A CHARACTERIZATION OF TREELESS THEORIES USING COLLAPSING INDISCERNIBLES

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Abstract. This note is based on a talk given at the RIMS model theory workshop held in Kyoto in December 2025. The talk focused on two main results: a characterization of treelessness via collapsing indiscernibility, and the fact that, under the assumption of weak-treelessness, NATP and NCTP coincide.

1. Preliminary and notations

1.1. Generalized indiscernibility.

Definition 1.1. Let $\mathcal{L}$ be a language. By the *age* of an $\mathcal{L}$-structure $\mathcal{I}$ ($\mathrm{age}_{\mathcal{L}}(\mathcal{I})$), we mean the class of finitely generated $\mathcal{L}$-structures that are isomorphic to $\mathcal{L}$-substructures of $\mathcal{I}$. Sometimes we omit the subscript $\mathcal{L}$ in $\mathrm{age}_{\mathcal{L}}(\mathcal{I})$ when it is clear from the context.

Fact 1.2. *Let $\mathcal{L}' \subseteq \mathcal{L}$ be languages, $\mathcal{I}, \mathcal{J}$ be $\mathcal{L}$-structures. If $\mathrm{age}_{\mathcal{L}}(\mathcal{I}) = \mathrm{age}_{\mathcal{L}}(\mathcal{J})$ then $\mathrm{age}_{\mathcal{L}'}(\mathcal{I}') = \mathrm{age}_{\mathcal{L}'}(\mathcal{J}')$, where $\mathcal{I}' := \mathcal{I}|_{\mathcal{L}'}, \mathcal{J}' := \mathcal{J}|_{\mathcal{L}'}$.*

Proof. Suppose $X \in \mathrm{age}_{\mathcal{L}_0}(\mathcal{I}_0)$. Then there exists a finite tuple $\bar{a} \in \mathcal{I}$ such that $\langle \bar{a} \rangle_{\mathcal{L}_0} \sim_{\mathcal{L}_0} X$. Since $\langle \bar{a} \rangle_{\mathcal{L}_1} \in \mathrm{age}_{\mathcal{L}_1}(\mathcal{I}_1)$, there exist $Y \in \mathrm{age}_{\mathcal{L}_1}(\mathcal{J}_1)$ such that $\langle \bar{a} \rangle_{\mathcal{L}_1} \sim_{\mathcal{L}_1} Y$. There exists $\bar{b} \in \mathcal{J}$ such that $Y \sim_{\mathcal{L}_1} \langle \bar{b} \rangle_{\mathcal{L}_1}$. Thus $X \sim_{\mathcal{L}_0} \langle \bar{a} \rangle_{\mathcal{L}_0} \sim_{\mathcal{L}_0} \langle \bar{a} \rangle_{\mathcal{L}_1}|_{\mathcal{L}_0} \sim_{\mathcal{L}_0} Y|_{\mathcal{L}_0} \sim_{\mathcal{L}_0} \langle \bar{b} \rangle_{\mathcal{L}_1}|_{\mathcal{L}_0} \sim_{\mathcal{L}_0} \langle \bar{b} \rangle_{\mathcal{L}_0} \in \mathrm{age}_{\mathcal{L}_0}(\mathcal{J}_0)$. Thus $X \in \mathrm{age}_{\mathcal{L}_0}(\mathcal{J}_0)$. □

Fact 1.3. *Let $\mathcal{L}, \mathcal{L}'$ be languages and $\mathcal{I}, \mathcal{J}$ $\mathcal{L}'$-structures. Suppose $\mathrm{age}(\mathcal{I}) = \mathrm{age}(\mathcal{J})$ and they have modeling property. Let T be an $\mathcal{L}$-structure and $\mathbb{M}$ a sufficiently saturated model of T. Then for any $(a_\eta)_{\eta \in \mathcal{I}} \subseteq \mathbb{M}$, there exists $(b_\eta)_{\eta \in \mathcal{J}} \subseteq \mathbb{M}$ such that*

(i) for all finite tuple $\bar{\eta}$ in $\mathcal{J}$, if $\mathbb{M} \models \varphi(\bar{b}_{\bar{\eta}})$, then there exists $\bar{\nu} \sim_{\mathcal{L}'} \bar{\eta}$ such that $\mathbb{M} \models \varphi(\bar{a}_{\bar{\eta}})$,

(ii) $(b_\eta)_{\eta \in J}$ is $\mathcal{L}'$-indiscernible.

Remark 1.4. Let $\mathcal{L}_0 \subseteq \mathcal{L}_1$ be languages and $\mathcal{I}_1, \mathcal{J}_1$ $\mathcal{L}_1$-structures with underlying sets I, J and assume $\mathrm{age}_{\mathcal{L}_1}(\mathcal{I}_1) = \mathrm{age}_{\mathcal{L}_1}(\mathcal{J}_1)$ and they have modeling property with respect to $\mathcal{L}_1$. Let $\mathcal{L}$ be a language, T an $\mathcal{L}$-theory, $\mathbb{M}$ a sufficiently saturated model of T. Suppose that every $\mathcal{L}_1$-indiscernible $\mathcal{I}_1$-indexed set is $\mathcal{L}_0$-indiscernible. If $(b_\eta)_{\eta \in \mathcal{J}_1}$ is $\mathcal{L}_1$-indiscernible, then it is $\mathcal{L}_0$-indiscernible.

1.2. Tree languages.

Notation 1.5. We will use the following languages for tree structures

$$\mathcal{L}_{0,P} := \{\trianglelefteq, <_{\mathrm{lex}}, \wedge, P\} \qquad \mathcal{L}_{0,P^-} := \{\trianglelefteq, <_{\mathrm{lex}}, P\}$$
$$\mathcal{L}_0 := \{\trianglelefteq, <_{\mathrm{lex}}, \wedge\} \qquad \mathcal{L}_{0^-} := \{\trianglelefteq, <_{\mathrm{lex}}\}$$
$$\mathcal{L}_\delta := \{\Delta, <_{\mathrm{lex}}\} \qquad \mathcal{L}_{<_{\mathrm{lex}}} := \{<_{\mathrm{lex}}\}$$

where $\trianglelefteq, <_{\mathrm{lex}}$ are binary relation symbols, $\wedge$ is a binary function symbol, P is a unary relation symbol, and δ is a quaternary relation symbol.

Let $\square \in \{0{,}P;\ 0{,}P^-;\ 0;\ 0^-;\ \delta;\ <_{\mathrm{lex}}\}$ (we use semicolons to separate the items, since '$0{,}P$' and '$0{,}P^-$' already contain commas). For an $\mathcal{L}_\square$-structure $\mathcal{I}$ and tuples $\bar{\eta}, \bar{\nu}$ in $\mathcal{I}$, we write $\bar{\eta} \sim_\square \bar{\nu}$ if $\bar{\eta}$ and $\bar{\nu}$ have the same quantifier-free type in $\mathcal{L}_\square$. We also say $(a_\eta)_{\eta \in \mathcal{I}}$ is $\square$-*indiscernible* if it is $\mathcal{L}_\square$-indiscernible for each $\square$. Depending on the context, we may also refer to $0{,}P$; $0{,}P^-$; 0; and 0^--indiscernibility as treetop, treetop$^-$, strong, and strong$^-$-indiscernibility, respectively

Notation 1.6. For each ordinals α and β, let β^α be the set of all functions from α to β. Let $\beta^{<\alpha} = \bigcup_{\beta<\alpha} \beta^\beta$ and $\beta^{\leq\alpha} = \bigcup_{\beta\leq\alpha} \beta^\beta$.

We give a partial order $\trianglelefteq$ on $\beta^{<\alpha}$ and $\beta^{\leq\alpha}$ by $\eta \trianglelefteq \nu \leftrightarrow \eta \subseteq \nu$. $\eta \wedge \nu$ is the greatest element ξ such that $\xi \trianglelefteq \eta$ and $\xi \trianglelefteq \nu$. We write $\eta \perp \nu$ if $\eta \not\trianglelefteq \nu$ and $\nu \not\trianglelefteq \eta$. We write $\mathrm{len}(\eta) = \mathrm{dom}(\eta)$. We write $\eta <_{\mathrm{lex}} \nu$ if $\eta \triangleleft \nu$, or $\eta \perp \nu$ and $\eta(\mathrm{len}(\eta \wedge \nu)) < \nu(\mathrm{len}(\eta \wedge \nu))$. In $\beta^{\leq \alpha}$, we write $P(\eta)$ if $\eta \in \beta^{\alpha}$. For $\eta_0, \eta_1, \eta_2, \eta_3 \in \beta^{\leq \alpha}$, we write $\Delta(\eta_0, \eta_1, \eta_2, \eta_3)$ if and only if $\eta_0 \wedge \eta_1 \trianglelefteq \eta_2 \wedge \eta_3$.

Notation 1.7. [7] For a linear order I and $a \in I$, let $I_{a<} := \{x \in I : a < x\}$. Similarly we define $I_{a\leq}, I_{<a}, I_{\leq a}, I_{a<<b} := \{x \in I : a < x < b\}, I_{a\leq<b}, I_{a<\leq b}$, and $I_{a\leq\leq b}$ for $a < b \in I$.

We will use the symbol '$-\infty$' and the extended order on $\{-\infty\} \cup I$ given by $-\infty < a$ for all $a \in I$. Note that $I_{-\infty<<a} = I_{<a}$ and $I_{-\infty<\leq a} = I_{\leq a}$.

For a linear order I and an ordinal β, let

$$\begin{aligned}\mathcal{T}_{I,\beta} :=& \{\eta : I_{a<} \to \beta :: a \in I, |\{x \in I_{a<} : \eta(x) \neq 0\}| < \aleph_0\} \\ &\cup \{\eta : I \to \omega :: |\{x \in I : \eta(x) \neq 0\}| < \aleph_0\}.\end{aligned}$$

If $\beta = \omega$, then we omit β and simply write $\mathcal{T}_I$. If $I = \emptyset$, then $\mathcal{T}_I := \{\emptyset\}$.

Let $\mathcal{T}^+_{I,\beta} := \{\eta : I \to \beta :: |\{x \in I : \eta(x) \neq 0\}| < \aleph_0\}$ and $\mathcal{T}^-_{I,\beta} = \mathcal{T}_{I,\beta} \setminus \mathcal{T}^+_{I,\beta}$. We let $\mathcal{T}^+_\emptyset = \mathcal{T}_\emptyset = \{\emptyset\}$ and $\mathcal{T}^-_\emptyset = \emptyset$. If I has the greatest element a, then $\emptyset$ can be regarded as a function from $I_{>a}$ to β. Thus $\emptyset$ becomes the least element of $\mathcal{T}_{I,\beta}$ with respect to the order $\trianglelefteq$ which will be defined below. If I has no greatest element, then $\mathcal{T}_{I,\beta}$ has no least element.

The interpretation of the symbols is similar to the cases of $\beta^{<\alpha}$ and $\beta^{\leq\alpha}$. We give a partial order $\trianglelefteq$ on $\mathcal{T}_{I,\beta}$ by $\eta \trianglelefteq \nu \leftrightarrow \eta \subseteq \nu$. For $\eta, \nu \in \mathcal{T}_{I,\beta}$, let $\eta \wedge \nu$ be the greatest element ξ such that $\xi \trianglelefteq \eta$ and $\xi \trianglelefteq \nu$. By finite support (the condition $|\{x \in I_{>a} : \eta(x) \neq 0\}| < \omega$), $\wedge$ is well-defined. We write $\eta \perp \nu$ if $\eta \ntrianglelefteq \nu$ and $\nu \ntrianglelefteq \eta$. We write $\mathrm{len}(\eta) = a$ if $\mathrm{dom}(\eta) = I_{>a}$. If $\eta \in \mathcal{T}^+_I$, then we write $\mathrm{len}(\eta) = -\infty$. We write $\eta <_{\mathrm{lex}} \nu$ if $\eta \triangleleft \nu$, or $\eta \perp \nu$ and $\eta(\mathrm{len}(\eta \wedge \nu)) < \nu(\mathrm{len}(\eta \wedge \nu))$. We write $P(\eta)$ if $\eta \in \mathcal{T}^+_{I,\beta}$. For $\eta_0, \eta_1, \eta_2, \eta_3 \in \mathcal{T}_{I,\beta}$, we write $\Delta(\eta_0, \eta_1, \eta_2, \eta_3)$ if and only if $\eta_0 \wedge \eta_1 \trianglelefteq \eta_2 \wedge \eta_3$.

2. Strong/weak-treeless and meet-eliminating theories

Definition 2.1. The following definitions are from [8].

(i) [8, Definition 3.11 and Proposition 3.12] We say T is *treeless* if for every treetop indiscernible $(a_\eta)_{\eta \in \omega^{\leq\omega}}$, $(a_\eta)_{\eta\in\omega^\omega}$ is $<_{\mathrm{lex}}$-indiscernible over $a_\emptyset$.
(ii) [8, Question 3.14] We say T is *weakly treeless* or *weak-treeless* if for every treetop indiscernible $(a_\eta)_{\eta \in \omega^{\leq\omega}}$, $(a_\eta)_{\eta\in\omega^\omega}$ is $<_{\mathrm{lex}}$-indiscernible over $\emptyset$.

Remark 2.2. If T is treeless, then it is weakly treeless.

Remark 2.3. The following are equivalent.

(i) T is weakly treeless.
(ii) Every δ-indiscernible $(a_\eta)_{\eta\in\omega^\omega}$ is $<_{\mathrm{lex}}$-indiscernible.

Definition 2.4.

(i) We say T is *strongly treeless* or *strong-treeless* if every treetop indiscernible $(a_\eta)_{\eta \in \omega^{\leq\omega}}$ is treetop$^-$ indiscernible.
(ii) We say T is *meet-eliminating* or T has *meet-elimination* if every strongly indiscernible $(a_\eta)_{\eta\in\omega^{<\omega}}$ is strong$^-$-indiscernible.

Remark 2.5. If T is strongly treeless, then it is treeless.

Remark 2.6. If T is strongly treeless, then it is meet-eliminating.

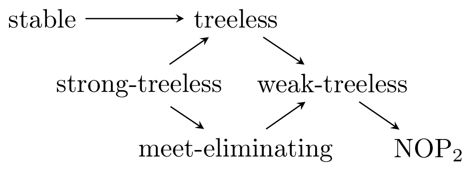
Remark 2.7. If T is meet-eliminating, then it is weakly treeless.

Fact 2.8. [8, Corollary 5.4] *If T is stable, then it is treeless.*

Definition 2.9. [12] We say a formula $\varphi(x, y, z)$ has OP_2 if there exist $(a_i)_{i<\omega}$ and $(b_i)_{i<\omega}$ such that for any increasing $f : \omega \to \omega$ and $X_f := \{(i, j) \in \omega \times \omega : j \geq f(i)\}$, there exists c_f such that $X_f = \{(i, j) \in \omega \times \omega :\models \varphi(c_f, a_i, b_j)\}$. We say $\varphi(x, y, z)$ is NOP_2 if it does not have OP_2. We say a theory is NOP_2 if every formula is NOP_2.

Fact 2.10. [6, Proposition A.4] *If T is weakly treeless, then it is NOP_2.*

So we have the following diagram.



3. $\mathcal{C}$-less theories

Notation 3.1. Let $\mathcal{L}_{\mathcal{C}_\prec} := \{C, <_{\text{lex}}\}$ where C is a ternary relation symbol. We write $C(\eta, \nu, \xi)$ if $\eta \wedge \nu \wedge \xi \lhd \nu \wedge \xi$, for any η, ν, ξ in $\beta^{\leq \alpha}$ and $\mathcal{T}_{I,\beta}$.

Definition 3.2. We say T is *$\mathcal{C}$-less* if every $\mathcal{C}_\prec$-indiscernible sequence $(a_\eta)_{\eta \in 2^\omega}$ is $<_{\text{lex}}$-indiscernible.

Remark 3.3. For any theory, $(a_\eta)_{\eta \in \omega^\omega}$ is δ-indiscernible if and only if it is $\mathcal{C}_\prec$-indiscernible.

Proof. For any $\eta, \nu, \xi, \zeta \in \omega^\omega$, we have

$$\omega^\omega \models \Delta(\eta, \nu, \xi, \zeta) \leftrightarrow \neg C(\xi, \eta, \nu) \wedge \neg C(\zeta, \eta, \nu).$$

Thus if $(a_\eta)_{\eta \in \omega^\omega}$ is δ-indiscernible, then it is $\mathcal{C}_\prec$-indiscernible.

Conversely, for any $\eta, \nu, \xi \in \omega^\omega$, we have

$$\omega^\omega \models C(\eta, \nu, \xi) \leftrightarrow \Delta(\eta, \nu, \nu, \xi) \wedge \Delta(\eta, \xi, \nu, \xi) \wedge \neg\Delta(\nu, \xi, \eta, \nu) \wedge \neg\Delta(\nu, \xi, \eta, \xi).$$

Thus if $(a_\eta)_{\eta \in \omega^\omega}$ is $\mathcal{C}_\prec$-indiscernible, then it is δ-indiscernible. □

Remark 3.4. T is weakly treeless if and only if every $\mathcal{C}_\prec$-indiscernible sequence $(a_\eta)_{\eta \in \omega^\omega}$ is $<_{\text{lex}}$-indiscernible.

Proof. By Remark 3.3. □

Remark 3.5. If T is weakly treeless, then it is $\mathcal{C}$-less.

Proof. Suppose $(a_\eta)_{\eta \in 2^\omega}$ is $\mathcal{C}_\prec$-indiscernible. Let f be a map from $\omega^{\leq \omega}$ to $2^{\leq \omega}$ such that

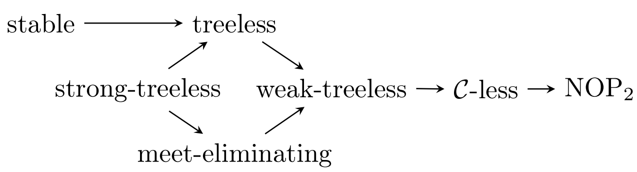
(i) $f(\emptyset) = \emptyset$,
(ii) $f(\eta^\frown\langle i \rangle) = f(\eta)^\frown\langle 1 \rangle^{i+1\frown}\langle 0 \rangle$ for each $\eta \in \omega^{<\omega}$ and $i < \omega$,
(iii) $f(\eta) = \bigcup_{n<\omega} f(\eta|_n)$ for each $\eta \in \omega^\omega$.

Then $f(\eta) \in 2^\omega$ for all $\eta \in \omega^\omega$. Let $b_\eta := a_{f(\eta)}$ for each $\eta \in \omega^\omega$. It is easy to check that $f(\bar\eta) \sim_{\mathcal{C}_\prec} f(\bar\nu)$ for all $\bar\eta, \bar\nu \in \omega^\omega$ with $\bar\eta \sim_{\mathcal{C}_\prec} \bar\nu$. Thus $(b_\eta)_{\eta \in \omega^\omega}$ is $\mathcal{C}_\prec$-indiscernible and hence it is $<_{\text{lex}}$-indiscernible by Remark 3.4.

If $(a_\eta)_{\eta \in 2^\omega}$ is not $<_{\text{lex}}$-indiscernible, then there exist $\bar\eta \sim_{<_{\text{lex}}} \bar\nu$ in 2^ω such that $\bar a_{\bar\eta} \not\equiv \bar a_{\bar\nu}$. We have an $\mathcal{L}$-formula $\varphi(\bar x)$ such that $\models \varphi(\bar a_{\bar\eta}) \wedge \neg\varphi(\bar a_{\bar\nu})$. Note that by regarding 2^ω as a subset of ω^ω, we can say that $f(\bar\xi) \sim_{\mathcal{C}_\prec} \bar\xi$ for all $\bar\xi \in 2^\omega$. Thus $f(\bar\eta\bar\nu) \sim_{\mathcal{C}} \bar\eta\bar\nu$ and hence $\models \varphi(\bar b_{\bar\eta}) \wedge \neg\varphi(\bar b_{\bar\nu})$. But we have $\bar\eta \sim_{<_{\text{lex}}} \bar\nu$, which implies $\models \varphi(\bar b_{\bar\eta}) \leftrightarrow \varphi(\bar b_{\bar\nu})$, a contradiction. □

Fact 3.6. [3] *If T is $\mathcal{C}$-less, then it is NOP$_2$.*

So we have the following diagram.



4. Treeless is strong-treeless

Fact 4.1. [8, Proposition 3.12] *The following are equivalent.*

(i) T is treeless.
(ii) For any infinite linear order I, if $(b_\eta)_{\eta\in\mathcal{T}_I}$ is treetop indiscernible, then $(b_\eta)_{\nu\trianglelefteq\eta\in\mathcal{T}_I^+}$ is $<_{lex}$-indiscernible over b_ν for any $\nu\in\mathcal{T}_I^-$ with $|\{a\in I : a<\mathrm{len}(\nu)\}|=\omega$.
(iii) For some infinite linear order I, if $(b_\eta)_{\eta\in\mathcal{T}_I}$ is treetop indiscernible, then $(b_\eta)_{\nu\trianglelefteq\eta\in\mathcal{T}_I^+}$ is $<_{lex}$-indiscernible over b_ν for any $\nu\in\mathcal{T}_I$ with $|\{a\in I : a<\mathrm{len}(\nu)\}|=\omega$.

Remark 4.2. The following are equivalent.

(i) T is strongly treeless.
(ii) For any infinite linear order I, if $(b_\eta)_{\eta\in\mathcal{T}_I}$ is treetop indiscernible, then $(b_\eta)_{\eta\in\mathcal{T}_I}$ is treetop$^-$ indiscernible.
(iii) For some infinite linear order I, if $(b_\eta)_{\eta\in\mathcal{T}_I}$ is treetop indiscernible, then $(b_\eta)_{\eta\in\mathcal{T}_I}$ is treetop$^-$ indiscernible.

Proof. By Remark 1.4. □

Notation 4.3. Let I be a linear order, J a linear order with the greatest element $b_\emptyset\in J$, ι an order preserving map from $\{-\infty\}\cup J$ to $\{-\infty\}\cup I$. For each $\eta\in\mathcal{T}_J$, let η_ι be a map from $I_{\iota(\mathrm{len}(\eta))<\le\iota(b_\emptyset)}$ to ω such that

$$\eta_\iota(a)=\begin{cases}\eta(b) & \text{if } a=\iota(b) \text{ for some } \mathrm{len}(\eta)<b\\ 0 & \text{otherwise}\end{cases}$$

for each $a\in I_{\iota(\mathrm{len}(\eta))<\le\iota(b_\emptyset)}$. Note that $\emptyset_\iota=\emptyset$

Remark 4.4. Let I be a linear order, J a linear order with the greatest element $b_\emptyset\in J$, ι an order preserving map from $\{-\infty\}\cup J$ to $\{-\infty\}\cup I$, and ν an element of $\mathcal{T}_I$ such that $\mathrm{len}(\nu)=\iota(b_\emptyset)$. Then the map $f:\mathcal{T}_J\to\mathcal{T}_I$ sending η to $\eta_\iota\cup\nu$ is an $\mathcal{L}_0$-embedding. If $\iota(-\infty)=-\infty$ additionally, then f is an $\mathcal{L}_{0,P}$-embedding.

Lemma 4.5. *Let I be a dense linear order and $\eta_0,\dots,\eta_{n-1},\nu\in\mathcal{T}_I^-$. If $\eta_0,\dots,\eta_{n-1}$ form an antichain, $\mathrm{len}(\eta_0)=\cdots=\mathrm{len}(\eta_{n-1})$, and $\nu\trianglelefteq\bigwedge_{i<n}\eta_i$, then there exists an $\mathcal{L}_0$-embedding $f:\mathcal{T}_{\omega^+}\to\mathcal{T}_I$ such that $\eta_0,\dots,\eta_{n-1}\in f(\mathcal{T}_{\omega^+}^+)$ and $f(\emptyset)=\nu$. If $\mathrm{len}(\eta_0)=-\infty$ additionally, then f is an $\mathcal{L}_{0,P}$-embedding.*

Proof. Since I is dense, we can find an order preserving map from $\{-\infty\}\cup\omega^+$ to I such that $\iota(-\infty)=\mathrm{len}(\eta_0)$, $\iota(\omega)=\mathrm{len}(\nu)$, and $\{a\in I_{\mathrm{len}(\eta_0)<\le\mathrm{len}(\nu)} : \eta_i(a)\neq 0 \text{ for some } i<n\}\subseteq\mathrm{Im}(\iota)$. For each $\eta\in\mathcal{T}_{\omega^+}$, let $f(\eta):=\eta_\iota\cup\nu$. Then f is an $\mathcal{L}_0$-embedding from $\mathcal{T}_{\omega^+}$ to $\mathcal{T}_I$. Clearly $\eta_0,\dots,\eta_{n-1}\in f(\mathcal{T}_{\omega^+}^+)$ and $f(\emptyset)=\nu$. If $\mathrm{len}(\eta_0)=-\infty$, then f is an $\mathcal{L}_{0,P}$-embedding by Remark 4.4. □

Lemma 4.6. *Let I be a dense linear order without the least element, $n<\omega$, $\eta_0\in\mathcal{T}_I$ and $\bar\eta:=(\eta_{i,j})_{i,j<n}\in\mathcal{T}_I$ such that*

(i) $\eta_0\trianglelefteq\eta_{i,j}$ for all $i,j<n$
(ii) $\eta_{i,0}\in\mathcal{T}_I^-$ for all $i<n$,
(iii) $\eta_{i,0}\not\trianglelefteq\eta_{i',0}$ for all $i,i'<n$,
(iv) $\eta_{i,0}\trianglelefteq\eta_{i,j}$ for all $i,j<n$.

Then there exist $\eta_0'\in\mathcal{T}_I$ and $\bar\eta':=(\eta_{i,j}')_{i,j<n}\in\mathcal{T}_I$ such that $\eta_0'\bar\eta'\sim_{0,P}\eta_0\bar\eta$ and $\mathrm{len}(\eta_{i,0}')=\mathrm{len}(\eta_{i',0}')$ for all $i,i'<n$.

Proof. Let $\eta_0':=\eta_0$. Let $a^*:=\max_{i<n}\{\mathrm{len}(\eta_{i,0})\}$ and $\eta_{i,0}':=\eta_{i,0}\cup\{(a,0):a^*<a\le\mathrm{len}(\eta_{i,0})\}$ for each $i<n$. For each $i<n$, we can give an order preserving map ι_i from $\{-\infty\}\cup\omega^+$ to $\{-\infty\}\cup I$ such that $\iota_i(-\infty)=-\infty$, $\iota_i(\omega)=\mathrm{len}(\eta_{i,0})$, and $\mathrm{Im}(\iota_i)\supseteq\{\mathrm{len}(\eta_{i,j}):j<n\}\cup\{a\in I_{\le\mathrm{len}(\eta_{i,0})}:\eta_{i,j}(a)\neq 0 \text{ for some } j<n\}$. Let $f_i:\mathcal{T}_{\omega^+}\to\mathcal{T}_i;\eta\to\eta_{\iota_i}\cup\eta_{i,0}$. Then f_i is an $\mathcal{L}_{0,P}$-embedding by Remark 4.4 and $\{\eta_{i,j}\}_{j<n}\subseteq\mathrm{Im}(f_i)$.

Now we give an order preserving map ι_i' from $\{-\infty\}\cup\omega^+$ to $\{\infty\}\cup I$ such that $\iota_i'(-\infty)=-\infty$ and $\iota_i'(\omega)=\mathrm{len}(\eta_{i,0}')$. Then $f_i':\mathcal{T}_{\omega^+}\to\mathcal{T}_I;\eta\mapsto\eta_{\iota_i'}\cup\eta_{i,0}'$ is an $\mathcal{L}_{0,P}$-embedding. For each $i,j<n$ with $0<j$, let $\eta_{i,j}':=f_i'(f_i^{-1}(\eta_{i,j}))$. Then $\bar\eta'\sim_{0,P}\bar\eta$ where $\bar\eta':=(\eta_{i,j}')_{i,j<n}$. Clearly $\mathrm{len}(\eta_{i,0}')=\mathrm{len}(\eta_{i',0}')$ for all $i,i'<n$. □

Lemma 4.7. *Let I be a dense linear order without the least element, $n < \omega$, $\bar{\eta} := (\eta_0, ..., \eta_{n-1}) \in \mathcal{T}_I$. For each $\nu \in \mathcal{T}_I$, there exists $\bar{\eta}' \sim_{0,P} \bar{\eta}$ such that $\bigwedge \bar{\eta}' \trianglerighteq \nu$.*

Proof. Let ι be an order preserving map from $\{-\infty\} \cup \omega^+$ to $\{-\infty\} \cup I$ such that $\iota(-\infty) = -\infty$, $\iota(\omega) = \mathrm{len}(\bigwedge \bar{\eta})$, and $\mathrm{Im}(\iota) \supseteq \{\mathrm{len}(\eta_i) : i < n\} \cup \{a \in I : a \leq \mathrm{len}(\bigwedge \bar{\eta}), \eta_i(a) \neq 0 \text{ for some } i < n\}$. Let ι' be an order preserving map from $\{-\infty\} \cup \omega^+$ to $\{-\infty\} \cup I$ such that $\iota'(-\infty) = -\infty$ and $\iota'(\omega) = \mathrm{len}(\nu)$.

Let $f : \mathcal{T}_{\omega^+} \to \mathcal{T}_I; \eta \mapsto \eta_\iota \cup \bigwedge \bar{\eta}$ and $f' : \mathcal{T}_{\omega^+} \to \mathcal{T}_I; \eta \mapsto \eta'_\iota \cup \nu$. For each $i < n$, let $\eta'_i := f'(f^{-1}(\eta_i))$. Then $\bar{\eta}' \sim_{0,P} \bar{\eta}$ and $\bigwedge \bar{\eta}' \trianglerighteq \nu$. □

Lemma 4.8. *Let I be a dense linear order without the least element and f be an $\mathcal{L}_0$-embedding from $\mathcal{T}_{\omega^+}$ to $\mathcal{T}_I^-$ with $f(\emptyset) \rhd \nu$ for some $\nu \in \mathcal{T}_I$. Let $n < \omega$ and $\{\eta_{i,j}\}_{i,j<n} \in \mathcal{T}_I$ such that*

(i) $\eta_{i,0} \in \mathcal{T}_I^-$ for all $i < n$,
(ii) $\eta_{i,0} \in f(\mathcal{T}_{\omega^+}^+)$ for all $i < n$,
(iii) $\eta_{i,j} \rhd \eta_{i,0}$ for all $i < n$ and $0 < j < n$.

Then there exist $m < \omega$ and a family of $\mathcal{L}_0$-embeddings $(g_\eta)_{\eta \in \mathcal{T}_{\omega^+}}$ from $\mathcal{T}_{m^+}$ to $\mathcal{T}_I$ such that

(iv) $g_\eta(\emptyset) = f(\eta)$ for all $\eta \in \mathcal{T}_{\omega}^+$,
(v) $g_\eta(\zeta) = f(\eta)$ for all $\eta \in \mathcal{T}_{\omega^+}^-$,
(vi) $g_\eta(\mathcal{T}_{m^+}^+) \subseteq \mathcal{T}_I^+$ for all $\eta \in \mathcal{T}_{\omega^+}^+$,
(vii) $g_\eta(\mathcal{T}_{m^+}) \subseteq \mathcal{T}_I^-$ for all $\eta \in \mathcal{T}_{\omega^+}^-$,
(viii) $g_\eta(\zeta) \lhd g_\nu(\emptyset)$ for all $\eta \lhd \nu \in \mathcal{T}_{\omega^+}$,
(ix) $\eta_{i,j} \in g_{f^{-1}(\eta_{i,0})}(\mathcal{T}_{m^+})$ for all $i < n$ and $0 < j < n$,

where $\zeta \in \mathcal{T}_{m^+}^+$ is the map from m^+ to ω such that $\zeta(l) = 0$ for all $l < m^+$.

Proof. Let $m := |\{a \in I : \eta_{i,j}(a) \neq 0 \text{ for some } i, j < n\}| + 2$. By finite support, $m < \omega$. We construct $(g_\eta)_{\eta \in \mathcal{T}_{\omega^+}}$ as follow.

If $\eta = f^{-1}(\eta_{i,0})$ for some $i < n$, then choose any order preserving map ι_η from $\{-\infty\} \cup m^+$ to I such that $\iota_\eta(-\infty) = -\infty$, $\iota_\eta(m) = \mathrm{len}(\eta_{i,0})$, and

$$\{a \in I_{\leq \mathrm{len}(\eta_{i,0})} : \eta_{i,j}(a) \neq 0 \text{ for some } 0 < j < n\} \subseteq \mathrm{Im}(\iota_\eta).$$

For each $\xi \in \mathcal{T}_{m^+}$, define $g_\eta(\xi) := \xi_{\iota_\eta} \cup f(\eta)$.

If $\eta \in \mathcal{T}_{\omega^+}^+ \setminus \{f^{-1}(\eta_{0,0}), ..., f^{-1}(\eta_{n-1,0})\}$, then choose any order preserving map $\iota_\eta : \{-\infty\} \cup m^+ \to I$ such that $\iota_\eta(-\infty) = -\infty$ and $\iota_\eta(m) = \mathrm{len}(f(\eta))$. For each $\xi \in \mathcal{T}_{m^+}$, define $g_\eta(\xi) := \xi_{\iota_\eta} \cup f(\eta)$.

If $\eta \in \mathcal{T}_{\omega^+}^-$, then choose any order preserving map $\iota_\eta : \{-\infty\} \cup m^+ \to I$ such that $\iota_\eta(-\infty) = \mathrm{len}(f(\eta))$, $\iota_\eta(m) < \mathrm{len}(f(\eta^-))$ and $f(\eta)(a) = 0$ for all $\mathrm{len}(f(\eta)) < a \leq \iota_\eta(m)$, where $\eta^- := \eta \setminus \{(\mathrm{len}(\eta)+1, \eta(\mathrm{len}(\eta)+1))\}$. For each $\xi \in \mathcal{T}_{m^+}$, let $g_\eta(\xi) := \xi_\iota \cup f(\eta)|_{I_{\iota_\eta(m)<}}$. Then $(g_\eta)_{\eta \in \mathcal{T}_{\omega^+}}$ is a family of $\mathcal{L}_0$-embeddings from $\mathcal{T}_{m^+}$ to $\mathcal{T}_I$ satisfies (iv), (v), (vi), (vii), (viii), and (ix). □

Lemma 4.9. *Let $\mathcal{L}$ be a language, T an $\mathcal{L}$-theory, $\mathbb{M}$ a sufficiently saturated model of T, I a dense linear order without the least element, and $m < \omega$. Suppose that $(b_\eta)_{\eta \in \mathcal{T}_I} \subseteq \mathbb{M}$ is treetop indiscernible, f is an $\mathcal{L}_0$-embedding from $\mathcal{T}_{\omega^+}$ to $\mathcal{T}_I^-$, and $(g_\eta)_{\eta \in \mathcal{T}_{\omega^+}}$ is a family of $\mathcal{L}_0$-embeddings from $\mathcal{T}_{m^+}$ to $\mathcal{T}_I$ such that*

(i) $g_\eta(\emptyset) = f(\eta)$ for all $\eta \in \mathcal{T}_{\omega^+}^+$,
(ii) $g_\eta(\zeta) = f(\eta)$ for all $\eta \in \mathcal{T}_{\omega^+}^-$,
(iii) $g_\eta(\mathcal{T}_{m^+}^+) \subseteq \mathcal{T}_I^+$ for all $\eta \in \mathcal{T}_{\omega^+}^+$,
(iv) $g_\eta(\mathcal{T}_{m^+}) \subseteq \mathcal{T}_I^-$ for all $\eta \in \mathcal{T}_{\omega^+}^-$,
(v) $g_\eta(\zeta) \lhd g_\nu(\emptyset)$ for all $\eta \lhd \nu \in \mathcal{T}_{\omega^+}$,

where $\zeta \in \mathcal{T}_{m^+}^+$ is the map from m^+ to ω such that $\zeta(n) = 0$ for all $n < m^+$. Let $c_\eta := (b_{g_\eta(\xi)})_{\xi \in \mathcal{T}_{m^+}}$ for each $\eta \in \mathcal{T}_{\omega^+}$. Then $(c_\eta)_{\eta \in \mathcal{T}_{\omega^+}}$ is treetop indiscernible.

Proof. Suppose $\bar{\eta} := (\eta_0, ..., \eta_{d-1}), \bar{\nu} := (\nu_0, ..., \nu_{d-1}) \in \mathcal{T}_{\omega^+}$ and $\bar{\eta} \sim_{0,P} \bar{\nu}$ in $\mathcal{T}_{\omega^+}$. Then we have

(vi) $\mathcal{T}_{\omega^+} \models \sigma(\bar{\eta}) \leftrightarrow \sigma(\bar{\nu})$ for any quantifier free $\mathcal{L}_{0,P}$-formula $\sigma(\bar{x})$.

Since we assume $(b_\eta)_{\eta \in \mathcal{T}_I}$ is treetop indiscernible and choose $\bar{\eta}, \bar{\nu}$ arbitrary, it is enough to show that for any $e < \omega$, $i_0, ..., i_{e-1} < d$, $\xi_0, ..., \xi_{e-1} \in \mathcal{T}_{m^+}$, and an atomic $\mathcal{L}_{0,P}$-formula $\sigma(\bar{x})$, if

$$\mathcal{T}_I \models \sigma(g_{\eta_{i_0}}(\xi_0), ..., g_{\eta_{i_{e-1}}}(\xi_{e-1})),$$

then

$$\mathcal{T}_I \models \sigma(g_{\nu_{i_0}}(\xi_0), ..., g_{\nu_{i_{e-1}}}(\xi_{e-1})).$$

Note that every $\mathcal{L}_{0,P}$-term generated by $\{g_{\eta_{i_l}}(\xi_l)\}_{l<e}$ can be written as a meet of at most two elements of $\{g_{\eta_{i_l}}(\xi_l)\}_{l<e}$. Moreover, for any $X \subseteq \{i_0, ..., i_{e-1}\}$ and $i_j, i_k \in X$, $\bigwedge_{i_l \in X} g_{\eta_{i_l}}(\xi_l) = g_{\eta_{i_j}}(\xi_j) \wedge g_{\eta_{i_k}}(\xi_k)$ if and only if $\bigwedge_{i_l \in X} g_{\nu_{i_l}}(\xi_l) = g_{\nu_{i_j}}(\xi_j) \wedge g_{\nu_{i_k}}(\xi_k)$ by (vi). Thus we only have to consider the following 4 cases.

Case a: $\mathcal{T}_I \models P(g_{\eta_{i_j}}(\xi_j) \wedge g_{\eta_{i_k}}(\xi_k))$ for some $j, k < e$
Case b: $\mathcal{T}_I \models g_{\eta_{i_j}}(\xi_j) \wedge g_{\eta_{i_k}}(\xi_k) \trianglelefteq g_{\eta_{i_l}}(\xi_l) \wedge g_{\eta_{i_m}}(\xi_m)$ for some $j, k, l, m < e$
Case c: $\mathcal{T}_I \models g_{\eta_{i_j}}(\xi_j) \wedge g_{\eta_{i_k}}(\xi_k) <_{\text{lex}} g_{\eta_{i_l}}(\xi_l) \wedge g_{\eta_{i_m}}(\xi_m)$ for some $j, k, l, m < e$
Case d: $\mathcal{T}_I \models g_{\eta_{i_j}}(\xi_j) \wedge g_{\eta_{i_k}}(\xi_k) = g_{\eta_{i_l}}(\xi_l) \wedge g_{\eta_{i_m}}(\xi_m)$ for some $j, k, l, m < e$

One can easily check each case by dividing them into some subcases with respect to $\text{qftp}_{0,P}(\eta_{i_j}\eta_{i_k}\eta_{i_l}\eta_{i_m})$, and using (vi). □

Notation 4.10. Let I be a dense linear order without the least element. For $n < \omega$, $\bar{\eta} := (\eta_0, ..., \eta_{n-1}) \in \mathcal{T}_I$, and $m < n$, $U^m_{\bar{\eta}} := \{i < n : \eta_m \triangleleft \eta_i$ and there is no $j < n$ such that $\eta_m \triangleleft \eta_j \triangleleft \eta_i\}$.

Definition 4.11. Let I be a dense linear order without the least element. We say $\bar{\eta} := (\eta_0, ..., \eta_{n-1}) \in \mathcal{T}_I$ is *good* if

(i) $\eta_i \wedge \eta_j \neq \eta_k$ for all distinct $i, j, k < n$,
(ii) for each $m < n$, $\eta_i \wedge \eta_j = \eta_k \wedge \eta_l$ for all $i, j, k, l \in U^m_{\bar{\eta}}$ with $i \neq j$ and $k \neq l$.

Lemma 4.12. *Let I be a dense linear order without the least element and $\bar{\eta}, \bar{\nu} \in \mathcal{T}_I$. If $\bar{\eta} \sim_{0,P^-} \bar{\nu}$ and they are good, then $\bar{\eta} \sim_{0,P} \bar{\nu}$.*

Proof. It is easy to check that $\text{cl}_{0,P}(\bar{\eta}) \sim_{0,P} \text{cl}_{0,P}(\bar{\nu})$. □

Lemma 4.13. *Let T be a treeless theory. If I is a dense linear order without least element, A is a small set, $(b_\eta)_{\eta \in \mathcal{T}_I}$ is treetop indiscernible over A, $\bar{\eta} := (\eta_0, ..., \eta_{n-1}) \in \mathcal{T}_I$. Suppose that $\eta_0 \triangleleft \eta_i$ for all $0 < i < n$. Then there exists $\bar{\eta}' \sim_{0,P^-} \bar{\eta}$ such that $\bar{b}_{\bar{\eta}'} \equiv_A \bar{b}_{\bar{\eta}}$ and $\eta'_0 \triangleleft \eta'_i \wedge \eta'_j = \eta'_k \wedge \eta'_l$ for all $i, j, k, l \in U^0_{\bar{\eta}'}$ with $i \neq j$ and $k \neq l$.*

Proof. By Lemma 4.7 we may assume there exists $\nu \in \mathcal{T}_I$ such that $\nu \triangleleft \eta_0$. We may assume $U^0_{\bar{\eta}} = m \setminus \{0\}$ for some $m \leq n$ and $\eta_1 <_{\text{lex}} \cdots <_{\text{lex}} \eta_{m-1}$. For each $0 < i < m$, choose any $\eta^*_i \in \mathcal{T}^-_I$ such that $\eta_i \wedge \eta_j \triangleleft \eta^*_i \triangleleft \eta_i$ for all $0 < j < m$ with $j \neq i$. By Lemma 4.6, we may assume $\text{len}(\eta^*_i) = \text{len}(\eta^*_j)$ for all $0 < i, j < m$. By Lemma 4.5, there is an $\mathcal{L}_0$-embedding $f : \mathcal{T}_{\omega^+} \to \mathcal{T}_I$ such that $\eta^*_1, ..., \eta^*_{m-1} \in f(\mathcal{T}^+_{\omega^+})$ and $f(\emptyset) = \eta_0$. By Lemma 4.8, there exist $m < \omega$ and a family of $\mathcal{L}_0$-embeddings $(g_\eta)_{\eta \in \mathcal{T}_{\omega^+}}$ from $\mathcal{T}_{m^+}$ to $\mathcal{T}_I$ such that

(i) $g_\eta(\emptyset) = f(\eta)$ for all $\eta \in \mathcal{T}^+_\omega$,
(ii) $g_\eta(\zeta) = f(\eta)$ for all $\eta \in \mathcal{T}^-_{\omega^+}$,
(iii) $g_\eta(\mathcal{T}^+_{m^+}) \subseteq \mathcal{T}^+_I$ for all $\eta \in \mathcal{T}^+_{\omega^+}$,
(iv) $g_\eta(\mathcal{T}_{m^+}) \subseteq \mathcal{T}^-_I$ for all $\eta \in \mathcal{T}^-_{\omega^+}$,
(v) $g_\eta(\zeta) \triangleleft g_\nu(\emptyset)$ for all $\eta \triangleleft \nu \in \mathcal{T}_{\omega^+}$,
(vi) $\eta_j \in \bigcup_{0<i<m} g_{f^{-1}(\eta^*_i)}(\mathcal{T}_{m^+})$ for all $0 < j < n$,

where $\zeta \in \mathcal{T}^+_{m^+}$ is the map from m^+ to ω such that $\zeta(l) = 0$ for all $l < m^+$. By Lemma 4.9, $(c_\eta)_{\eta \in \mathcal{T}_{\omega^+}}$ is treetop indiscernible over A where $c_\eta := (b_{g_\eta(\xi)})_{\xi \in \mathcal{T}_{m^+}}$ for each $\eta \in \mathcal{T}_{\omega^+}$.

For each $0 < i < m$, let ζ_i be a map from ω^+ to ω such that $\zeta_i(0) = i$ and $\zeta_i(n) = 0$ for all $0 < n < \omega^+$. Note that for each $0 < j < n$, there exist unique $0 < i_j < m$ and $\xi_j \in \mathcal{T}_{m^+}$ such that $\eta_j = g_{f^{-1}(\eta^*_{i_j})}(\xi_j)$. Let $\eta'_j := g_{\zeta_{i_j}}(\xi_j)$ for each $0 < j < n$. Let $\eta'_0 := \eta_0$. Since T is treeless, we have

$$c_{\zeta_1} ... c_{\zeta_{m-1}} \equiv_{Ac_\emptyset} c_{f^{-1}(\eta^*_1)} ... c_{f^{-1}(\eta^*_{m-1})},$$

in particular,

$$b_{\eta'_1} ... b_{\eta'_{n-1}} \equiv_{Ab_{\eta'_0}} b_{\eta_1} ... b_{\eta_{n-1}},$$

and hence

$$b_{\eta'_0} b_{\eta'_1} ... b_{\eta'_{n-1}} \equiv_A b_{\eta'_0} b_{\eta_1} ... b_{\eta_{n-1}} = b_{\eta_0} b_{\eta_1} ... b_{\eta_{n-1}}.$$

Clearly $\bar{\eta}' \sim_{0,P^-} \bar{\eta}$, $U^0_{\bar{\eta}'} = m \setminus \{0\}$, and $\eta'_0 \triangleleft \eta'_i \wedge \eta'_j = f(\zeta^-)$ for all $i, j \in U^0_{\bar{\eta}'}$ with $i \neq j$, where ζ^- is the map from $\omega^+ \setminus \{0\}$ to ω such that $\text{Im}(\zeta^-) = \{0\}$. □

Lemma 4.14. *Let T be a treeless theory. If I is a dense linear order without least element, A is a small set, $(b_\eta)_{\eta\in\mathcal{T}_I}$ is treetop indiscernible over A, $\bar{\eta} := (\eta_0, ..., \eta_{n-1}) \in \mathcal{T}_I$. Then there exists good $\bar{\eta}' \in \mathcal{T}_I$ such that $\bar{\eta}' \sim_{0,P^-} \bar{\eta}$ and $\bar{b}_{\bar{\eta}'} \equiv_A \bar{b}_{\bar{\eta}}$.*

Proof. By repeating Lemma 4.13. Note that $(b_\eta)_{\nu\trianglelefteq\eta\in\mathcal{T}_I}$ is treetop indiscernible over $A\cup\{b_\eta\}_{\eta\ntrianglelefteq\nu}$ for each $\nu \in \mathcal{T}_I$. If there is no $i < n$ such that $\eta_i \trianglelefteq \eta_j$ for all $j < n$, then put $\eta_n := \bigwedge_{i<n} \eta_i$ and apply Lemma 4.13 on $(\eta_0, ..., \eta_n)$. □

Theorem 4.15. *If T is treeless, then it is strongly treeless.*

Proof. Let I be a dense linear order without the least element. Let $(b_\eta)_{\eta\in\mathcal{T}_I}$ be treetop indiscernible, $\bar{\eta}, \bar{\nu} \in \mathcal{T}_I$, and suppose $\bar{\eta} \sim_{0,P^-} \bar{\nu}$. Then by Lemma 4.14, there exist good $\bar{\eta}', \bar{\nu}'$ such that $\bar{\eta}' \sim_{0,P^-} \bar{\eta}$, $\bar{\nu}' \sim_{0,P^-} \bar{\nu}$, $b_{\bar{\eta}'} \equiv b_{\bar{\eta}}$, and $b_{\bar{\nu}'} \equiv b_{\bar{\nu}}$. By Lemma 4.12, $\bar{\eta}' \sim_{0,P} \bar{\nu}'$. Thus we have $b_{\bar{\eta}} \equiv b_{\bar{\eta}'} \equiv b_{\bar{\nu}'} \equiv b_{\bar{\nu}}$. By Remark 4.2, T is strongly treeless. □

So we have the following diagram

$$\text{stable} \to \text{treeless} \to \text{meet-eliminating} \to \text{weak-treeless} \to \mathcal{C}\text{-less} \to \text{NOP}_2,$$

and a characterization of treelessness using collapsing indiscernible as below.

Corollary 4.16. *The following are equivalent.*

- *(i) T is treeless.*
- *(ii) Every treetop indiscernible $(a_\eta)_{\eta\in\omega^{\leq\omega}}$ is treetop$^-$ indiscernible.*

Proof. By Remark 2.5 and Theorem 4.15 □

Corollary 4.17. Th(C) *is treeless if and only if* Th$(G(C))$ *(Mekler group) is treeless.*

5. Weakly treeless NATP theories are NCTP

Remark 5.1. 'right-combs' in [10, Definition 1.1] are 'increasing comb' in the sense of [11, Definition 2.4]. Similarly, 'left-combs' are 'descending combs'.

Definition 5.2 (Equivalent to [10, Definition 1.1])**.** A theory T has k-CTP if there is a binary tree $(a_\eta)_{\eta\in 2^{<\omega}}$ and a formula $\varphi(x, y)$ such that for any $\eta \in 2^\omega$, $\{\varphi(x, a_{\eta\lceil n}) : n < \omega\}$ is k-inconsistent but for any descending comb $X \subseteq 2^{<\omega}$, $\{\varphi(x, a_\eta) : \eta \in X\}$ is consistent. T has CTP if it has k-CTP for some $k < \omega$.

Theorem 5.3. *If T is weakly treeless and NATP, then it is NCTP.*

Proof. Suppose T is weakly treeless, NATP, but has CTP. Then we have a formula $\varphi(x, y)$, $(a_\eta)_{\eta\in 2^{<\omega}}$, and $k < \omega$ such that

- (i) $\{\varphi(x, a_\eta)\}_{\eta\in X}$ is consistent if $X \subseteq 2^{<\omega}$ is a descending comb,
- (ii) $\{\varphi(x, a_\eta)\}_{\eta\in X}$ is inconsistent if $X \subseteq 2^{<\omega}$ is a $\trianglelefteq$-chain with length k.

Let f be a map from $\omega^{<\omega}$ to $2^{<\omega}$ defined by

$$f(\eta) = \begin{cases} \emptyset & \text{if } \eta = \emptyset \\ f(\nu)^\frown(0^i)^\frown(1) & \text{if } \eta = \nu^\frown(i) \text{ for some } i \in \omega \end{cases}$$

and $b_\eta := a_{f(\eta)}$ for each $\eta \in \omega^{<\omega}$. Note that $f(\eta) \trianglelefteq f(\nu)$ for all $\eta, \nu \in \omega^{<\omega}$ with $\eta \trianglelefteq \nu$. Thus $\{\varphi(x, b_\eta)\}_{\eta\in X}$ is inconsistent if $X \subseteq \omega^{<\omega}$ is a $\trianglelefteq$-chain with length k. Also note that if $X \subseteq \omega^{<\omega}$ is an increasing comb, then $f(X) := \{f(\eta) : \eta \in X\} \subseteq 2^{<\omega}$ is a descending comb. Thus $\{\varphi(x, b_\eta)\}_{\eta\in X}$ is consistent if $X \subseteq \omega^{<\omega}$ is an increasing comb. By compactness and modelling property, we can find strongly indiscernible $(c_\eta)_{\eta\in\omega^{\leq\omega}}$ such that

- (iii) $\{\varphi(x, c_\eta)\}_{\eta\in X}$ is consistent if $X \subseteq \omega^{\leq\omega}$ is an increasing comb,
- (iv) $\{\varphi(x, c_\eta)\}_{\eta\in X}$ is inconsistent if $X \subseteq \omega^{\leq\omega}$ is a $\trianglelefteq$-chain with length k.

By [1, Lemma 3.5], $(c_\eta)_{\eta\in\omega^\omega}$ is δ-indiscernible since $(c_\eta)_{\eta\in\omega^{\leq\omega}}$ is strongly indiscernible. Since T is weakly treeless, $(c_\eta)_{\eta\in\omega^\omega}$ is $<_{lex}$-indiscernible. For any antichain $X \subseteq \omega^{\leq\omega}$, there exists an antichain $Y \subseteq \omega^\omega$ such that $X \sim_{L_0} Y$. And for any antichain $Y \subseteq \omega^\omega$, there exists an increasing comb $Z \subseteq \omega^\omega$ such that $(c_\eta)_{\eta\in Y} \equiv (c_\eta)_{\eta\in Z}$ since $(c_\eta)_{\eta\in\omega^\omega}$ is $<_{lex}$-indiscernible. Thus we have

- (v) $\{\varphi(x, c_\eta)\}_{\eta\in X}$ is consistent if $X \subseteq \omega^{\leq\omega}$ is an antichain.

By (iv) and (v), T has k-ATP. But then, T has ATP by [1, Lemma 3.20]. We have a contradiction. □

6. Questions

Question 6.1. If T is meet-eliminating, then $\mathop{\smile\!\!\!\!\!\!\mid}^{GS}$ satisfies symmetry and base monotonicity??

Question 6.2. Suppose T is weakly treeless and NCTP. Is T NBTP? [9, Definition 5.1]

Question 6.3. [8, Question 3.14] It is known that weak treeless implies treeless in NIP [8, Proposition 3.13]. Does weak treeless imply treeless in general? Is there a condition that makes weak treeless to imply treeless? Or is there an example of non-treeless theory having weak treeless?

Question 6.4. [8, Question 3.15] L_δ consists of only two relation symbols Δ and $<_{lex}$, where Δ is quaternary. Considering this, we can check if the following weakened version of Artem's question is true. Namely, does (*) imply weak treeless? where

(*) If $(a_\eta)_{\eta\in\omega^\omega}$ is δ-indiscernible, $\eta_0 <_{lex} \eta_1 <_{lex} \eta_2 <_{lex} \eta_3$, and $\nu_0 <_{lex} \nu_1 <_{lex} \nu_2 <_{lex} \nu_3$, then $a_{\eta_0}a_{\eta_1}a_{\eta_2}a_{\eta_3} \equiv a_{\nu_0}a_{\nu_1}a_{\nu_2}a_{\nu_3}$.

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