Hrushovski Construction and Chromatic Numbers

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1 Introduction

This is a note related to a joint work with Akito Tsuboi and Koitaro Nakaura [3]. We have shown that any generic graphs obtained by Hrushovski predimension construction have a finite chromatic number. But depending on the coefficient of the predimension function, the generic graph can have arbitrarily large chromatic number. On the other hand, assuming that an infinite graph G has a stable theory with U-rank one and the chromatic number of G is infinite then there is an infinite clique in some elementary extension of G. With the fact that there are generic graphs with arbitrarily large finite chromatic numbers obtained by Hrushovski construction, we can construct an infinite graph with an infinite chromatic number such that there are no infinite cliques in any elementary extensions of it.

Also, we will discuss about chromatic numbers of "projective planes" constructed by Baldwin using Hrushovski construction.

2 Preliminaries

Definition 1. Let G be a (simple) graph. A map $G \to C$ is called a *coloring* of G if whenever $p, q \in G$ are adjacent then $f(p) \neq f(q)$.

The *chromatic number* of G, written $\chi(G)$, is a smallest cardinal number κ such that there is a coloring $f : G \to C$ with $|C| = \kappa$.

Let $L = \{R\}$ where R is intended to represent an edge relation.

Let α be a real number with $0 < \alpha < 1$.

We assume that each R is symmetric and irreflexive on any structure we consider.

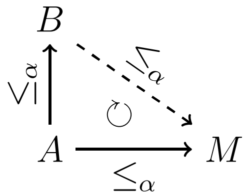
For a finite L-structure A, let $R(A) = \{u \in [A]^2 \mid A \models R(u)\}$ and let $\delta_\alpha(A) = |A| - \alpha|R(A)|$. $A \leq_\alpha B$ if $A \subseteq X \subseteq_{\mathrm{fin}} B$ implies $\delta(A) \leq \delta(X)$. $A <_\alpha B$ if $A \subsetneq X \subseteq_{\mathrm{fin}} B$ implies $\delta_\alpha(A) < \delta_\alpha(X)$.

Put $\mathbf{K}_\alpha = \{A : \text{finite} \mid \emptyset < A\}$.

Definition 2. Suppose $\mathbf{K} \subseteq \mathbf{K}_\alpha$.

A countable L-structure M is called a *Fraïssé-Hrushovski limit* of $(\mathbf{K}, \leq_\alpha)$ if

- $A \subseteq_{\mathrm{fin}} M \Rightarrow$ there is B such that $A \subseteq B \subseteq_{\mathrm{fin}} M$ and $B \leq_\alpha M$;
- $A \subset_{\mathrm{fin}} M \Rightarrow A \in \mathbf{K}$;
- for any A, B in $\mathbf{K}$ with $A \leq_\alpha B$ and $A \leq_\alpha M$,



A *Fraïssé-Hrushovski limit* of $(\mathbf{K}, <_\alpha)$ is defined similarly. A Fraïssé-Hrushovski limit is also called a *generic* structure of a given class.

3 Chromatic Numbers of Hrushovski-Fraïssé Limit

The following theorems hold [3].

Theorem 3. *Let M be a Fraïssé-Hrushovski limit of $(\mathbf{K}, \leq_\alpha)$ or $(\mathbf{K}, <_\alpha)$ based on $\delta(A) = |A| - \alpha R(A)$. If $k > 2/\alpha$ then $A \in \mathbf{K}$ implies $\chi(A) \leq k$. Therefore $\chi(M)$ is finite. If $\alpha > 2/3$ then $\chi(M) = 3$.*

Theorem 4. *Let M be a Fraïssé-Hrushovski limit of $(\mathbf{K}, \leq_\alpha)$ or $(\mathbf{K}, <_\alpha)$ based on $\delta(A) = |A| - \alpha R(A)$. If $k > 2/\alpha$ then $A \in \mathbf{K}$ implies $\chi(A) \leq k$. Therefore $\chi(M)$ is finite. If $\alpha > 2/3$ then $\chi(M) = 3$.*

Definition 5. A class $(K, \leq_\alpha)$ has FAP if $A \leq_\alpha B$, $A \leq_\alpha C$ then $B \oplus_A C \in K$. A class $(K, <_\alpha)$ has FAP if $A <_\alpha B$, $A <_\alpha C$ then $B \oplus_A C \in K$.

Theorem 6. *Let $n > 0$ be an integer. There is $\varepsilon > 0$ such that if $(K, \leq_\alpha)$ has FAP with $0 < \alpha < \varepsilon$ then $\chi(M) \geq n$ for the generic structure M of $(K, \leq_\alpha)$. Similar statement holds for $(K, <_\alpha)$.*

4 Graphs With an Infinite Chromatic Number

Let M be a Fraïssé-Hrushovski limit of $(\mathbf{K}, \leq_\alpha)$. Then $\mathrm{Th}(M)$ is stable. If α is an irrational number then $\mathrm{Th}(M)$ is strictly stable.

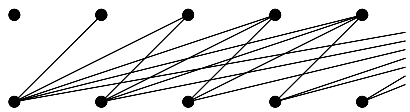
Let M' be a Fraïssé-Hrushovski limit of $(\mathbf{K}, <_\alpha)$. Then $\mathrm{Th}(M)$ is unstable if α is rational and is stable if α is irrational.

Note that if α is irrational then $A \leq_\alpha B$ if and only if $A <_\alpha B$.

There are generic structures M of $(K, <_\alpha)$ with arbitrarily large chromatic numbers. We can make M triangle-free and stable by choosing irrational numbers α and using control functions. One may hope to construct a graph with infinite chromatic number which is stable but has no infinite clique in any elementary extensions with this method. But unfortunately, we get unstable structures with this method.

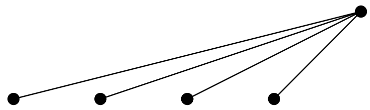
Proposition 7. *Let α_n $(n < \omega)$ be real numbers with $0 < \alpha_n < 2/n$. Suppose each $(K_n, \leq_{\alpha_n})$ is a non-trivial (i.e., there is a graph with an edge) amalgamation class with FAP. Let M_n be the Fraïssé-Hrushovski limit of $(K_n, \leq_{\alpha_n})$ for each n. Let M be a graph such that $M_n \subset M$ for each $n < \omega$. Then $Th(M)$ is unstable.*

Proof. We show that the following figure appears in some elementary extension of M.



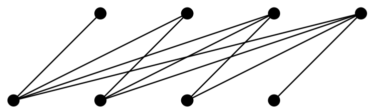
That is, there will be a sequence of two points $\{a_i, b_i\}_{i<\omega}$ such that a_i and b_j are adjacent if and only if $i < j$.

Since each K_n has FAP and non-trivial, we can freely amalgamate a graph with two points and one edge over a single point in K_n. Hence, stars of any sizes belong to K_n.



Now, for each n, there are infinitely many α_i with $0 < \alpha_i < 1/n$. For such α_i, the set of n leaves is closed in the star with $n+1$ points because $1 - n\alpha_n > 0$. Therefore we can make a free amalgams of copies of the star with $n+1$ points over the leaves.

For example, suppose $\alpha < 1/4$ with $\alpha = \alpha_k$. Then the following figures belong to K_k by FAP.



Now, suppose that $\alpha_k < 1/n$. Then in the similar way, we can see that there is a figure in K_k consists of a sequence of two points $\{a_i, b_i\}_{i<n}$ such that a_i and b_j are adjacent if and only if $i < j$. Therefore, $Th(M)$ has the order property. □

5 Baldwin's Projective Plane

J. T. Baldwin constructed countably infinite projective planes using Hrushovski construction. The class of finite graphs consists of 4-cycle-free graphs and it does not have FAP. The coefficient of the predimension function has to be 1/2. Therefore, the chromatic number has to be at most 5. But it seems that the chromatic number is 3. Since the generic structure must be a projective plane, any edge is a part of a triangle. On the other hand, there are no 4 cycles. It seems very difficult to have finite graphs with chromatic number 4 or 5 in the class. It is likely that there is a class of finite graphs with 4-cycle-free such that the chromatic number of the generic graph is 3.

References

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[2] E. Hrushovski, A stable $\aleph_0$-categorical pseudoplane, Preprint, 1988.

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