DEPENDENT DIVIDING AND INDISCERNIBILITY

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1. Introduction

In this note, we discuss results and future work related to [Tak26]. In the paper [Tak26], we explored the consequences of dependent dividing on the sub-additivity of Shelah's $\kappa^n_{inp}(T)$. In doing so, we defined the notion of a sequence of tuples being *NIP-indiscernible over* $(B; C)$, where B, C are some sets. This definition captures the idea of a sequence being indiscernible with respect to NIP formulas. In this note, we extend this definition of "indiscernible with respect to NIP formulas" to "indiscernible with respect to some collection of formulas T_0" where T_0 could be a collection of partitioned formulas that is not necessarily the collection of all NIP formulas. We then state questions and ideas related to this concept, in the context of T_0 being the collection of stable formulas in relation to stable forking. In particular, we ask if this type of idea can be used to generalize a result in stable groups to simple groups with stable forking.

2. Preliminaries

We start by introducing the key definitions of NTP_2 theories, dependent dividing, and stable forking. Note that throughout this paper, we do not distinguish between a singleton and a tuple.

Definition 2.1. *A formula $\phi(x, y)$ has TP_2 if there is an array $(b_{\alpha,i})_{\alpha,i<\omega}$ such that $\{\phi(x, b_{\alpha,i})\}_{i<\omega}$ is 2-inconsistent for every $\alpha < \omega$ and $\{\phi(x, b_{\alpha,f(\alpha)})\}_{\alpha<\omega}$ is consistent for any $f : \omega \to \omega$. Otherwise, we say that $\phi(x, y)$ is NTP_2. A theory T is NTP_2 if every formula is NTP_2.*

Note that every simple theory and NIP theory is NTP_2. The guiding principle in the exploration of NTP_2 theories is: "NTP_2 = NIP + Simple." The following definitions and conjectures exemplify this principle.

Definition 2.2. *We say that a theory T has dependent dividing if given models M, N with $M \preceq N$ and $p(x) \in S(N)$ dividing over M, then there is an NIP formula $\phi(x; y)$ and $c \in N$ such that $\phi(x, c) \in p(x)$ and $\phi(x, c)$ divides over M.*

Definition 2.3. *We say that a theory T has stable forking if given sets A, B with $A \subseteq B$ and $p(x) \in S(B)$ forking over A, then there is a stable formula $\phi(x; y)$ and $b \in B$ such that $\phi(x, b) \in p(x)$ and $\phi(x, b)$ forks over A.*

Fact 2.4 ([Che14], Proposition 4.14)**.** *If T has dependent dividing (or even just NTP_2 dividing), then T is NTP_2. If T has stable forking (or even just simple dividing), then T is simple.*

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The converses of these facts are important conjectures. The stable forking conjecture states that every simple theory has stable forking, and the dependent dividing conjecture states that every NTP_2 theory has dependent dividing.

In [Tak26], we investigated the implications of dependent dividing on Shelah's $\kappa^n_{inp}(T)$. We include the definition of $\kappa^n_{inp}(T)$, burden, and dp-rank here for completeness.

Definition 2.5. *An inp-pattern (inp stands for independent partition) in $p(x)$ of depth κ consists of $(b_{\alpha,i})_{\alpha<\kappa,i<\omega}$, $(\phi_\alpha(x,y_\alpha))_{\alpha<\kappa}$ and $k_\alpha<\omega$ such that*

- *$\{\phi_\alpha(x,b_{\alpha,i})\}_{i<\omega}$ is k_α-inconsistent, for each $\alpha<\kappa$.*
- *$\{\phi_\alpha(x,b_{\alpha,f(\alpha)})\}_{\alpha<\kappa}\cup p(x)$ is consistent, for any $f:\kappa\to\omega$.*

We define the burden of $p(x)$, denoted $bdn(p)$, to be the supremum of the depths of all inp-patterns in $p(x)$. By $bdn(a/C)$ we mean $bdn(tp(a/C))$.

We also define $\kappa^n_{inp}(T)$ to be the minimum cardinal κ such that there is no inp-pattern of depth κ in the type $\{x=x\}$, where $|x|=n$.

Note that $\kappa^n_{inp}(T)$ essentially expresses the same idea as $\mathrm{bdn}(\{x=x\})$, where $|x|=n$; for example, if $\kappa^n_{inp}(T)<\omega$, then $\kappa^n_{inp}(T)=\mathrm{bdn}(\{x=x\})+1$. The burden is a notion of a cardinal-valued rank or dimension of a type, introduced in [Adl07]. Burden is a notion of a rank that is suitable for NTP_2 theories, since a theory T is NTP_2 iff $\mathrm{bdn}(a/C)<|T|^+$ for every tuple a and set C [Che14, Lemma 3.2]. Here is a related but different notion of a rank, called the dp-rank.

Definition 2.6. *The dp-rank of a partial type $p(x)$ over C, denoted $dp(p(x))$, is the supremum of κ for which there exist $d\vDash p(x)$ and $\{I_\alpha:\alpha<\kappa\}$, mutually indiscernible sequences over C, such that for all $\alpha<\kappa$, I_α is not indiscernible over Cd. By $dp(a/C)$ we mean $dp(tp(a/C))$.*

Note that in NIP theories, the burden of a type agrees with the dp-rank of the type. The definition of dp-rank was introduced in [She14], and dp-rank is suitable for NIP theories, since a theory T is NIP iff $\mathrm{dp}(a/C)<|T|^+$ for every tuple a and set C [Sim15, Observation 4.13].

The key observation in [Tak26] is the following: under the assumption of a slightly stronger version of dependent dividing (which we call existentially NIP dividing),[1], $\kappa^n_{inp}(T)$ coincides with the value of $\kappa^n_{inp}(T)$ where witnessing formulas are NIP. This allows us to adopt the proof strategy for the sub-additivity of dp-rank in [KOU11] to prove the sub-additivity of $\kappa^n_{inp}(T)$, i.e., $\kappa^{n+m}_{inp}(T)+1\leq\kappa^n_{inp}(T)+\kappa^m_{inp}(T)$, via the use of NIP-indiscernible sequences. Now, we introduce the definition of an NIP-indiscernible sequence and, more generally, an T_0-indiscernible sequence in the following section.

3. NIP-indiscernibility and its generalizations

First, we introduce the definitions of NIP-indiscernibility and T_0-indiscernibility.

Given some sequence $I=\langle a_i:i\in\mathcal{I}\rangle$ and some sets B and C, we say that I is *NIP-indiscernible over* $(B;C)$ if for every NIP formula $\phi(x_0,\dots,x_n,y;z)$ with $|x_0|=\dots=|x_n|=|a_i|$, tuple $b\in B$ of length $|y|$, tuple $c\in C$ of length $|z|$, and $i_0<\ldots<i_n$ and $j_0<\ldots<j_n$ from $\mathcal{I}$, we have

$$\vDash\phi(a_{i_0},\dots a_{i_n},b;c)\leftrightarrow\phi(a_{j_0},\dots,a_{j_n},b;c).$$

[1] Existentially NIP dividing is the same as Definition 2.2 except $\phi(x;y)$ comes from a collection T^{opp}_{NIP} of NIP formulas that are closed under boolean combinations and existential quantifiers on the left side, i.e., if $\phi(x_1,x_2;y)\in T^{opp}_{\mathrm{NIP}}$, then $\exists x_2\phi(x_1,x_2;y)\in T^{opp}_{\mathrm{NIP}}$ (assuming T has such a collection T^{opp}_{NIP}). Then, $\kappa^n_{inp}(T)$ coincides with $\kappa^n_{inp}(T)$ where the witnessing formula is from T_{NIP}, and we use T_{NIP}-indiscernible sequence, as in Section 3, to prove the sub-additivity. This is to be updated in the paper [Tak26].

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Note that any formula $\phi(x_0, \ldots, x_n, y)$ with no parameter variables is NIP, so if I is NIP-indiscernible over $(B; C)$, then it is indiscernible over B. We say that I is *NIP-indiscernible over* C if it is NIP-indiscernible over $(\emptyset; C)$. We say that a collection of sequences $\{I_\alpha : \alpha < \kappa\}$ is *mutually NIP-indiscernible over* $(B; C)$ if each I_α is NIP-indiscernible over $(BI_{\beta_{\beta \neq \alpha}}; C)$. Similarly, a collection of sequences $\{I_\alpha : \alpha < \kappa\}$ is *mutually NIP-indiscernible over* C if each I_α is NIP-indiscernible over $(I_{\beta_{\beta \neq \alpha}}; C)$.

We can reasonably expand the definition of NIP-indiscernibility to the following. Let $T_0 \subseteq T$ be some collection of partitioned formulas that is closed under boolean combinations. Given some sequence $I = \langle a_i : i \in \mathcal{I} \rangle$ and some sets B and C, we say that I is T_0*-indiscernible over* $(B; C)$ if for every formula $\phi(x_0, \ldots, x_n, y; z)$ in T_0 with $|x_0| = \cdots = |x_n| = |a_i|$, tuple $b \in B$ of length $|y|$, tuple $c \in C$ of length $|z|$, and $i_0 < \ldots < i_n$ and $j_0 < \ldots < j_n$ from $\mathcal{I}$, we have

$$\models \phi(a_{i_0}, \ldots a_{i_n}, b; c) \leftrightarrow \phi(a_{j_0}, \ldots, a_{j_n}, b; c).$$

If every formula $\phi(x_0, \ldots, x_n, y)$ with no parameter variables is in T_0, then T_0-indiscernibility over $(B; C)$ implies indiscernibility over B. We say that I is T_0*-indiscernible over* C if it is T_0-indiscernible over $(\emptyset; C)$. We say that a collection of sequences $\{I_\alpha : \alpha < \kappa\}$ is *mutually* T_0*-indiscernible over* $(B; C)$ if each I_α is T_0-indiscernible over $(BI_{\beta_{\beta \neq \alpha}}; C)$. Similarly, a collection of sequences $\{I_\alpha : \alpha < \kappa\}$ is *mutually* T_0*-indiscernible over* C if each I_α is T_0-indiscernible over $(I_{\beta_{\beta \neq \alpha}}; C)$.

Based on this definition, we can define the following:

Definition 3.1. *Assume that* T *has a collection of NIP formulas* T_{NIP} *that is closed under boolean combinations and existential quantifiers on the right side, i.e., if* $\phi(x; y_1, y_2) \in T_{NIP}$*, then* $\exists y_2 \phi(x; y_1, y_2) \in T_{NIP}$*. Fix such a collection* T_{NIP}*.*

Let $p(x)$ *be a (partial) type over* C*. We define the NIP dp-rank of* $p(x)$*, denoted NIP-dp(*$p(x)$*), to be the supremum of* κ *for which there exist* $d \models p(x)$ *and* $\{I_\alpha : \alpha < \kappa\}$*, mutually* T_{NIP}*-indiscernible sequences over* C *such that for all* $\alpha < \kappa$*,* I_α *is not* T_{NIP}*-indiscernible over* Cd*.*

In [Tak26], we prove the sub-additivity of $\kappa^n_{inp}(T)$ under the hypothesis of existentially NIP dividing, by showing that bdn($\{x = x\}$) is equivalent to NIP-dp($\{x = x\}$) and that NIP-dp($p(x)$) is sub-additive. Based on this result, it is natural to ask whether a similar strategy can be used to prove results about burden in simple theories, under the assumption of stable forking. We explore this question in the following section.

4. Stable forking and Stable-indiscernibility

First, we introduce the definition of weight and its connection to the burden.

Definition 4.1. *Assume that* T *is simple. Given a sequence* $(b_i)_{i \in I}$ *of tuples and some set* C*, we say* $(b_i)_{i \in I}$ *is* C*-independent if* $b_i \mathop{\perp\!\!\!\perp}_C \{b_j : j \neq i\}$ *for all* $i \in I$*.*

Given some set C *and* $p(x) \in S(C)$*, define the weight of* $p(x)$ *to be the supremum over cardinals* κ *for which there is some* $B \supseteq C$*, a realization* $a \models p(x)$ *and a* B*-independent sequence* $(b_i)_{i<\kappa}$ *such that* $a \mathop{\perp\!\!\!\perp}_C B$ *and* $a \not\mathop{\perp\!\!\!\perp}_B b_i$ *for all* $i < \kappa$*.*

Fact 4.2 ([Adl07], Proposition 8)**.** *In simple theories, the burden of a partial type is the supremum of the weights of its complete extensions.*

Based on Fact 4.2 and the equivalence of bdn($\{x = x\}$) and NIP-dp($\{x = x\}$) under the assumption of existentially NIP dividing, we ask whether stable forking implies the equivalence between weight and the weight whose witnessing formulas are stable.

Question 4.3. *Assume that T has possibly a stronger version of stable forking, such as existentially stable forking. How is the burden/weight related to the weight whose witnessing formulas are stable?*

As a potential application of the study of weight and burden, here is a fact about stable groups in relation to the weight and the SU-rank. We first remind the reader of the definition of SU-rank.

Definition 4.4. *Let T be a simple theory. For a type $p(x) \in S(A)$, we define $SU(p) \geq \alpha$ by recursion on α:*

1. *$SU(p) \geq 0$ for all types p,*
2. *$SU(p) \geq \beta + 1$ if p has a forking extension q with $SU(q) \geq \beta$,*
3. *$SU(p) \geq \lambda$ for a limit ordinal λ if $SU(p) \geq \beta$ for all $\beta < \lambda$.*

The $SU(p)$ is the maximum α such that $SU(p) \geq \alpha$. If there is no maximum, we say $SU(p) = \infty$.

Let $<_\infty$ denote the partial order on groups given by: $H <_\infty K$ if $H \leq K$ and $[K : H] = \infty$.

Fact 4.5 ([CP16], Theorem 1.4)**.** *If G is stable then the following are equivalent.*

1. *G is superstable of finite SU-rank.*
2. *G has finite weight and no infinite $<_\infty$-chains of definable subgroups.*
3. *G has finite weight and no infinite $<_\infty$-chains of definable normal subgroups.*

In a conversation with the author, Gabe Conant asked if an analogue Fact 4.5 is true in simple theories. An answer to Question 4.3 may help answer this question under the assumption of stable forking.

Question 4.6. *Assume T has stable forking (hence simple by Fact 2.4). Is the following equivalent?*

1. *G is supersimple of finite SU-rank.*
2. *G has finite weight and no infinite $<_\infty$-chains of definable subgroups.*
3. *G has finite weight and no infinite $<_\infty$-chains of definable normal subgroups.*

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