Deformations of codimension two singularities and their geometric properties

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Abstract

We consider two parameters deformations of codimension two singularities and investigate differential geometric properties. To this end, we give a normal form representing the deformation using only diffeomorphisms on the source space and isometries on the target space.

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1 Introduction

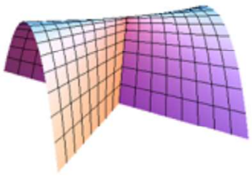
Singularities of smooth map-germs exhibit a rich variety of behaviors under deformation. In particular, S_2 singularities correspond to the appearance or disappearance of S_1 singularities and the Whitney umbrella. Therefore, it is natural to consider such deformations when studying these singularities. In this paper, we present a normal form for the deformations of codimension-two singularities using a diffeomorphism-germ on the source space and an isometry-germ on the target space. Furthermore, using this form, we investigate the differential geometric properties of the deformations of S_2 singularities.

Singularities of smooth map-germs have long been studied under equivalence given by coordinate changes in both the source and the target. This equivalence relation is called $\mathcal{A}$-equivalence. The classification of simple singularities appearing on surfaces in three-dimensional space with respect to $\mathcal{A}$-equivalence is given in [5]. For reference, we present the classification of singularities of codimension up to two, which constitute the focus of this paper, a full classification of simple singularities is given in [5, Theorem 1.1].

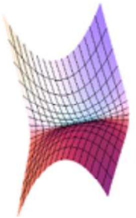
Fact 1.1. [5, Theorem 1.1] *Let* $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ *be a map-germ with a singular point at the origin, and assume its codimension is less than or equal to two. Then* f *is* $\mathcal{A}$*-equivalent to one of the following:*

The following theorem is known for the Whitney umbrella.

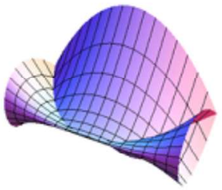
Germ	Codimension	Name
(u, v^2, uv)	0	Whitney umbrella (S_0)
$(u, v^2, v(\pm u^2 + v^2))$	1	$S_1^{\pm}$
$(u, v^2, v(u^3 + v^2))$	2	S_2
$(u, v^2, v(u^2 \pm v^4))$	2	$B_2^{\pm}$
$(u, uv + v^5, v^3)$	2	H_2



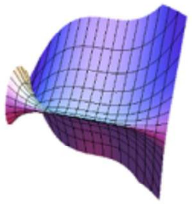
S_0(Whitney umbrella)



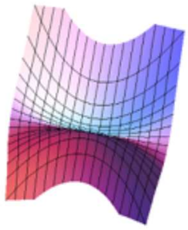
S_1^+



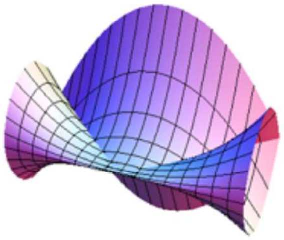
S_1^-



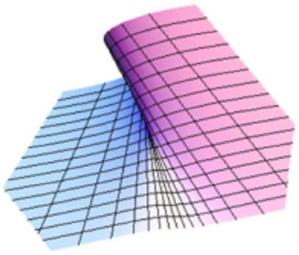
S_2



B_2^+



B_2^-



H_2

Theorem 1.2. [6] *Let* $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ *be a Whitney umbrella. Then there exist a diffeomorphism-germ* $\varphi : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ *and* $T \in SO(3)$ *such that*

$$T \circ f \circ \varphi^{-1}(u, v) = \Big(u, uv + \frac{b_3}{6}v^3 + O(4), \tag{1.1}$$
$$\frac{1}{2}(a_{20}u^2 + 2a_{11}uv + a_{02}v^2) + \frac{1}{6}(a_{30}u^3 + 3a_{21}u^2v + 3a_{12}uv^2 + a_{03}v^3) + O(4)\Big)$$

with a_{20}, a_{11}, a_{02}, a_{30}, a_{21}, a_{12}, a_{03}, $b_3 \in \boldsymbol{R}$. *By an orientation-preserving diffeomorphism-germ on the source space, we can take* $a_{02} > 0$. *Here,* $O(n)$ *stands for the terms whose degrees are equal to or greater than* n

The coefficients a_{20}, a_{11}, a_{02}, a_{30}, a_{21}, a_{12}, a_{03}, b_3 are the differential geometric invariants of the Whitney umbrella, and these invariants can be used to reveal the differential geometric properties of the Whitney umbrella instead of the Gaussian curvature and the mean curvature, which have lost their meanings at the Whitney umbrella.

Let $f : (\boldsymbol{R}^2, p) \to (\boldsymbol{R}^n, 0)$ $(n = 2, 3)$ be a map-germ with $\operatorname{rank} df_p = 1$. A vector $\eta \in T_p\boldsymbol{R}^2$ is called a *null vector* if it is a generator of $\operatorname{Ker} df_p$. A vector field η on the source space is an *extended null vector field* if $q \in S(f)$ then η_q is a null vector, where stands for the set for singular points of f. If $\operatorname{rank} df_p = 1$, then there exist a neighborhood U of p such that $\operatorname{rank} df_q = 1$ for any $q \in S(f) \cap U$ and there exist an extended null vector field on U. An extended null vector field is also called a *null vector field* for short.

2 Normal forms of codimension two singularities including deformations

In this section, we consider two-parameter deformations of germs whose 2-jets are $\mathcal{A}$-equivalent to (u, v^2, uv) or $(u, v^2, 0)$.

Definition 2.1. A map-germ $f : (\boldsymbol{R}^2 \times \boldsymbol{R} \times \boldsymbol{R}, 0) \to (\boldsymbol{R}^3, 0)$ is a 2-*parameters deformation of* $g : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$, if $f(u, v, 0, 0) = g(u, v)$ and $f(0, 0, s, t) = (0, 0, 0)$.

In this definition, the parameter $s \in \boldsymbol{R}$ and $t \in \boldsymbol{R}$ as the third component and fourth component of the source space is called the *deformation parameters.* We define an equivalence relation between two deformations preserving the deformation parameters.

Definition 2.2. Let $f_1, f_2 : (\boldsymbol{R}^2 \times \boldsymbol{R} \times \boldsymbol{R}, 0) \to (\boldsymbol{R}^3, 0)$ be deformations of $g : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$. Then f_1 and f_2 are *equivalent as deformations* if there exist an orientation preserving diffeomorphism-germs $\varphi : (\boldsymbol{R}^2 \times \boldsymbol{R}, 0) \to (\boldsymbol{R}^2 \times \boldsymbol{R}, 0)$ with the form

$$\varphi(u, v, s, t) = (\varphi_1(u, v, s, t), \varphi_2(u, v, s, t), \varphi_3(s, t), \varphi_4(s, t)), \tag{2.1}$$

and $\psi : (\boldsymbol{R}^3, 0) \to (\boldsymbol{R}^3, 0)$ such that $\psi \circ f_1 \circ \varphi^{-1}(u, v, s) = f_2(u, v, s)$ holds.

Since φ_3 and φ_4 are defined to depend only on s and t, this allows changes in the deformation parameters themselves while preventing any effect on the other parameters. As the third and fourth components of the source space corresponds to the deformation parameters, this definition means that an $\mathcal{A}$-equivalence preserving the deformation parameters is defined.

Example 2.3. Let $f_{s,t}$ be a deformation of an S_2 singularity defined by

$$f_{s,t} : (\boldsymbol{R}^2 \times \boldsymbol{R} \times \boldsymbol{R}, 0) \ni (u, v, s, t) \mapsto (u, v^2, v(u^3 + v^2) + sv + tuv) \in (\boldsymbol{R}^3, 0).$$

We show the deformation of $f_{s,t}$ in Figure 1.

Theorem 2.4. *Let* $f : (\boldsymbol{R}^2 \times \boldsymbol{R} \times \boldsymbol{R}, 0) \to (\boldsymbol{R}^3, 0)$ *be a deformation of* $g : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ *such that the* 2*-jet of* g *is* $\mathcal{A}$*-equivalent to* $(u, v^2, 0)$ *or* (u, v^2, uv)*. Then there exist an orientation preserving diffeomorphism-germ* $\varphi : (\boldsymbol{R}^2 \times \boldsymbol{R} \times \boldsymbol{R}, 0) \to (\boldsymbol{R}^2 \times \boldsymbol{R} \times \boldsymbol{R}, 0)$ *with the form* (2.1)*,* $T \in SO(3)$ *and the functions* $f_{21}, f_{31} \in C^\infty(1, 1), f_{24}, f_{35} \in C^\infty(2, 1), f_{33}, f_{34} \in C^\infty(3, 1), f_{32} \in C^\infty(4, 1)$ *such that*

$$\begin{aligned} {f_{\mathrm{n}}}^{s,t} &= T \circ f \circ \varphi^{-1}(u, v, s, t) \\ &= \Big(u, u^2 f_{21}(u) + v^2 + us f_{24}(u, s, t) + ut f_{25}(u, t), \\ &\quad u^2 f_{31}(u) + v^2 f_{32}(u, v, s, t) + v f_{33}(u, s, t) + us f_{34}(u, s, t) + ut f_{35}(u, t)\Big), \end{aligned} \tag{2.2}$$

where $f_{32}(0, 0, 0, 0) = f_{33}(0, 0, 0) = 0$*. If the* 2*-jet of* g *is* $\mathcal{A}$*-equivalent to* $(u, v^2, 0)$*, then* $(f_{33})_u(0, 0, 0) = 0$ *holds, and if it is* $\mathcal{A}$*-equivalent to* (u, v^2, uv)*, then* $(f_{33})_u(0, 0, 0) > 0$ *holds. If* $(f_{33})_u(0, 0, 0) = 0$*, then* $(f_{33})_{uuu}(0, 0, 0)\ 0$ *holds.*

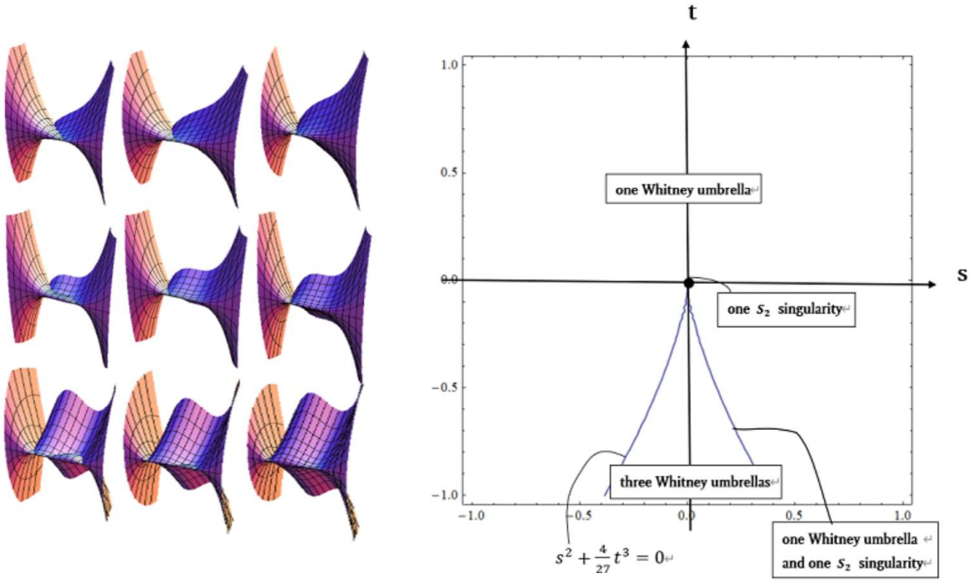


Figure 1: A Deformation of an S_2 singularity (The top row are $(s,t)=(-2/3\sqrt{3},1)$, $(0,1)$, $(2/3\sqrt{3},1)$, the middle row are $(-2/3\sqrt{3},0)$, $(0,0)$, $(2/3\sqrt{3},0)$, the bottom row are $(-2/3\sqrt{3},-1)$, $(0,-1)$, $(2/3\sqrt{3},-1)$ from left to right)

In Theorem 2.4, the given f and ${f_{\mathrm{n}}}^{s,t}$ are equivalent as deformations (Definition 2.2), and they have the same differential geometric properties.

By [3], we see that f of the form (2.2) with these conditions is an S_2 singularity if and only if

$$f_{32}(0,0,0,0)=f_{33}(0,0,0)=(f_{33})_u(0,0,0)=(f_{33})_{uu}(0,0,0)=0 \tag{2.3}$$

and

$$(f_{32})_v(0,0,0,0)(f_{33})_{uuu}(0,0,0)\neq 0, \tag{2.4}$$

holds, and a $B_2^{\pm}$ singularity if and only if

$$f_{32}(0,0,0,0)=f_{33}(0,0,0)=(f_{33})_u(0,0,0)=(f_{32})_v(0,0,0,0)=0, \tag{2.5}$$

$$(f_{33})_{uu}(0,0,0)\neq 0, \tag{2.6}$$

$$3(f_{32})_{vvv}(0,0,0,0)(f_{33})_{uu}(0,0,0)-5(f_{32})_{uv}(0,0,0,0)^2\neq 0 \tag{2.7}$$

hold. The ±-sign of the $B_2^{\pm}$ singularity is determined by the sign of last formula of (2.7). We remark that we may assume $(f_{33})_{uuu}(0,0,0)>0$ for an S_2 singularity by rotations on the source and the target spaces. The form ${f_{\mathrm{n}}}^{s,t}$ satisfying (2.3) and (2.4) is called the *normal form of the deformations of an S_2 singularity* and satisfying (2.5), (2.6) and (2.7) is called the *normal form of the deformations of a B_2 singularity*. The set of singular points of ${f_{\mathrm{n}}}^{s,t}$ is

$$S({f_{\mathrm{n}}}^{s,t})=\{(u,v)\,|\,v=0, f_{33}(u,s,t)=0\}.$$

Here, we set

$$f_{33}(u,s,t)=s+tu+u^2f_{332}(s,t)+u^3f_{333}(u,s,t). \tag{2.8}$$

In Theorem 2.4, the given f and ${f_{\mathrm{n}}}^{s,t}$ are equivalent as deformations (Definition 2.2). Since we only use elements of $SO(3)$ as a diffeomorphism-germ on the target space, they have the same differential geometric properties. In what follows f_{33} is written as form (2.8). Moreover, the uniqueness of the normal form holds.

3 Geometry on deformations of S_2 singularities

In this section, we consider geometry on the geometry on deformations of S_2 singularities using (2.2), (2.3) and (2.4). We set $f = {f_{\mathrm{n}}}^{s,t}$ (see (2.2)).

3.1 Location of singular point

To determine the location of the singular point, we fix s and t, and solve for u such that $f_{33}(u,s,t) = 0$. Since we assume (2.8) and $(f_{33})_s(0,0,0) \neq 0$, we can write $s = a(u,t)$ that solves $f_{33}(u,s,t) = 0$ for s. We set $u = x$ in $a(u,t)$ and substitute $a(x,t)$ into s in f_{33}. Define $F(u,x,t) = f_{33}(u, a(x,t), t)$. Then we have

$$F = a(x,t) + tu + u^2 f_{332}(a(x,t),t) + u^3 f_{333}(u, a(x,t), t). \tag{3.1}$$

Since $F|_{x=u} = 0$, we see F has the form $(u-x)(g_0(x,t) + ug_1(x,t) + u^2 g_2(u,x,t))$ for $g_0, g_1 \in C^\infty(2,1)$ and $g_2 \in C^\infty(3,1)$. Then we see

$$\begin{aligned} F = {} & -xg_0(x,t) + u(g_0(x,t) - xg_1(x,t)) \\ & \qquad + u^2(g_1(x,t) - xg_2(u,x,t)) + u^3 g_2(u,x,t). \end{aligned} \tag{3.2}$$

Comparing the terms of u in (3.1) and (3.2), then we have $g_0(x,t) - xg_1(x,t) = t$. Since $f_{332}(0,0,0) = 0$, we have $g_1(0,0) = 0$. Thus we obtain

$$g_1(x,t) = xg_{11}(x,t) + tg_{12}(x,t)$$

for some functions g_{11} and g_{12} and

$$g_0(x,t) = t + x(xg_{11}(x,t) + tg_{12}(x,t)).$$

Moreover,

$$\begin{aligned} F/(u-x) &= g_0(x,t) + ug_1(x,t) + u^2 g_2(u,x,t) \\ &= g_2(u,x,t)\Big(\big(u + \frac{g_1(x,t)}{2g_2(u,x,t)}\big)^2 - \frac{1}{4g_2(u,x,t)^2}\big((xg_{11}(x,t) + tg_{12}(x,t))^2 \\ &\quad -4g_2(u,x,t)(x(xg_{11}(x,t) + tg_{12}(x,t)) + t)\big)\Big) \\ &= g_2(u,\tilde{x}\tilde{t},-\tilde{t}^2)\Big(\big(u + \frac{g_1(\tilde{x}\tilde{t},-\tilde{t}^2)}{2g_2(u,\tilde{x}\tilde{t},-\tilde{t}^2)}\big)^2 \\ &\quad - \frac{\tilde{t}^2}{4g_2(u,\tilde{x}\tilde{t},-\tilde{t}^2)^2}\Big((\tilde{x}g_{11}(\tilde{x}\tilde{t},-\tilde{t}^2) - \tilde{t}g_{12}(\tilde{x}\tilde{t},-\tilde{t}^2))^2 \\ &\quad - 4g_2(u,\tilde{x}\tilde{t},-\tilde{t}^2)\big(\tilde{x}(\tilde{x}g_{11}(\tilde{x}\tilde{t},-\tilde{t}^2) - \tilde{t}g_{12}(\tilde{x}\tilde{t},-\tilde{t}^2)) - 1\big)\Big), \end{aligned}$$

and we set $x = \tilde{x}\tilde{t}$ and $t = -\tilde{t}^2$. Since f is a deformation of an S_2 singularity, $(f_{33})_{uuu}(0,0,0) \neq 0$, and this is equivalent to $g_2(0,0,0) \neq 0$. Since $g_2(0,0,0) \neq 0$, we know u has three solutions for $F = 0$ in a neighborhood of $(\tilde{t},\tilde{x}) = (0,0)$. If $(u,v) \in S(f)$, then $v = 0$ and u depends on $\tilde{t}$ and $\tilde{x}$. We set this function $u(\tilde{t},\tilde{x})$. Then we obtain locations of singular points with respect to the setting $x = \tilde{x}\tilde{t}$ and $t = -\tilde{t}^2$.

Theorem 3.1. *Let f be a deformation of an S_2 singularity given in* (2.2)*. If $(u,v) \in S(f)$ and $f_{333}(0,0,0) > 0$, then $u(\tilde{t},\tilde{x})$ can be expanded as follows*:

$$u(\tilde{t},\tilde{x}) = \tilde{t}\tilde{x}, \tag{3.3}$$

$$\begin{aligned} u(\tilde{t},\tilde{x}) = &+\frac{1}{f_{3330}}\tilde{t} + \frac{1}{2f_{3330}^4}\Big(f_{3330}^2(f_{332})_t - (f_{333})_u\Big)\tilde{t}^2 - \frac{1}{2}\tilde{t}\tilde{x} \\ &+ \frac{1}{8f_{3330}^7}\Big(f_{3330}^4(f_{332})_t^2 - 6f_{3330}^2(f_{332})_t(f_{333})_u + 5(f_{333})_u^2 \\ &- 2f_{3330}^2(f_{333})_{uu} + 4f_{3330}^4(f_{333})_t\Big)\tilde{t}^3 + \frac{1}{4f_{3330}^3}\Big(f_{3330}^2(f_{332})_t \\ &+ (f_{333})_u\Big)\tilde{t}^2\tilde{x} - \frac{3}{8}f_{3330}\tilde{t}\tilde{x}^2 + O_{\tilde{t},\tilde{x}}(4) \end{aligned} \tag{3.4}$$

or

$$\begin{aligned} u(\tilde{t},\tilde{x}) = &-\frac{1}{f_{3330}}\tilde{t} + \frac{1}{2f_{3330}^4}\Big(f_{3330}^2(f_{332})_t - (f_{333})_u\Big)\tilde{t}^2 - \frac{1}{2}\tilde{t}\tilde{x} \\ &- \frac{1}{8f_{3330}^7}\Big(f_{3330}^4(f_{332})_t^2 - 6f_{3330}^2(f_{332})_t(f_{333})_u + 5(f_{333})_u^2 \\ &- 2f_{3330}^2(f_{333})_{uu} + 4f_{3330}^4(f_{333})_t\Big)\tilde{t}^3 - \frac{1}{4f_{3330}^3}\Big(f_{3330}^2(f_{332})_t \\ &+ (f_{333})_u\Big)\tilde{t}^2\tilde{x} + \frac{3}{8}f_{3330}\tilde{t}\tilde{x}^2 + O_{\tilde{t},\tilde{x}}(4), \end{aligned} \tag{3.5}$$

where $f_{333}(0) = f_{3330}^2$, and all functions are evaluated at $u = v = s = t = 0$. The formula (3.5) *is obtained* (3.4) *by $\tilde{t}$ to $-\tilde{t}$ and $\tilde{x}$ to $-\tilde{x}$.*

Since Whitney umbrellas appear in deformations of the S_2 singularities, we introduce formulas for invariants of a Whitney umbrella.

Theorem 3.2. [4] *Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a Whitney umbrella. For coordinate system (u,v) satisfying* $\mathrm{Ker}df_0 = \langle \partial v \rangle$ *and $|f_u, f_{uv}, f_{vv}| > 0$, the invariants a_{02}, a_{20}, a_{11} in Theorem* 1.2 can be written as:

$$a_{20} = \frac{|f_u \times f_{vv}|}{4|f_u|^3|f_u, f_{uv}, f_{vv}|^2}(|f_u, f_{uu}, f_{vv}|^2 + 4|f_u, f_{uv}, f_{vv}||f_u, f_{uv}, f_{uu}|),$$

$$a_{11} = \frac{1}{2|f_u||f_u, f_{uv}, f_{vv}|^2}\left(2|f_u, f_{uv}, f_{vv}|\begin{vmatrix} f_u \cdot f_u & f_u \cdot f_{uv} \\ f_{vv} \cdot f_u & f_{vv} \cdot f_{uv} \end{vmatrix} - |f_u \times f_{vv}|^2|f_u, f_{uu}, f_{vv}|\right),$$

$$a_{02} = \frac{|f_u||f_u \times f_{vv}|^3}{|f_u, f_{uv}, f_{vv}|^2}.$$

As we saw in the above, the singular point of $f = {f_{\mathrm{n}}}^{s,t}$ is $(u(\tilde{t},\tilde{x}),0)$. Considering the above invariants a_{20}, a_{11}, a_{02} at $(u(\tilde{t},\tilde{x}),0)$, one can regard the invariants a_{20}, a_{11}, a_{02} as functions of $\tilde{t}$ and $\tilde{x}$. Let $a_{20}(\tilde{t},\tilde{x})$, $a_{11}(\tilde{t},\tilde{x})$, $a_{02}(\tilde{t},\tilde{x})$ denote these functions. We remark that

$$\lim_{\tilde{t},\tilde{x}\to 0} |f_u, f_{uv}, f_{vv}|(u(\tilde{t},\tilde{x}),0) = 0.$$

Thus these functions generally diverge at the $S_1^{\pm}$ and S_2 singularities. More precisely, we obtain the following theorem. For each of the three singular points (3.3), (3.4) and (3.5), the invariants can be computed as follows. The invariants a_{20}, a_{11}, a_{02} correspond to the cases (3.3) (first one) and (3.4) (second one), respectively. The case of (3.5) is obtained (3.4) by setting $\tilde{t}$ to $-\tilde{t}$ and $\tilde{x}$ to $-\tilde{x}$.

Theorem 3.3. *The functions $a_{20}(\tilde{t},\tilde{x}), a_{11}(\tilde{t},\tilde{x})$ and $a_{02}(\tilde{t},\tilde{x})$ can be expanded as follows:*

$$\begin{aligned}
&a_{20}(\tilde{t},\tilde{x})\\
=&\frac{1}{\tilde{t}^4}\Big(\frac{f_{31}^2}{2} + O_{\tilde{t},\tilde{x}}(2)\Big),\\
&\frac{1}{\tilde{t}^4}\Big(\frac{f_{31}^2}{2} - \frac{f_{31}}{2f_{3330}^2}\Big(f_{31}(f_{333})_u + f_{3330}^2(-6f_{31}' + f_{31}(f_{332})_t + 2f_{21}(f_{32})_u)\Big)\tilde{t} + \frac{3}{2}f_{3330}f_{31}^2\tilde{x}\\
&+ O_{\tilde{t},\tilde{x}}(2)\Big),\\
&a_{11}(\tilde{t},\tilde{x})\\
=&\frac{1}{\tilde{t}^4}\Big(\frac{f_{31}}{2} + O_{\tilde{t},\tilde{x}}(2)\Big),\\
&\frac{1}{\tilde{t}^4}\Big(\frac{f_{31}}{2} - \frac{1}{2f_{3330}^3}\Big(f_{31}(f_{333})_u + f_{3330}^2\big(f_{31}(f_{332})_t + f_{21}(f_{32})_u - 3f_{31}'\big)\Big)\tilde{t} + \frac{3}{2}f_{3330}f_{31}\tilde{x}\\
&+ O_{\tilde{t},\tilde{x}}(2)\Big),\\
&a_{02},(\tilde{t},\tilde{x})\\
=&\frac{1}{\tilde{t}^4}\Big(\frac{1}{2} + O_{\tilde{t},\tilde{x}}(2)\Big),\\
&\frac{1}{\tilde{t}^4}\Big(\frac{1}{2} - \frac{1}{2f_{3330}^2}\Big(f_{3330}^2(f_{332})_t + (f_{333})_u\Big)\tilde{t} + \frac{3}{2}f_{3330}\tilde{x} + O_{\tilde{t},\tilde{x}}(2)\Big),
\end{aligned}$$

where all functions are evaluated at $u = v = s = t = 0$.

3.2 Focal conics

Definition 3.4. Let $f : (\boldsymbol{R}^2,(u_0,v_0)) \to \boldsymbol{R}^3$ be a map-germ with $\operatorname{rank} df_{(u_0,v_0)} = 1$. Then the *focal set* of f at (u_0,v_0) is defined by

$$\{x \in \boldsymbol{R}^3 | D_u^x(u_0,v_0) = D_v^x(u_0,v_0) = 0, D_{uu}^x(u_0,v_0)D_{vv}^x(u_0,v_0) - D_{uv}^x(u_0,v_0)^2 = 0\},$$

where D^x is the distance squared function $D^x(u,v) = \frac{1}{2}|x-f|^2$.

The focal set is located on the normal plane of the image of $df_{(u_0,v_0)}$. Since the focal set is a conic for the Whitney umbrella, the focal set is called the *focal conic.*The classification for the focal conic of a Whitney umbrella is as follows:

Proposition 3.5. [2, Proposition 3.4] *Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a Whitney umbrella. Then the focal conic of f at 0 is an ellipse if and only if $a_{20} < 0$, a hyperbola if and only if $a_{20} > 0$, and a parabola if and only if $a_{20} = 0$, where we assume $a_{02} > 0$.*

We have the following corollary for focal conics appearing in deformations of S_2 singularities.

Corollary 3.6. *In the deformations of S_2 singularities with $f_{31}(0) \neq 0$, all focal conics on the Whitney umbrella are hyperbolas for $\tilde{t}$ sufficiently close to 0.*

3.3 Curvature line

Let $g : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a map-germ satisfying $j^2g(0,0) = (u, v^2, 0)$ or (u, v^2, uv). The normal form of this map-germ is given in [6], and is reformulated in the following theorem.

Theorem 3.7. *Let $g : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a map-germ satisfying $j^2g(0,0) = (u, v^2, 0)$ or (u, v^2, uv). Then there exist an orientation preserving diffeomorphism-germ $\varphi : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ and $T \in SO(3)$ and the functions g_{21}, g_{31}, $g_{331} \in C^\infty(1,1), g_{32} \in C^\infty(2,1)$ such that*

$$T \circ f \circ \varphi^{-1}(u,v) = (u, u^2g_{21}(u) + v^2, u^2g_{31}(u) + v^2g_{32}(u,v) + uvg_{331}(u)), \tag{3.6}$$

where $g_{32}(0,0) = 0$. If $j^2g(0,0)$ is $\mathcal{A}$-equivalent to $(u, v^2, 0)$, then $g_{331}(0) = 0$ and $j^2g(0,0)$ is $\mathcal{A}$-equivalent to (u, v^2, uv), then $g_{331}(0) > 0$.

Since we focus on Whitney umbrellas in this section, we assume $g_{331}(0) > 0$. Let $U \subset \boldsymbol{R}^2$ be an open set and (u, v) be a coordinate system on U. We consider

$$\omega = a(u,v)du^2 + 2b(u,v)dudv + c(u,v)dv^2,$$

where a, b and c are smooth functions. We call $\omega = 0$ a *binary differential equation.* Here, let us set $\delta = b^2 - ac$. This δ is called the *discriminant.* If $\delta > 0$ at $x \in U$, then $\omega(x,x) = 0$ defines two linearly independent directions in T_xU, if $\delta = 0$ at $x \in U$, then $\omega(x,x) = 0$ defines a single direction and if $\delta < 0$ at $x \in U$, then $\omega(x,x) = 0$ defines no direction. Here, a map-germ $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ is called a *frontal* if there exists a never vanishing map-germ $\nu : (\boldsymbol{R}^2, 0) \to \boldsymbol{R}^3$ such that for any $p \in (\boldsymbol{R}^2, 0)$ and $X_p \in T_p\boldsymbol{R}^2$, it holds that $df(X_p) \cdot \nu(p) = 0$. Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a frontal and ν be a unit normal vector along f. Let us set

$$\mathcal{M} = \{(u, v, [\alpha, \beta]) \in (\boldsymbol{R}^2, 0) \times \boldsymbol{R}P^1 | a\alpha^2 + 2b\alpha\beta + c\beta^2 = 0\}.$$

We assume $\mathcal{M}$ is a manifold. Following [1], let us set $p = \alpha/\beta$ on the set $\beta \neq 0$ and $\mathcal{F}(u, v, p) = ap^2 + 2bp + c$. We consider the vector field on $\mathcal{M}$

$$\xi(u,v,p) = p\mathcal{F}_p(u,v,p)\partial_u + \mathcal{F}_p(u,v,p)\partial_v - (p\mathcal{F}_u(u,v,p) + \mathcal{F}_v(u,v,p))\partial_p.$$

Considering the projection $\pi : \mathcal{M} \ni (u,v,p) \mapsto (u,v) \in \boldsymbol{R}^2$, we have $\omega(d\pi(\xi), d\pi(\xi)) = 0$. Therefore, $d\pi(\xi)$ is a solution of $\omega = 0$. We define $\omega_{\ell c}$ by

$$\omega_{\ell c} = (LF - EM)du^2 + (GL - EN)/2dudv + (MG - FN)dv^2,$$

where $E = |f_u|^2$, $G = f_u \cdot f_v$, $F = |f_v|^2$, $L = f_{uu} \cdot \nu$, $M = f_{uv} \cdot \nu$, $N = f_{vv} \cdot \nu$. The integral curves of solutions of $\omega_{\ell c} = 0$ are called the *curvature lines.* Although Whitney umbrella cannot define a unit normal vectors, it is known by the blowing-up, the unit normal vector can be extended as a function of (r, θ) for the Whitney umbrella [2]. We set $a = LF - EM$, $b = (GL - EN)/2$, $c = MG - FN$. Then $\tilde{a}(r,\theta) = a/r^2$, $\tilde{b}(r,\theta) = b/r^2$, $\tilde{c}(r,\theta) = c/r^2$ and using (3.6), they satisfy

$$\begin{aligned}\tilde{a}(r,\theta) =& = -\cos\theta\sin\theta((g'_{331}(0) - 3(g_{32})_v(0,0))\cos 2\theta \\ &\qquad + g'_{331}(0) + 3(g_{32})_v(0,0) + 2(g_{32})_u(0,0)\sin 2\theta) + O_r(1), \qquad (3.7)\\ \tilde{b}(r,\theta) =& 2\cos\theta g_{331}(0) + O_r(1), \qquad (3.8)\\ r\tilde{c}(r,\theta) =& r(-2\sin\theta g_{331}(0) + O_r(1)). \qquad (3.9)\end{aligned}$$

We set

$$\tilde{\mathcal{M}} = \{(r, \theta, [\tilde{\alpha}, \tilde{\beta}]) \in \boldsymbol{R} \times S^1 \times \boldsymbol{R}P^1 | \tilde{a}\tilde{\alpha}^2 + 2\tilde{b}\tilde{\alpha}\tilde{\beta} + r\tilde{c}\tilde{\beta}^2 = 0\}.$$

Theorem 3.8. *Under the standing assumptions of Section 3.3, $\tilde{\mathcal{M}}$ is a manifold for $r > 0$ sufficiently close to* 0.

We obtain a complete description of the possible curvature line configurations near the singular point. In this situation, we discuss about vector field on $\tilde{\mathcal{M}}$. The projection is $d\pi(\xi) = \tilde{\mathcal{F}}_{\tilde{p}}(\tilde{p}\partial_r + \partial_\theta)$.

Corollary 3.9. *On θ where $\tilde{a}(0,\theta) = 0$, the principal direction converges to $(\partial_\theta, \partial_r)$ for $\tilde{r} \to 0$. On the other hand, on θ where $\tilde{a}(0,\theta) \neq 0$, the principal direction converges to e $(\partial_\theta, -4\cos\theta g_{331}(0)\partial_r + \tilde{a}(0,\theta)\partial_\theta)$ for $\tilde{r} \to 0$.*

Therefore, the direction of convergence depends on whether

$$\begin{aligned}\tilde{a}(0,\theta) = \cos\theta\sin\theta((g'_{331}(0) - 3(g_{32})_v(0,0))\cos 2\theta \\ + g'_{331}(0) + 3(g_{32})_v(0,0) + 2(g_{32})_u(0,0)\sin 2\theta)\end{aligned}$$

is zero or not, making the invariants in Subsection 3.4 crucial.

3.4 Geometric invariants of Whitney umbrella

We see the third-order coefficients in (3.6) play a crucial role to investigate the curvature line around Whitney umbrella. First, we see the relation of coefficients in the form (3.6) with the coefficients in (1.1). We have

$$g_{21}(0) = \frac{a_{20}a_{02} - a_{11}^2}{2a_{02}}, \ g_{31}(0) = -\frac{a_{11}}{a_{02}}, \ g_{331}(0) = \left(\frac{2}{a_{02}}\right)^{\frac{1}{2}} \qquad (3.10)$$

and

$$
\begin{aligned}
g'_{21}(0) = & \ \frac{a_{03}a_{11}^3 - a_{02}\left(3a_{11}^2a_{12} - 3a_{02}a_{11}a_{21} + a_{02}^2a_{30}\right)}{6a_{02}^3}, \\
g'_{31}(0) = & \ \frac{3a_{03}a_{11}^2 - 6a_{02}a_{11}a_{12} + 3a_{02}^2a_{21} + a_{11}^3b_3}{6a_{02}^3}, \\
(g_{32})_u(0,0) = & \ \frac{a_{03} + 3a_{11}b_3}{3a_{02}^2}, \\
(g_{32})_v(0,0) = & \ -\frac{\sqrt{2}b_3}{3a_{02}^{3/2}}, \\
g'_{331}(0) = & \ \frac{-a_{03}a_{11} + a_{02}a_{12} - a_{11}^2b_3}{\sqrt{2}a_{02}^{5/2}}.
\end{aligned}
\tag{3.11}
$$

It is convenient to take an adapted, strongly adapted or most strongly adapted coordinate system as follows. Let the map-germ g be a germ satisfying $\xi g \times \eta\eta g \neq 0$ for a following adapted vector fields. A pair of vector fields (ξ, η) on $(\boldsymbol{R}^2, 0)$ is called *adapted* if it satisfies (a) and (b) at the origin, *strongly adapted* if it satisfies (a), (b) and (c) at the origin, *most strongly adapted* if it satisfies (a), (b), (c) and (d) at the origin,
(a) η is a null vector field,
(b) ξ is a transverse vector field to η,
(c) it hold that $\xi g \cdot \eta\eta g$=0,
(d) it hold that $\eta\xi g = \xi\eta g$.

We assume (ξ, η) is an adapted vector field. Then by (3.10) together with Proposition 3.2, we obtain

$$
\begin{aligned}
\widetilde{g_{21}} = & \ \frac{1}{2\,|g_\xi|^3|g_\xi \times g_{\eta\eta}|^3}\Bigg((g_\xi \cdot g_{\eta\eta})^2\Big((g_\xi \cdot g_{\xi\xi})\,(g_\xi \cdot g_{\eta\eta}) - (g_\xi \cdot g_{\xi\eta})^2\Big) \\
& - |g_\xi|^2\,(g_\xi \cdot g_{\eta\eta})\Big((g_\xi \cdot g_{\eta\eta})\,(g_{\xi\xi} \cdot g_{\eta\eta}) - 2(g_\xi \cdot g_{\xi\eta})\,(g_{\xi\eta} \cdot g_{\eta\eta}) + (g_\xi \cdot g_{\xi\xi})\,|g_{\eta\eta}|^2\Big) \\
& + (g_\xi \cdot g_\xi)^2\Big((g_{\xi\xi} \cdot g_{\eta\eta})\,|g_{\eta\eta}|^2 - |g_{\eta\eta}|^2\Big)\Bigg),
\end{aligned}
\tag{3.12}
$$

$$
\widetilde{g_{31}} = \frac{1}{2|g_\xi|^2|g_\xi \times g_{\eta\eta}|^3}X, \ \ \widetilde{g_{331}} = 2^{\frac{1}{2}}\frac{\det(g_\xi, g_{\eta\eta}, g_{\xi\eta})}{|g_\xi|^{1/2}|g_\xi \times g_{\eta\eta}|^{3/2}},
$$

where

$$
X = 2\det(g_\xi, g_{\xi\eta}, g_{\eta\eta})\det\begin{pmatrix} |g_\xi|^2 & g_\xi \cdot g_{\xi\eta} \\ g_{\eta\eta} \cdot g_\xi & g_{\eta\eta} \cdot g_{\xi\eta} \end{pmatrix} - |g_\xi \times g_{\eta\eta}|^2 \det(g_\xi, g_{\xi\xi}, g_{\eta\eta}).
$$

evaluated at $(u, v) = (0, 0)$ does not depend on the choice of adapted vector fields (ξ, η). If g is written by the right-hand side of (3.6), then it holds that

$$
\widetilde{g_{21}} = g_{21}(0), \ \widetilde{g_{31}} = g_{31}(0), \ \widetilde{g_{331}} = g_{331}(0).
$$

For other coefficients, we have the following theorem.

Theorem 3.10. *Let $g : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a map-germ. Let (ξ, η) be an adapted vecotr field. We assume $g_\xi \times g_{\eta\eta} \neq 0$. The value*

$$
\widetilde{g_{32\eta}} = \frac{2^{\frac{1}{2}}}{3|g_\xi||g_{\eta\eta}|^{\frac{5}{2}}}\left(\det(g_\xi, g_{\eta\eta}, g_{\eta\eta\eta}) - \frac{3\det(g_\xi, g_{\eta\eta}, g_{\xi\eta})(g_\xi \cdot g_{\eta\eta})}{|g_\xi|^2}\right)
\tag{3.13}
$$

evaluated at $(u,v)=(0,0)$ *does not depend on the choice of adapted vector fields* (ξ,η)*. The value*

$$\begin{aligned}\widetilde{g_{32\xi}} = {} & \frac{\det(g_\xi, g_{\eta\eta}, g_{\xi\eta})}{|g_{\eta\eta}|^2|g_\xi|^{\frac{8}{3}}}\Bigg(\frac{|g_\xi|^{\frac{2}{3}}\det(g_\xi, g_{\eta\eta}, g_{\xi\eta\eta})}{\det(g_\xi, g_{\eta\eta}, g_{\xi\eta})} - \frac{2g_\xi\cdot g_{\xi\eta}}{|g_\xi|^{\frac{4}{3}}} \\ & + \frac{|g_\xi|^{\frac{2}{3}}\det(g_\xi, g_{\xi\eta}, g_{\eta\eta\eta})}{3\det(g_\xi, g_{\eta\eta}, g_{\xi\eta})} - \frac{2|g_\xi|^{\frac{2}{3}}\det(g_\xi, g_{\eta\eta}, g_{\xi\xi})\det(g_\xi, g_{\eta\eta}, g_{\eta\eta\eta})}{3\det(g_\xi, g_{\eta\eta}, g_{\xi\eta})^2} \\ & + \frac{2\det(g_\xi, g_{\eta\eta}, g_{\eta\eta\eta})}{3|g_\xi|^{\frac{4}{3}}|g_{\eta\eta}|^2\det(g_\xi, g_{\eta\eta}, g_{\xi\eta})^2}X\Bigg)\end{aligned} \tag{3.14}$$

evaluated at $(u,v)=(0,0)$ *does not depend on the choice of strongly adapted vector fields* (ξ,η)*. The value*

$$\begin{aligned}\widetilde{g_{331\xi}} = {} & \frac{2^{\frac{3}{2}}\det(g_\xi, g_{\eta\eta}, g_{\xi\eta})}{|g_{\eta\eta}|^{\frac{3}{2}}|g_\xi|^3}\Bigg(\frac{\det(g_\xi, g_{\eta\eta}, g_{\xi\xi\eta})}{\det(g_\xi, g_{\eta\eta}, g_{\xi\eta})} - \frac{\det(g_\xi, g_{\eta\eta}, g_{\xi\xi})\det(g_\xi, g_{\xi\eta}, g_{\eta\eta\eta})}{2\det(g_\xi, g_{\eta\eta}, g_{\xi\eta})^2} \\ & - \frac{3\det(g_\xi, g_{\eta\eta}, g_{\xi\xi})\det(g_\xi, g_{\eta\eta}, g_{\xi\eta\eta})}{2\det(g_\xi, g_{\eta\eta}, g_{\xi\eta})^2} + \frac{(g_\xi\cdot g_{\xi\eta})\det(g_\xi, g_{\eta\eta}, g_{\xi\xi})}{|g_\xi|^2\det(g_\xi, g_{\eta\eta}, g_{\xi\eta})} \\ & + \frac{\det(g_\xi, g_{\eta\eta}, g_{\eta\eta\eta})\det(g_\xi, g_{\eta\eta}, g_{\xi\xi})^2}{2\det(g_\xi, g_{\eta\eta}, g_{\xi\eta})^3} + \frac{\det(g_\xi, g_{\xi\eta}, g_{\xi\eta\eta})}{\det(g_\xi, g_{\eta\eta}, g_{\xi\eta})} - \frac{g_\xi\cdot g_{\xi\xi}}{|g_\xi|^2} \\ & - \frac{\det(g_\xi, g_{\eta\eta}, g_{\eta\eta\eta})}{2|g_\xi|^4|g_{\eta\eta}|^4\det(g_\xi, g_{\eta\eta}, g_{\xi\eta})^3}X^2 \\ & + \frac{3}{|g_\xi|^{\frac{8}{3}}|g_{\eta\eta}|^2\det(g_\xi, g_{\eta\eta}, g_{\xi\eta})}X\Bigg(\frac{|g_\xi|^{\frac{2}{3}}\det(g_\xi, g_{\eta\eta}, g_{\xi\eta\eta})}{\det(g_\xi, g_{\eta\eta}, g_{\xi\eta})} - \frac{2g_\xi\cdot g_{\xi\eta}}{|g_\xi|^{\frac{4}{3}}} \\ & + \frac{|g_\xi|^{\frac{2}{3}}\det(g_\xi, g_{\xi\eta}, g_{\eta\eta\eta})}{3\det(g_\xi, g_{\eta\eta}, g_{\xi\eta})} - \frac{2|g_\xi|^{\frac{2}{3}}\det(g_\xi, g_{\eta\eta}, g_{\xi\xi})\det(g_\xi, g_{\eta\eta}, g_{\eta\eta\eta})}{3\det(g_\xi, g_{\eta\eta}, g_{\xi\eta})^2} \\ & + \frac{2\det(g_\xi, g_{\eta\eta}, g_{\eta\eta\eta})}{3|g_\xi|^{\frac{4}{3}}|g_{\eta\eta}|^2\det(g_\xi, g_{\eta\eta}, g_{\xi\eta})^2}X\Bigg)\Bigg)\end{aligned} \tag{3.15}$$

evaluated at $(u,v)=(0,0)$ *does not depend on the choice of most strongly adapted vector fields* (ξ,η)*.*

Here, g_ξ *denotes the directional derivative of* g *with respect to the vector field* ξ*. We note that* $g_{\xi_1\xi_2}=\xi_2\xi_1 g$*, for example for the directional derivative of vector fields* ξ_1 *and* ξ_2*.*

If g is written by the right-hand side of (3.6), then it holds that

$$\widetilde{g_{32\eta}} = (g_{32})_\eta(0,0),\ \widetilde{g_{32\xi}} = (g_{32})_\xi(0,0),\ \widetilde{g_{331\xi}} = (g_{331})_\xi(0).$$

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