nu}}_1\times\overline{\boldsymbol{\nu}}_2)(u,v)$, then $\{\overline{\boldsymbol{\nu}}_1,\overline{\boldsymbol{\nu}}_2,\overline{\boldsymbol{\nu}}_3\}$ is a moving frame along $\overline{\boldsymbol{x}}$.

Poincaré 3-disc $D^3=\{(x_1,x_2,x_3)\mid x_1^2+x_2^2+x_3^2<1\}$ is an open subset of Euclidean space. There is a diffeomorphism $\boldsymbol{\pi}$ from H^3 to D^3,

$$\boldsymbol{\pi}(x_1,x_2,x_3,x_4)=\left(\frac{x_2}{x_1+1},\frac{x_3}{x_1+1},\frac{x_4}{x_1+1}\right).$$

The inverse mapping of $\boldsymbol{\pi}$ is

$$\boldsymbol{\pi}^{-1}(x_1,x_2,x_3)=\frac{(1+x_1^2+x_2^2+x_3^2,2x_1,2x_2,2x_3)}{1-x_1^2-x_2^2-x_3^2}.$$

Proposition 4.2. *Let $(\boldsymbol{x},\boldsymbol{\nu}_1,\boldsymbol{\nu}_2):U\to H^3\times\Delta_5$ be a hyperbolic generalized framed surface. Denote $\boldsymbol{x}(u,v)=(x_1,x_2,x_3,x_4)(u,v)$, $\boldsymbol{\nu}_1(u,v)=(y_1,y_2,y_3,y_4)(u,v)$ and $\boldsymbol{\nu}_2(u,v)=(z_1,z_2,z_3,z_4)(u,v)$. Then $(\overline{\boldsymbol{x}},\overline{\boldsymbol{\nu}}_1,\overline{\boldsymbol{\nu}}_2):U\to D^3\times\Delta$ is a generalized framed surface, where*

$$\overline{\boldsymbol{x}}(u,v)=\boldsymbol{\pi}\circ\boldsymbol{x}(u,v)=\left(\frac{x_2}{x_1+1},\frac{x_3}{x_1+1},\frac{x_4}{x_1+1}\right)(u,v),$$

$$\overline{\boldsymbol{\nu}}_1(u,v)=\frac{(p_1,q_1,r_1)}{\sqrt{p_1^2+q_1^2+r_1^2}}(u,v),\qquad \overline{\boldsymbol{\nu}}_2(u,v)=\frac{(p_2,q_2,r_2)}{\sqrt{p_2^2+q_2^2+r_2^2}}(u,v),$$

$$p_1(u,v)=(x_2y_1-x_1y_2-y_2)(u,v),\qquad q_1(u,v)=(x_3y_1-x_1y_3-y_3)(u,v),$$

$$r_1(u,v)=(x_4y_1-x_1y_4-y_4)(u,v),\qquad p_2(u,v)=(x_2z_1-x_1z_2-z_2)(u,v),$$

$$q_2(u,v)=(x_3z_1-x_1z_3-z_3)(u,v),\qquad r_2(u,v)=(x_4z_1-x_1z_4-z_4)(u,v).$$

Proposition 4.3. *Let $(\overline{\boldsymbol{x}},\overline{\boldsymbol{\nu}}_1,\overline{\boldsymbol{\nu}}_2):U\to D^3\times\Delta$ be a generalized framed surface. Denote $\overline{\boldsymbol{x}}(u,v)=(x_1,x_2,x_3)(u,v)$, $\overline{\boldsymbol{\nu}}_1(u,v)=(y_1,y_2,y_3)(u,v)$ and $\overline{\boldsymbol{\nu}}_2(u,v)=(z_1,z_2,z_3)(u,v)$. Then $(\boldsymbol{x},\boldsymbol{\nu}_1,\boldsymbol{\nu}_2)$:*

$U \to H^3 \times \Delta_5$ *is a hyperbolic generalized framed surface, where*

$$\begin{aligned}
\boldsymbol{x}(u,v) &= \boldsymbol{\pi}^{-1} \circ \overline{\boldsymbol{x}}(u,v) = \frac{(1+x_1^2+x_2^2+x_3^2, 2x_1, 2x_2, 2x_3)}{1-x_1^2-x_2^2-x_3^2}(u,v),\\
\boldsymbol{\nu}_1(u,v) &= \frac{(p_1,q_1,r_1,s_1)}{\sqrt{-p_1^2+q_1^2+r_1^2+s_1^2}}(u,v),\ \boldsymbol{\nu}_2(u,v) = \frac{(p_2,q_2,r_2,s_2)}{\sqrt{-p_2^2+q_2^2+r_2^2+s_2^2}}(u,v),\\
p_1(u,v) &= 2(x_1y_1+x_2y_2+x_3y_3)(u,v),\ p_2(u,v) = 2(x_1z_1+x_2z_2+x_3z_3)(u,v),\\
q_1(u,v) &= ((1+x_1^2-x_2^2-x_3^2)y_1+2x_1x_2y_2+2x_1x_3y_3)(u,v),\\
q_2(u,v) &= ((1+x_1^2-x_2^2-x_3^2)z_1+2x_1x_2z_2+2x_1x_3z_3)(u,v),\\
r_1(u,v) &= (2x_1x_2y_1+(1-x_1^2+x_2^2-x_3^2)y_2+2x_2x_3y_3)(u,v),\\
r_2(u,v) &= (2x_1x_2z_1+(1-x_1^2+x_2^2-x_3^2)z_2+2x_2x_3z_3)(u,v),\\
s_1(u,v) &= (2x_1x_3y_1+2x_2x_3y_2+(1-x_1^2-x_2^2+x_3^2)y_3)(u,v),\\
s_2(u,v) &= (2x_1x_3z_1+2x_2x_3z_2+(1-x_1^2-x_2^2+x_3^2)z_3)(u,v).
\end{aligned}$$

4.2 Relations between hyperbolic generalized framed base surfaces and lightcone framed base surfaces

The submanifold Δ_4 in [1] is defined as $\Delta_4 = \{(\boldsymbol{\nu}_1,\boldsymbol{\nu}_2) \in LC^* \times LC^* \mid \langle \boldsymbol{\nu}_1,\boldsymbol{\nu}_2\rangle = -2\}$, where $LC^* = \{\boldsymbol{x} \in \mathbb{R}^3_1 \backslash \{\boldsymbol{0}\} \mid \langle \boldsymbol{x},\boldsymbol{x}\rangle = 0\}$.

Definition 4.4 ([5])**.** We call $(\widetilde{\boldsymbol{x}},\boldsymbol{\ell}^+,\boldsymbol{\ell}^-) : U \to \mathbb{R}^3_1 \times \Delta_4$ a *lightcone framed surface* if there exist smooth functions $\widetilde{\alpha},\widetilde{\beta} : U \to \mathbb{R}$ such that $(\widetilde{\boldsymbol{x}}_u \wedge \widetilde{\boldsymbol{x}}_v)(u,v) = \widetilde{\alpha}(u,v)\boldsymbol{\ell}^+(u,v) + \widetilde{\beta}(u,v)\boldsymbol{\ell}^-(u,v)$ for all $(u,v) \in U$. $\widetilde{\boldsymbol{x}} : U \to \mathbb{R}^3_1$ is a *lightcone framed base surface* if there exists $(\boldsymbol{\ell}^+,\boldsymbol{\ell}^-) : U \to \Delta_4$ such that $(\widetilde{\boldsymbol{x}},\boldsymbol{\ell}^+,\boldsymbol{\ell}^-)$ is a lightcone framed surface.

Consider the projections $\boldsymbol{\pi}_i : H^3 \to \mathbb{R}^3_1$ $(i = 2,3,4)$, where $\boldsymbol{\pi}_2(x_1,x_2,x_3,x_4) = (x_1,x_3,x_4)$, $\boldsymbol{\pi}_3(x_1,x_2,x_3,x_4) = (x_1,x_2,x_4)$ and $\boldsymbol{\pi}_4(x_1,x_2,x_3,x_4) = (x_1,x_2,x_3)$.

Proposition 4.5. *Let* $(\boldsymbol{x},\boldsymbol{\nu}_1,\boldsymbol{\nu}_2) : U \to H^3 \times \Delta_5$ *be a hyperbolic generalized framed surface. If* $\boldsymbol{\pi}_i \circ \boldsymbol{\nu}_3(u,v)$ *is spacelike for all* $(u,v) \in U$*, then* $\widetilde{\boldsymbol{x}} = \boldsymbol{\pi}_i \circ \boldsymbol{x} : U \to \mathbb{R}^3_1$ *is a lightcone framed base surface,* $i = 2,3,4$*.*

From $\mathbb{R}^3_1$ to H^3, we consider the maps $\boldsymbol{p}_i : \mathbb{R}^3_1 \to H^3$ $(i = 2,3,4)$, where

$$\begin{aligned}
\boldsymbol{p}_2(a,b,c) &= \left(a, \sqrt{-1+a^2-b^2-c^2}, b, c\right),\\
\boldsymbol{p}_3(a,b,c) &= \left(a, b, \sqrt{-1+a^2-b^2-c^2}, c\right),\\
\boldsymbol{p}_4(a,b,c) &= \left(a, b, c, \sqrt{-1+a^2-b^2-c^2}\right)
\end{aligned}$$

and $a^2-b^2-c^2 > 1$.

Proposition 4.6. *Let* $(\widetilde{\boldsymbol{x}},\boldsymbol{\ell}^+,\boldsymbol{\ell}^-) : U \to \mathbb{R}^3_1 \times \Delta_4$ *be a lightcone framed surface. Denote* $\widetilde{\boldsymbol{x}}(u,v) = (x_1,x_2,x_3)(u,v)$*. If* $(x_1^2-x_2^2-x_3^2)(u,v) > 1$ *for all* $(u,v) \in U$*, then* $\boldsymbol{x} = \boldsymbol{p}_i \circ \widetilde{\boldsymbol{x}} : U \to H^3$ *is a hyperbolic generalized framed base surface,* $i = 2,3,4$*.*

5 Examples

Two examples are provided as applications of hyperbolic generalized framed surfaces.

Example 5.1 (Corank one singularities)**.** *Let* $\boldsymbol{x} : (\mathbb{R}^2, (0,0)) \to (H^3, \boldsymbol{x}(0,0))$ *be given by*

$$\boldsymbol{x}(u,v) = \left(\sqrt{u^2 + f^2(u,v) + g^2(u,v) + 1}, u, f(u,v), g(u,v)\right),$$

where $f, g : U \to \mathbb{R}$ *are smooth functions with* $f_v(0,0) = g_v(0,0) = 0$. *By a direct calculation, we have*

$$\begin{aligned}(\boldsymbol{x} \wedge \boldsymbol{x}_u \wedge \boldsymbol{x}_v)(u,v) = \Bigg(& (g_v f - f_v g)(u,v) + u(f_v g_u - f_u g_v)(u,v), \\ & \frac{(1+u^2)(f_v g_u - f_u g_v)(u,v) + u(g_v f - f_v g)(u,v)}{\sqrt{u^2 + (f^2+g^2)(u,v) + 1}}, \\ & \frac{(g_v + g_v f^2 - f_v g f)(u,v) + u(f f_v g_u - f f_u g_v)(u,v)}{\sqrt{u^2 + (f^2+g^2)(u,v) + 1}}, \\ & \frac{(-f_v + g_v f g - f_v g^2)(u,v) + u(g f_v g_u - g f_u g_v)(u,v)}{\sqrt{u^2 + (f^2+g^2)(u,v) + 1}}\Bigg).\end{aligned}$$

There exist unit spacelike vectors

$$\boldsymbol{\nu}_1(u,v) = \frac{\sqrt{(g(u,v) - u g_u(u,v))^2 + g_u^2(u,v) + 1}}{\sqrt{p(u,v)}}(\overline{\boldsymbol{\nu}}_1 - k\overline{\boldsymbol{\nu}}_2)(u,v)$$

and

$$\boldsymbol{\nu}_2(u,v) = \frac{(u g_u(u,v) - g(u,v))\boldsymbol{x}(u,v) + (0, g_u(u,v), 0, -1)}{\sqrt{(g(u,v) - u g_u(u,v))^2 + g_u^2(u,v) + 1}}$$

such that $\langle \boldsymbol{x}, \boldsymbol{\nu}_1\rangle(u,v) = \langle \boldsymbol{x}, \boldsymbol{\nu}_2\rangle(u,v) = \langle \boldsymbol{\nu}_1, \boldsymbol{\nu}_2\rangle(u,v) = 0$ *and*

$$\begin{aligned}&(\boldsymbol{x} \wedge \boldsymbol{x}_u \wedge \boldsymbol{x}_v)(u,v) \\ &= \frac{g_v(u,v)\sqrt{p(u,v)}}{\sqrt{u^2 + (f^2+g^2)(u,v) + 1}\sqrt{(g(u,v) - u g_u(u,v))^2 + g_u^2(u,v) + 1}}\boldsymbol{\nu}_1(u,v) \\ &\quad + \frac{q(u,v)}{\sqrt{u^2 + (f^2+g^2)(u,v) + 1}\sqrt{(g(u,v) - u g_u(u,v))^2 + g_u^2(u,v) + 1}}\boldsymbol{\nu}_2(u,v),\end{aligned}$$

where

$$\begin{aligned}\overline{\boldsymbol{\nu}}_1(u,v) &= \Big((f(u,v) - u f_u(u,v))\sqrt{u^2 + (f^2+g^2)(u,v) + 1}, u f(u,v) - (1+u^2) f_u(u,v), \\ &\qquad 1 + f^2(u,v) - u f(u,v) f_u(u,v), f(u,v)g(u,v) - u g(u,v) f_u(u,v)\Big), \\ \overline{\boldsymbol{\nu}}_2(u,v) &= (u g_u(u,v) - g(u,v))\boldsymbol{x}(u,v) + (0, g_u(u,v), 0, -1), \\ k(u,v) &= \frac{-(gf + g_u f_u)(u,v) + u(g_u f + g f_u)(u,v) - u^2 f_u(u,v) g_u(u,v)}{(g(u,v) - u g_u(u,v))^2 + g_u^2(u,v) + 1}, \\ p(u,v) &= (f(u,v) - u f_u(u,v))^2 + (g(u,v) - u g_u(u,v))^2 + ((f_u g - g_u f)^2 + f_u^2 + g_u^2)(u,v) + 1, \\ q(u,v) &= g_v(u,v)(-(gf + g_u f_u)(u,v) + u(g_u f + g f_u)(u,v) - u^2 f_u(u,v) g_u(u,v)) \\ &\quad + f_v(u,v)((g(u,v) - u g_u(u,v))^2 + g_u^2(u,v) + 1).\end{aligned}$$

Thus, $(\boldsymbol{x},\boldsymbol{\nu}_1,\boldsymbol{\nu}_2)$ is a hyperbolic generalized framed surface with

$$\alpha(u,v)=\frac{g_v(u,v)\sqrt{p(u,v)}}{\sqrt{u^2+(f^2+g^2)(u,v)+1}\sqrt{(g(u,v)-ug_u(u,v))^2+g_u^2(u,v)+1}},$$
$$\beta(u,v)=\frac{q(u,v)}{\sqrt{u^2+(f^2+g^2)(u,v)+1}\sqrt{(g(u,v)-ug_u(u,v))^2+g_u^2(u,v)+1}}.$$

Example 5.2 (A hyperbolic ruled surface)**.** *Let $\boldsymbol{\gamma}:(-\pi,\pi]\to H^3$,*

$$\boldsymbol{\gamma}(u)=\left(\frac{13}{5},\frac{9\cos u-3\cos 3u}{5},\frac{9\sin u-3\sin 3u}{5},\frac{6\sqrt{3}\cos u}{5}\right)$$

be a smooth curve in H^3 and $\boldsymbol{\delta}:(-\pi,\pi]\to S^3_1$,

$$\boldsymbol{\delta}(u)=\frac{\left(-156\sin u,-97\sin 2u+18\sin 4u,-50\sin^2 u-144\sin^4 u+25,-36\sqrt{3}\sin 2u\right)}{5\sqrt{144\sin^2 u+25}}.$$

Then

$$\begin{aligned}\boldsymbol{x}(u,v)&=\cosh v\boldsymbol{\gamma}(u)+\sinh v\boldsymbol{\delta}(u)\\&=\left(\frac{13\cosh v}{5}-\frac{156\sin u\sinh v}{5\sqrt{144\sin^2 u+25}},\frac{(9\cos u-3\cos 3u)\cosh v}{5}-\frac{(97\sin 2u-18\sin 4u)\sinh v}{5\sqrt{144\sin^2 u+25}},\right.\\&\qquad\frac{(9\sin u-3\sin 3u)\cosh v}{5}-\frac{(50\sin^2 u+144\sin^4 u-25)\sinh v}{5\sqrt{144\sin^2 u+25}},\\&\qquad\left.\frac{6\sqrt{3}\cos u\cosh v}{5}-\frac{36\sqrt{3}\sin 2u\sinh v}{5\sqrt{144\sin^2 u+25}}\right)\end{aligned}$$

is a hyperbolic ruled surface generated by $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$, where $(u,v)\in(-\pi,\pi]\times\mathbb{R}$. By a direct calculation, we have

$$\begin{aligned}(\boldsymbol{x}\wedge\boldsymbol{x}_u\wedge\boldsymbol{x}_v)(u,v)=&\frac{-12\sqrt{3}\sin u\cosh v+\sinh v\sqrt{432\sin^2 u+75}}{5}\boldsymbol{\nu}_1(u,v)\\&+\frac{65\sinh v}{\sqrt{144\sin^2 u+25}}\boldsymbol{\nu}_2(u,v),\end{aligned}$$

where

$$\boldsymbol{\nu}_1(u,v)=\frac{\left(24\cos u,13\cos 2u,13\sin 2u,13\sqrt{3}\right)}{2\sqrt{144\sin^2 u+25}},$$
$$\boldsymbol{\nu}_2(u,v)=\left(0,-\frac{\sqrt{3}}{2}\cos 2u,-\frac{\sqrt{3}}{2}\sin 2u,\frac{1}{2}\right)$$

and $\langle\boldsymbol{x},\boldsymbol{\nu}_1\rangle(u,v)=\langle\boldsymbol{x},\boldsymbol{\nu}_2\rangle(u,v)=\langle\boldsymbol{\nu}_1,\boldsymbol{\nu}_2\rangle(u,v)=0$. Thus, $(\boldsymbol{x},\boldsymbol{\nu}_1,\boldsymbol{\nu}_2)$ is a hyperbolic generalized framed surface with

$$\alpha(u,v)=\frac{-12\sqrt{3}\sin u\cosh v+\sinh v\sqrt{432\sin^2 u+75}}{5},\ \beta(u,v)=\frac{65\sinh v}{\sqrt{144\sin^2 u+25}}$$

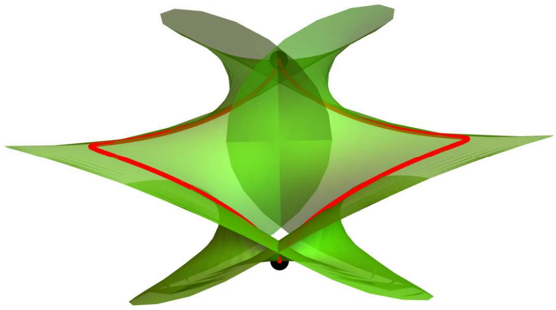

Figure 1: A curve $\boldsymbol{\gamma}$ and a hyperbolic ruled surface $\boldsymbol{x}$ projected to Poincaré 3-disc. The two black points are cross cap singularities of $\boldsymbol{x}$.

and the basic invariants

$$\begin{pmatrix} a_1(u,v) & b_1(u,v) & c_1(u,v) \\ a_2(u,v) & b_2(u,v) & c_2(u,v) \end{pmatrix} = \begin{pmatrix} -\frac{65\sinh v}{\sqrt{144\sin^2 u+25}} & \frac{-12\sqrt{3}\sin u\cosh v+\sinh v\sqrt{432\sin^2 u+75}}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} e_1(u,v) & f_1(u,v) & g_1(u,v) \\ e_2(u,v) & f_2(u,v) & g_2(u,v) \end{pmatrix} = \begin{pmatrix} 0 & \frac{65\cosh v}{144\sin^2 u+25} & \frac{12\sqrt{3}\sin u\sinh v-\cosh v\sqrt{432\sin^2 u+75}}{5} \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\alpha(0,0)=\beta(0,0)=0$, $(a_1,a_2,b_1,b_2)(0,0)=(0,0,0,0)$, $(c_1,c_2)(0,0)=(0,1)\neq(0,0)$ *and*

$$\left(\det\left(\boldsymbol{b}_u\ \boldsymbol{c}\right)\det\left(\boldsymbol{a}_v\ \boldsymbol{c}\right)-\det\left(\boldsymbol{b}_v\ \boldsymbol{c}\right)\det\left(\boldsymbol{a}_u\ \boldsymbol{c}\right)\right)(0,0)=12\sqrt{3}\neq 0,$$

we can obtain that $\boldsymbol{x}$ *is* $\mathcal{A}$*-equivalent to cross cap at* $(0,0)$ *by Proposition 3.11. A similar discussion leads to the conclusion that* $\boldsymbol{x}$ *is also* $\mathcal{A}$*-equivalent to cross cap at* $(\pi,0)$*, see Figure 1.*

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