On the open cell property in weakly o-minimal structures with the strong cell decomposition property

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Abstract

Under suitable conditions, o-minimal structures satisfy the open cell property. In this paper, we present results obtained concerning the open cell property in weakly o-minimal structures with the strong cell decomposition property.

Throughout this paper, "definable" means "definable possibly with parameters" and we assume that a structure $\mathcal{M} = (M, <, \dots)$ is a dense linear ordering $<$ without endpoints.

A subset A of M is said to be *convex* if $a, b \in A$ and $c \in M$ with $a < c < b$ then $c \in A$. Moreover, if $A = \emptyset$ or $\inf A, \sup A \in M \cup \{-\infty, +\infty\}$, then A is called an *interval* in M. We say that $\mathcal{M}$ is *o-minimal* (*weakly o-minimal*) if every definable subset of M is a finite union of points and intervals (definable convex sets), respectively. A theory T is said to be *weakly o-minimal* if every

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model of T is weakly o-minimal. The reader is assumed to be familiar with the fundamental results of o-minimality and weak o-minimality; see, for example, [2, 3, 8, 9].

For any subsets C, D of M, we write $C < D$ if $c < d$ whenever $c \in C$ and $d \in D$. A pair $\langle C, D \rangle$ of non-empty subsets of M is called a *cut* in M if $C < D, C \cup D = M$ and D has no lowest element. A cut $\langle C, D \rangle$ is said to be *definable* in $\mathcal{M}$ if the sets C, D are definable in $\mathcal{M}$. The set of all cuts definable in $\mathcal{M}$ will be denoted by $\overline{M}$. Note that we have $M = \overline{M}$ if $\mathcal{M}$ is o-minimal. We define a linear ordering on $\overline{M}$ by $\langle C_1, D_1 \rangle < \langle C_2, D_2 \rangle$ if and only if $C_1 \subsetneq C_2$. Then we may treat $(M, <)$ as a substructure of $(\overline{M}, <)$ by identifying an element $a \in M$ with the definable cut $\langle (-\infty, a], (a, +\infty) \rangle$.

We equip M ($\overline{M}$) with the *interval topology* (the open intervals form a base), and each product M^n ($\overline{M}^n$) with the corresponding product topology, respectively.

We recall the notion of definable functions from [9]. Let n be a positive integer and $A \subseteq M^n$ definable. A function $f : A \to \overline{M}$ is said to be *definable* if the set $\{\langle x, y \rangle \in M^{n+1} : x \in A,\ y < f(x)\}$ is definable. A function $f : A \to \overline{M} \cup \{-\infty, +\infty\}$ is said to be *definable* if f is a definable function from A to $\overline{M}$, $f(x) = -\infty$ for all $x \in A$, or $f(x) = +\infty$ for all $x \in A$.

We also recall the notion of strong cells from [9].

Definition 1. For every $m \in \mathbb{N}_+$, we define, by induction, strong cells $C \subseteq M^m$ and their completions $\overline{C} \subseteq \overline{M}^m$.

(1) Any singleton of M is a 0-strong cell in M and is equal to its completion.

(2) Any non-empty open convex definable subset of M is a 1-strong cell in M. If $C \subseteq M$ is a 1-strong cell, then we define its completion by $\overline{C} := \{x \in \overline{M} : (\exists a, b \in C)(a < x < b)\}$.

Let $m \in \mathbb{N}_+$, $k \leq m$ and suppose that we have already defined k-strong cells in M^m and their completions in $\overline{M}^m$.

(3) If $C \subseteq M^m$ is a k-strong cell and $f : C \to M$ is a continuous definable function which has a unique continuous extension $\overline{f} : \overline{C} \to \overline{M}$, then the graph of f, denoted by $\Gamma(f)_C$, is a k-strong cell in M^{m+1} and its completion in $\overline{M}^{m+1}$ is defined as $\Gamma(\overline{f})_{\overline{C}}$.

(4) If $C \subseteq M^m$ is a k-strong cell and $f, g : C \to \overline{M} \cup \{\pm\infty\}$ is continuous definable functions such that f, g have continuous extensions $\overline{f}, \overline{g} : \overline{C} \to \overline{M} \cup \{\pm\infty\}$, where $\overline{f}(\overline{a}) < \overline{g}(\overline{a})$ for any $\overline{a} \in \overline{C}$, then the set $(f, g)_C := \{\langle \overline{a}, b\rangle \in C \times M : f(\overline{a}) < b < g(\overline{a})\}$ is called a $(k+1)$-strong cell in M^{m+1}. The completion of $(f, g)_C$ in $\overline{M}^{m+1}$ is defined as $(\overline{f}, \overline{g})_{\overline{C}} := \{\langle \overline{a}, b\rangle \in \overline{C} \times \overline{M} : \overline{f}(\overline{a}) < b < \overline{g}(\overline{a})\}$.

(5) Every k-strong cell is of the form given in (1) through (4). We say that $C \subseteq M^m$ is a strong cell in M^m if there exists a non-negative integer k such that C is a k-strong cell in M^m.

The notion of strong cells in weakly o-minimal settings is consistent with that of cells in o-minimal settings.

A function $f : C \to \overline{M}$ is said to be a *cell-defining function* if $\Gamma(f)_C$ is a strong cell or there is a definable function $g : C \to \overline{M}$ so that $(f, g)_C$ or $(g, h)_C$ is a strong cell.

A strong cell $C \subseteq M^m$, $m \geq 2$, is called a *refined strong cell* if each of the cell-defining functions appearing in its definition assumes all its values in one of the following sets: M, $\overline{M} \setminus M$, $\{-\infty\}$, or $\{+\infty\}$. Refined strong cells in M coincide with strong cells in M. Note that the definitions of strong cells in [9] and [10] are not identical. A strong cell in [10] is called a refined strong cell in [9]. In this paper, we follow the terminology of [9].

Let $C \subseteq M^m$ be a strong cell and $f : C \to \overline{M}$ definable. The function f is said to be *strongly continuous* if it has a unique continuous extension $\overline{f} : \overline{C} \to \overline{M}$. A function which is identically equal to $-\infty$ or $+\infty$ and whose domain is a strong cell is also said to be *strongly continuous*.

Definition 2. Let $\mathcal{M} = (M, <, \ldots)$ be a weakly o-minimal structure. For each positive integer n, we inductively define a *strong cell decomposition* (or a *decomposition into strong cells*) in M^n of a non-empty definable set $A \subseteq M^n$.

(1) If $A \subseteq M$ is a non-empty definable set and $\mathcal{D} = \{C_1, \ldots, C_k\}$ is a partition of A into strong cells in M, then $\mathcal{D}$ is called a *decomposition of A into strong cells* in M.

(2) Suppose that $A \subseteq M^{n+1}$ is a non-empty definable set and $\mathcal{D} = \{C_1, \dots, C_k\}$ is a partition of A into strong cells in M^{n+1}. Then $\mathcal{D}$ is called a *decomposition of A into strong cells* in M^{n+1} if $\{\pi(C_1), \dots, \pi(C_k)\}$ is a decomposition of $\pi(A)$ into strong cells in M^n, where $\pi : M^{n+1} \to M^n$ is the projection on the first n coordinates.

Definition 3. Let $\mathcal{M} = (M, <, \dots)$ be a weakly o-minimal structure and n a positive integer. Suppose that $A, B \subseteq M^n$ are definable sets, $A \neq \emptyset$ and $\mathcal{D}$ is a decomposition of A into strong cells in M^n. We say that $\mathcal{D}$ *partitions* B if for each strong cell $C \in \mathcal{D}$, we have either $C \subseteq B$ or $C \cap B = \emptyset$.

Definition 4. A weakly o-minimal structure $\mathcal{M} = (M, <, \dots)$ is said to have the *strong cell decomposition property* if for any positive integers k, n and any definable sets $A_1, \dots, A_k \subseteq M^n$, there exists a decomposition of M^n into strong cells partitioning each of the sets $A_1, \dots, A_k$.

For any $n \in \mathbb{N}_+$ and any refined strong cell $C \subseteq M^n$, we denote by R_C an n-ary relational symbol. We interpret R_C in $\overline{M}^n$ as $\overline{C}$, the completion of C. According to Section 2 of [10], the structure $\overline{\mathcal{M}} := (\overline{M}, <, (R_C : C \text{ is a refined strong cell}))$ is o-minimal and is called the canonical o-minimal extension of $\mathcal{M}$. If $X \subseteq \overline{M}^m$ is a set definable in $\overline{\mathcal{M}}$, then $X \cap M^m$ is definable in $\mathcal{M}$. If additionally $Y \subseteq M^m$ is definable in $\mathcal{M}$, then $X \cap Y$ is definable in $\mathcal{M}$.

We next recall the notion of open cell property from [1]. An o-minimal structure $\mathcal{M} = (M, <, \dots)$ is said to have the *open cell property* if every non-empty definable open subset of M^n is a union of finitely many open cells. A weakly o-minimal structure $\mathcal{M} = (M, <, \dots)$ is said to have the *open cell property* if every non-empty definable open subset of M^n is a union of finitely many open strong cells.

In o-minimal settings, the following results hold.

Theorem 5 ([11, Theorem 1.3])**.** *Let $\mathcal{M} = (M, <, +, \cdot, \dots)$ be an o-minimal expansion of a real closed field. Then any definable, bounded open subset of M^n is a union of finitely many open cells.*

Remark 6. Wilkie's aforementioned result requires boundedness. For example, suppose that $\mathcal{M} = (M, <, +, \cdot, \dots)$ is an o-minimal expansion of a real closed field. Let $C_1 = \{\langle x, y\rangle \in M^2 : x > 0, y < 1/x\}$, $C_2 = \{\langle x, y\rangle \in M^2 : x < 0, y < -1/x\}$, and $C_3 = \{\langle x, y\rangle \in M^2 : x = 0\}$. Then $C_1 \cup C_2 \cup C_3$ is a definable open subset of M^2. However C_3 is not open in M^2.

Theorem 7 ([1, Theorem 2]). *Let $\mathcal{M} = (M, <, \dots)$ be an o-minimal structure. If $\mathcal{M}$ admits CE-cell decomposition property, then $\mathcal{M}$ has the open cell property.*

Let $\mathcal{M} = (M, <, +, \cdots)$ be an o-minimal expansion of an ordered group. We say that $\mathcal{M}$ is *semi-bounded* if there exists no definable bijection between a bounded interval and an unbounded interval [4].

Theorem 8 ([5, Theorem 1.1]). *Let $\mathcal{M} = (M, <, +, \dots)$ be an o-minimal expansion of an ordered group. If $\mathcal{M}$ is semi-bounded, then $\mathcal{M}$ has the open cell property.*

Let $\mathcal{M} = (M, <, +, \dots)$ be a weakly o-minimal expansion of an ordered group $(M, <, +)$. Then, the weakly o-minimal structure $\mathcal{M}$ is said to be *non-valuational* if for any definable cut $\langle C, D\rangle$ we have $\inf\{d - c : c \in C,\ d \in D\} = 0$.

Fact 9 ([9, Corollary 2.16]). *Let $\mathcal{M} = (M, <, +, \dots)$ be a weakly o-minimal expansion of an ordered group $(M, <, +)$. Then the following conditions are equivalent.*

(1) $\mathcal{M}$ is non-valuational.

(2) $\mathcal{M}$ has the strong cell decomposition property.

In [7], we prove the following results.

Theorem 10 ([7]). *Suppose that $\mathcal{M} = (M, <, +, \cdot, \dots)$ is a non-valuational weakly o-minimal expansion of a real closed field. Let U be a definable bounded open subset of M^n. Then, there exists a finite collection of open strong cells in M^n whose union is U.*

Theorem 11 ([7]). *Consider a non-valuational weakly o-minimal expansion* $\mathcal{M} = (M, <, +, \dots)$ *of an ordered group. Let* U *be a definable open subset of* M^n *and assume that* $\overline{\mathcal{M}}$ *is semi-bounded. Then there exist finitely many open strong cells whose union is* U*.*

We obtained an improvement of Theorem 11 inspired by a private communication with Fujita [6].

Theorem 12 ([7]). *Let* $\mathcal{M} = (M, <, +, \dots)$ *be a non-valuational weakly o-minimal expansion of an ordered group. Then the following three conditions are equivalent.*

(1) For any bounded definable set $I \subseteq M$ *and for every* $\mathcal{M}$*-definable function* $f : I \to \overline{M}$, $f(I)$ *is bounded.*

(2) The canonical o-minimal extension $\overline{\mathcal{M}}$ *of* $\mathcal{M}$ *is semi-bounded.*

(3) Any $\mathcal{M}$*-definable open set is a finite union of open strong cells.*

References

[1] S. Andrews, Definable open sets as finite unions of definable open cells, Notre Dame J. Form. Log. **51** (2010), no. 2, 247-251.

[2] M. Coste, An introduction to o-minimal geometry, Dottorato di Ricerca in Matematica, Dip. Mat. Univ. Pisa, Istituti Editoriali e Poligrafici Internazionali (2000).

[3] L. van den Dries, Tame topology and o-minimal structures, London Mathematical Society Lecture Notes Series, 248, Cambridge University Press, Cambridge, (1998).

[4] M. Edmundo, Structure theorems for o-minimal expansions of groups, Ann. Pure Appl. Logic **102** (2000), no. 1-2, 159-181.

[5] M. Edmundo, P. Eleftheriou and L. Prelli, Coverings by open cells, Arch. Math. Logic **53** (2014), 307-325.

[6] M. Fujita, Private communication, (2025).

[7] T. Kawakami and H. Tanaka, Open cell property in weakly o-minimal structures, preprint (2026).

[8] D. Macpherson, D. Marker and C. Steinhorn, Weakly o-minimal structures and real closed fields, Trans. Amer. Math. Soc. **352** (2000), 5435-5483.

[9] R. Wencel, Weakly o-minimal non-valuational structures, Ann. Pure Appl. Logic **154** (2008), 139-162.

[10] R. Wencel, On the strong cell decomposition property for weakly o-minimal structures. Math. Log. Quart. **59** (2013), 452-470.

[11] A. Wilkie, Covering open definable sets by open cells, In: M. Edmundo, D. Richardson, A. J. Wilkie (eds.) O-minimal Structures, Proceedings of the RAAG Summer School Lisbon 2003, Lecture Notes in Real Algebraic and Analytic Geometry, Cuvillier Verlag, (2005).