# On distal expansions

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Typically, NIP structures consist of stable parts and linear orders. Distality is a stability-theoretic property introduced by Simon [18] to characterize NIP theories that lack nontrivial stable parts. Beyond providing a finer classification of first-order theories, it implies good combinatorial properties of definable graphs, as demonstrated in [3] and [2].

Several authors have investigated which expansions of distal structures remain distal. Hieronymi and Nell [9] provides examples of both distal and non-distal expansions of divisible ordered abelian groups. For instance, the expansion of the real field by $2^{\mathbb{Z}} = \{2^z \mid z \in \mathbb{Z}\}$ is distal, while the expansion by $2^{\mathbb{Q}}$ is not distal. Tong's recent result [20] states that the expansion of the ordered group of integers by a congruence-periodic sparse sequence is distal.

This note announces some results on this theme, given in [14]. We consider expansions of distal structures by a unary subset that appears as the image of a projection map. Our first result provides a sufficient condition for such an expansion to remain distal. Based on this criterion, we proved that $(\mathbb{Z}; <, +, R)$ is distal for an almost sparse sequence $R$. Our proof generalizes Tong's result by omitting the assumption of congruence periodicity for $R$, Furthermore, we demonstrates that $(\mathbb{Q}_p; +, \cdot, p^{\mathbb{Z}})$ and $(\mathbb{Q}_p; +, \cdot, p^{\mathbb{Z}}, p^R)$ are distal. The latter one provides an example of NIP expansions of the $p$-adic field without the rationality of the Poincaré series.

First, we describe our distality criterion. Fix a complete theory $T$ with infinite models, and let $I$ and $J$ be sufficiently saturated dense linear orders without endpoints. We denote their concatenation by $I + J$ (i.e., $i < j$ for all $i \in I$ and $j \in J$). We take elements from a monster model of $T$.

**Definition 1.** $T$ is *distal* if the following holds: For any sequence $(\overline{a_k})_{k\in I+(c)+J}$ and any singleton $b$, if:

- $(\overline{a_k})_{k\in I+(c)+J}$ is indiscernible,
- $(\overline{a_k})_{k\in I+J}$ is indiscernible over $b$,

then $(\overline{a_k})_{k\in I+(c)+J}$ is indiscernible over $b$. A structure is distal if its theory is distal.

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Distal theories have NIP ([8]).

We describe our sufficient condition of distality for an expansion $\widetilde{\mathcal{N}} = (\mathcal{N}, R)$ of a structure $\mathcal{N}$, where $R$ is a unary subset of $N$. Let $\mathcal{R}$ be a structure on $R$ which is $\emptyset$-definable in $\widetilde{\mathcal{N}}$. Let $\mathcal{F}_N$ (resp. $\mathcal{F}_R$) be a set of functions that are $\emptyset$-definable in $\mathcal{N}$ (resp. $\mathcal{R}$). Let $\lambda : N \to R$ be a function $\emptyset$-definable in $\widetilde{\mathcal{N}}$. We extend each $h \in \mathcal{F}_R$ to a function in $\mathcal{N}$ by defining $h(\overline{x}) = h(\lambda(x_1), \dots, \lambda(x_n))$. We set $\mathcal{F} = \mathcal{F}_N \cup \mathcal{F}_R \cup \{\lambda\}$. We assume that $\widetilde{\mathcal{N}}$ is a monster model, as distality is a property of theories rather than structures.

Informally, $\widetilde{\mathcal{N}}$ *admits variable separation* if, for any sequence $(\overline{a_k})_{k \in I+(c)+J}$ and any $b$ satisfying the assumptions in Definition 1, and for any $\mathcal{F}$-term $t(\overline{x}, y)$, $t(\overline{a_k}, b)$ can be expressed as $u(\overline{r}(\overline{a_k}), \overline{s}(b))$ for any $k \in I' + (c) + J'$ by shrinking $I$ and $J$, where $u(\overline{z}, \overline{w})$ is an $\mathcal{F}_N$-term and $\overline{r}(\overline{x}) = (r_1(\overline{x}), \dots, r_m(\overline{x}))$ and $\overline{s}(y) = (s_1(y), \dots, s_n(y))$ are $\mathcal{F}$-terms. We also impose $\overline{s}(b) \in R$ if $t$ is of the form $h(\overline{t})$ for some $h \in \mathcal{F}_R \cup \{\lambda\}$.

**Theorem 2.** *$\widetilde{\mathcal{N}}$ is distal if the following conditions hold:*

1. *$\mathcal{N}$ and $\mathcal{R}$ are distal.*
2. *For any $M \subseteq N$ with $\langle M \rangle_{\mathcal{F}} = M$ and any $b, b' \in R$, if $\mathrm{tp}^{\mathcal{R}}(b/M \cap R) = \mathrm{tp}^{\mathcal{R}}(b'/M \cap R)$, then $\mathrm{tp}^{\widetilde{\mathcal{N}}}(b/M) = \mathrm{tp}^{\widetilde{\mathcal{N}}}(b'/M)$.*
3. *For any $M \subseteq N$ with $\langle M \rangle_{\mathcal{F}} = M$ and any $b, b' \in N$, if $\mathrm{tp}^{\mathcal{N}}(b/M) = \mathrm{tp}^{\mathcal{N}}(b'/M)$ and $\lambda(x) \in M$ for any $x \in \langle bM \rangle_{\mathcal{F}_N}$, then $\mathrm{tp}^{\widetilde{\mathcal{N}}}(b/M) = \mathrm{tp}^{\widetilde{\mathcal{N}}}(b'/M)$.*
4. *$\widetilde{\mathcal{N}}$ admits variable separation.*

We apply this criterion to an expansion of Presburger arithmetic $(\mathbb{Z}; <, +)$ by an almost sparse sequence. Let $R = (r_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence of integers. Let $S$ and $S^{-1}$ be the successor and predecessor functions on $R$.

**Definition 3.** An *operator* on $R$ is a term of the form $A(x) = \sum_{i=-n}^{m} z_i S^i(x)$, where $z_i \in \mathbb{Z}$. For operators $A(x)$ and $B(x)$, we write $A >_{ae} B$ if $A(a) > B(a)$ for all but finitely many $a \in R$. The relations $A <_{ae} B$ and $A =_{ae} B$ are defined similarly.

**Definition 4.** $R$ is *almost sparse* if the following conditions hold:

1. For any operators $A(x)$ and $B(x)$, one of $A >_{ae} B$, $A <_{ae} B$, or $A =_{ae} B$ holds.
2. For any operators $A(x)$ and $B(x)$ such that $A >_{ae} B$, there exists an integer $\Delta \geq 0$ such that $A(S^{\Delta}(x)) >_{ae} B(S^{\Delta}(x)) + x$.

Examples of almost sparse sequences include $(2^n)_{n \in \mathbb{N}}$, $(n!)_{n \in \mathbb{N}}$, and the Fibonatti sequence. We henthforth assume that $R$ is almost sparse. Following Semënov [16], Lambotte and Point [15, 11, 10] established important results on the structure $(\mathbb{Z}; <, +, R)$, such as quantifier elimination and NIP. Tong

[20] showed that this structure is distal under the assumption of congruence periodicity of $R$. We omit this additional assumption.

We define the function $\lambda$ on $\mathbb{Z}$ as follows: $\lambda(x) = \max\{r \in R \mid r \leq x\}$ if $x \geq r_0$ and $\lambda(x) = r_0$ otherwise. Let $\mathcal{N} = (\mathbb{Z}; <, +)$, and let $\mathcal{R} = (R; <, (\cdot \equiv_n i)_{n,i})$, where $\cdot \equiv_n i$ is a unary predicate for elements of residue $i$ modulo $n$. Also, we set $\mathcal{F}_N = \{+, -\}$ and $\mathcal{F}_R = \{S, S^{-1}\}$. The distality of $(\mathbb{Z}; <, +)$ is well-known [1]. Furthermore, $\mathcal{R} = (R^{\mathcal{U}}; <, (\cdot \equiv_n i)_{n,i})$ is a so-called colored order, and hence distal [17, 18]. Conditions (2) and (3) of Theorem 2 are shown by back-and-forth argument. Variable separation is given as follows: For a a $\{+, -, \lambda, S, S^{-1}\}$-term $t(\overline{x}, y)$ , there exist $\{+, -, \lambda, S, S^{-1}\}$-terms $u(\overline{x})$ and $r(y)$ such that $t(\overline{a_k}, b) = u(\overline{a_k}) + r(b)$. Therefore, we may apply Theorem 2 to conclude that $(\mathbb{Z}; <, +, R)$ is distal.

We then discuss expansions of the $p$-adic field. Let $p^{\mathbb{Z}} = \{p^z \mid z \in \mathbb{Z}\}$ and $p^R = \{p^r \mid r \in R\}$. Mariaule [12, 13] studied the structure $(\mathbb{Q}_p; +, \cdot, p^{\mathbb{Z}})$ and provided quantifier elimination and NIP result.

**Definition 5.** We define the map $\pi : \mathbb{Q}_p \to p^{\mathbb{Z}}$ as follows: $\pi(a) = p^{v_p(a)}$ if $a \neq 0$, and $\pi(0) = 1$. We write $a <_v b$ for $a, b \in p^{\mathbb{Z}}$ if $v_p(a) < v_p(b)$. We define $\lambda$, $S$, and $S^{-1}$ on $p^{\mathbb{Z}}$ such that $(p^{\mathbb{Z}}; <_v, \cdot, \lambda, S, S^{-1})$ is isomorphic to $(\mathbb{Z}; <, +, \lambda, S, S^{-1})$.

To prove the distality of $(\mathbb{Q}_p; +, \cdot, p^{\mathbb{Z}})$, we set $\mathcal{N} = (\mathbb{Q}_p; +, \cdot)$, $\mathcal{R} = (p^{\mathbb{Z}}; <_v, \cdot)$, $\lambda = \pi$, $\mathcal{F}_N = \{+, -, \cdot, ^{-1}\}$, and $\mathcal{F}_R = \emptyset$. Conditions (2) and (3) of Theorem 2 for this structure is derived from quantifier elimination. For $(\mathbb{Q}_p; +, \cdot, p^{\mathbb{Z}}, p^R)$, we set $\mathcal{N} = (\mathbb{Q}_p; +, \cdot, p^{\mathbb{Z}})$, $\mathcal{R} = (p^R; <_v, (\cdot \equiv_n i)_{n,i})$, $\mathcal{F}_N = \{+, -, \cdot, ^{-1}, \pi\}$, and $\mathcal{F}_R = \{S, S^{-1}\}$. These two conditions are shown by back-and-forth argument. Variable separation for $(\mathbb{Q}_p; +, \cdot, p^{\mathbb{Z}})$ and $(\mathbb{Q}_p; +, \cdot, p^{\mathbb{Z}}, p^R)$ is as follows: For any term $t(\overline{x}, y)$, there exist terms $\alpha_i(\overline{x})$, $\beta_i(\overline{x})$, $r_i(y)$, $s_i(y)$ $(1 \leq i \leq m)$ such that:

$$t(\overline{a_k}, b) = \frac{\sum_i \alpha_i(\overline{a_k}) r_i(b)}{\sum_i \beta_i(\overline{a_k}) s_i(b)}.$$

The $p$-adic field $(\mathbb{Q}_p; +, \cdot)$ is distal [7, 18], which allows us to apply Theorem 2 to conclude that $(\mathbb{Q}_p; +, \cdot, p^{\mathbb{Z}})$ and $(\mathbb{Q}_p; +, \cdot, p^{\mathbb{Z}}, p^R)$ are distal. The latter expansion provides an example of an NIP expansion of the $p$-adic field without the rationality of the Poincaré series.

**Definition 6.** For a subset $A$ of $\mathbb{Z}_p^n$ and $m \geq 0$, let $N_m$ be the cardinality of the set $\{(x_1 \bmod p^m, \ldots, x_n \bmod p^m) \in (\mathbb{Z}/p^m\mathbb{Z})^n \mid \overline{x} \in A\}$. The *Poincaré series* of $A$ is the formal power series $P_A(T) = \sum_{m=0}^{\infty} N_m T^m$.

Denef [5] proved that the Poincaré series of any set $A \subseteq \mathbb{Z}_p^n$ definable in the pure $p$-adic field $(\mathbb{Q}_p; +, \cdot)$ is a rational function. This rationality extends to any dp-minimal extension of the $p$-adic field by combining Simon and Walsberg [19] (dp-minimal $\Rightarrow$ P-minimal) and Kovacsics and Leenknegt [4] (P-minimal $\Rightarrow$ rationality). While Denef [6] showed that $(\mathbb{Q}_p; +, \cdot, p^{\mathbb{Z}})$ also admits rationality, this does not hold for all NIP expansions.

**Proposition 7.** *The Poincaré series of* $p^R$ *is not a rational function.*

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