# Connected components of sets definable in almost o-minimal structures and d-minimal structures

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概要

This note gives an intuitive explanation on the complicated proof of the author's results in [5] on connected components of sets definable in almost o-minimal structures and d-minimal structures.

## 1 Motivation

First we fix notations. Let $\mathrm{Con}(T)$ is the collection of connected components of a topological space $T$. $\mathfrak{R}$ is always an expansion of the ordered set of reals $\mathbb{R}$. 'Definable' means 'definable in $\mathfrak{R}$ with parameters'.

If $\mathfrak{R}$ is o-minimal, every connected component of a definable set is definable. This is an easy corollary of definable cell decomposition theorem in o-minimal structures. However, it is not the case in slightly wilder structures as indicated in the following two examples.

*Example* 1.1 ([2]). In $(\mathbb{R}, +, \mathbb{N})$, a connected component of a definable set is not necessarily definable. However, every connected component of a set definable in $(\mathbb{R}, +, \mathbb{N}, \cdot|_{\mathbb{N}})$ is definable.

*Example* 1.2 ([7]). In $(\mathbb{R}, +, \cdot, 2^{\mathbb{Z}})$, a connected component of a definable set is not necessarily definable.

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Both $(\mathbb{R}, +, \mathbb{N})$ and $(\mathbb{R}, +, \mathbb{N}, \cdot|_{\mathbb{N}})$ are almost o-minimal structures. $(\mathbb{R}, +, \cdot, 2^{\mathbb{Z}})$ is a d-minimal structure by [3]. Almost o-minimal structures and d-minimal structures are considered to be tame and their properties are investigated in several papers such as [4, 6]. Throughout, for simplicity, let $\mathcal{P}$ denote the property of $\mathfrak{R}$ that a connected component of a definable set is definable. Example 1.1 says that the almost o-minimal structure $(\mathbb{R}, +, \mathbb{N})$ does not possess $\mathcal{P}$, but its almost o-minimal expansion $(\mathbb{R}, +, \mathbb{N}, \cdot|_{\mathbb{N}})$ enjoys property $\mathcal{P}$. In this paper, we consider the following question:

**Does every almost o-minimal structure $\mathcal{R}$ have an almost o-minimal expansion $\mathfrak{R}^\natural$ of $\mathfrak{R}$ enjoying property $\mathcal{P}$?**

Concerning Example 1.2, we can also consider the following questions:

**Does every d-minimal structure $\mathcal{R}$ have a d-minimal expansion $\mathfrak{R}^\natural$ of $\mathfrak{R}$ enjoying property $\mathcal{P}$?**

In [5], the author solves these problems affirmatively under that extra assumption $\mathfrak{R}$ is an expansion of either the ordered group of reals or the ordered field of reals. The proof in [5] is complicated, and this note gives an intuitive and illustrative explanation on it.

## 2 Almost o-minimal case: the multi-cell decomposition implies the main theorem

The multi-cell decomposition theorem [4, Theorem 4.22] says that every definable sets is decomposed into finitely many multi-cells. Intuitively speaking, a multi-cell is a countable union of disjoint cells arranged neatly. Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection forgetting the last coordinate. Remarkable properties of a multi-cell $C \subseteq \mathbb{R}^n$ are

1. $\pi(C)$ is a multi-cell and
2. for every $\mathcal{C} \subseteq \mathrm{Con}(C)$, $\pi(\bigcup_{D\in\mathcal{C}} D) = \bigcup_{D\in\mathcal{C}} \pi(D)$.

More precisely, a definable subset $X$ of $\mathbb{R}^n$ is a *multi-cell* if it satisfies the following conditions:

- If $n = 1$, either $X$ is a discrete definable set or all connected components of the definable set $X$ are open intervals.

- If $n > 1$, the projection image $\pi(X)$ is a multi-cell and, for any connected component $Y$ of $X$, $\pi(Y)$ is a connected component of $\pi(X)$ and $Y$ is one of the following forms:

$$\begin{aligned} &Y = \pi(Y) \times \mathbb{R}, \\ &Y = \{(x, y) \in \pi(Y) \times \mathbb{R} \mid y = f(x)\}, \\ &Y = \{(x, y) \in \pi(Y) \times \mathbb{R} \mid y > f(x)\}, \\ &Y = \{(x, y) \in \pi(Y) \times \mathbb{R} \mid y < g(x)\} \text{ and} \\ &Y = \{(x, y) \in \pi(Y) \times \mathbb{R} \mid f(x) < y < g(x)\} \end{aligned}$$

  for some continuous functions $f$ and $g$ defined on $\pi(Y)$ with $f < g$.

The left figure of Figure 1 conceptually illustrates a multi-cell, but the right one doesn't.

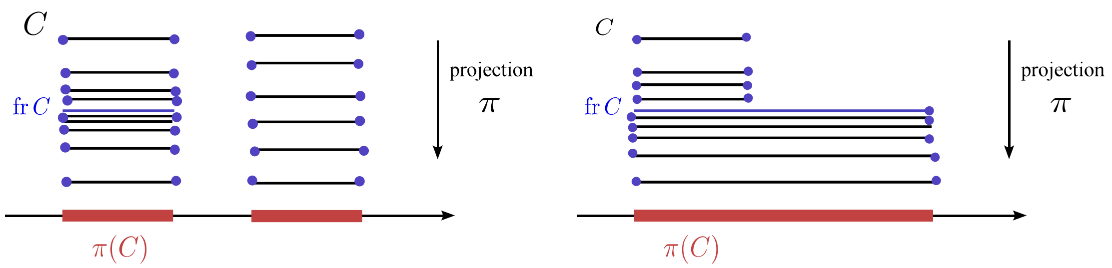


図 1　Multi-cell case versus Non-multi-cell

Using [4, Theorem 4.22] and these properties, it is not a hard task to prove that, for every set $X \subseteq \mathbb{R}^n$ definable in $\mathfrak{R}$ and every $\mathcal{C} \subseteq \mathrm{Con}(X)$, there exist finite many sets $Y_1, \ldots, Y_m \subseteq \mathbb{R}^{n-1}$ definable in $\mathfrak{R}$ and $\mathcal{C}_i \subseteq \mathrm{Con}(Y_i))$ such that

$$\pi(\bigcup_{C \in \mathcal{C}} C) = \bigcup_{i=1}^{m} \bigcup_{C \in \mathcal{C}_i} C.$$

Using this fact, we can prove the following theorem:

**Theorem 2.1.** *If $\mathfrak{R}$ is an almost o-minimal expansion of the ordered group of reals, there exists an almost o-minimal expansion $\mathfrak{R}^\natural$ of $\mathfrak{R}$ such that, for every definable set $X$ and a subfamily $\mathcal{C}$ of $\mathrm{Con}(X)$, $\bigcup_{C \in \mathcal{C}} C$ is definable in $\mathfrak{R}^\natural$. In particular, $\mathfrak{R}^\natural$ enjoys property $\mathcal{P}$.*

This theorem is an affirmative answer to the first problem in Section 1.

## 3 D-minimal case: how to decompose a definable set into finitely many multi-cells

How about d-minimal case? Does $(\mathbb{R}, +, \cdot, 2^{\mathbb{Z}})$ have a d-minimal expansion possessing property $\mathcal{P}$? The answer is affirmative.

**Theorem 3.1.** *If $\mathfrak{R}$ is a d-minimal expansion of the ordered field of reals, there exists a d-minimal expansion $\mathfrak{R}^\natural$ of $\mathfrak{R}$ such that, for every definable set $X$ and a subfamily $\mathcal{C}$ of* $\operatorname{Con}(X)$*, $\bigcup_{C \in \mathcal{C}} C$ is definable in $\mathfrak{R}^\natural$. In particular, $\mathfrak{R}^\natural$ enjoys property $\mathcal{P}$.*

In order to prove the above theorem, we need a d-minimal version of multi-cell decomposition theorem. In the proof of the multi-cell decomposition theorem, the hardest case is the case where, for every $x \in \pi(M)$, the fiber $M_x := \{y \in \mathbb{R} \mid (x, y) \in M\}$ has an empty interior. We concentrate on this case in this section. The author showed that, if a definable set $M$ satisfies some first-order technical conditions, it is a multi-cell. Some of the notions necessary for describing such conditions come from Thamrongthanyalak's study [8], and the others are the author's inventions. If $M$ is a multi-cell, every connected component of $X$ is the graph of a continuous function defined on a connected component of $\pi(X)$. So, the notion of connected graphs described below is introduced.

**Definition 3.2.** A connected subset $C$ of $\mathbb{R}^n$ is a *connected graph* on a definable subset $V$ of $\mathbb{R}^{n-1}$ if $C$ is the graph of a continuous function $f_C : V \to \mathbb{R}$. For a connected graph $C$ on $V$, we put

$$C^+ := \{(x, t) \in V \times \mathbb{R} \mid t > f_C(x)\} \text{ and}$$
$$C^- := \{(x, t) \in V \times \mathbb{R} \mid t < f_C(x)\}.$$

If $X$ is a connected set with $\pi(X) \subseteq V$ and $X \cap C = \emptyset$, we have either $X \subseteq C^+$ or $X \subseteq C^-$. Intuitively speaking, $X$ is above $C$ or below $C$. We denote $X >_V \underline{C}$ in the former case and $X <_V \underline{C}$ in the latter case. Underlines below $C$ are used to clarify which is a connected graph.

Differently from the almost o-minimal case, $\pi^{-1}(\pi(M)) \cap \operatorname{fr}(M)$ is not necessarily empty even if $M$ is a multi-cell, where $\operatorname{fr}(M)$ is the frontier of $M$ in $\mathbb{R}^n$. However, if $M$ is a multi-cell, it is expected that $\pi^{-1}(\pi(M)) \cap \operatorname{fr}(M)$ is a union of multi-cells whose images under $\pi$ coincide with $\pi(M)$. This is the reason why the notion of well-ordered pair was introduced.

**Definition 3.3.** A definable submanifold $M$ of $\mathbb{R}^n$ is *pseudo-special* if $\pi(M)$ is a submanifold of $\mathbb{R}^{n-1}$ and, for every $x \in M$, there exists an open box $B$ in $\mathbb{R}^n$ containing $x$ such that $\pi|_{B \cap M}$ is a homeomorphism onto $\pi(B) \cap \pi(M)$.

Let $M$ and $P$ be definable subsets of $\mathbb{R}^n$. We say that $(M, P)$ is a *well-ordered pair* if the following conditions are satisfied:

(a) $M$ is pseudo-special.
(b) $\pi(P) = \pi(M)$;
(c) $P \subseteq \operatorname{fr}(M) \cap \pi^{-1}(\pi(M))$;
(d) $P$ is a multi-cell.

If $(M, P)$ is a well-ordered pair, for every connected component $C$ of $P$ and every connected component $X$ of $M$, we have either $X >_V \underline{C}$ or $X <_V \underline{C}$, where $V = \pi(X) = \pi(P)$. Figure 2 illustrates a non-multi-cell, and $(C, \operatorname{fr}(C) \cap \pi^{-1}(\pi(C)))$ is not a well-ordered pair.

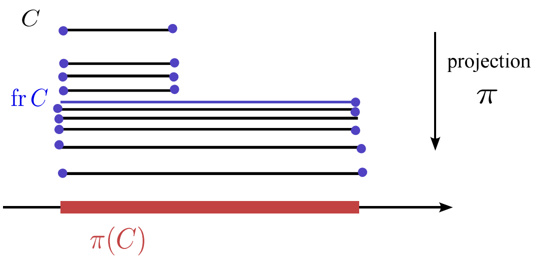


図 2　Non multi-cell case

See Figure 3. $M$ is not obviously a multi-cell. The closure $\operatorname{cl}(C^+ \cap M)$ does not contain $C$. We should consider the closure of the part of $M$ above $C$ and below $C$ separately. To do this, we consider $\operatorname{Adh}^+(M, P)$ and $\operatorname{Adh}^-(M, P)$.

For a well-ordered pair $(M, P)$, every connected component of the multi-cell $P$ is a connected graph. Let $\operatorname{Adh}^+(M, P)$ be the set of points $z \in P$ such that $z \in \operatorname{fr}(M \cap C^+)$, where $C \in \operatorname{Con}(P)$ with $z \in C$. We define $\operatorname{Adh}^-(M, P)$ in the same

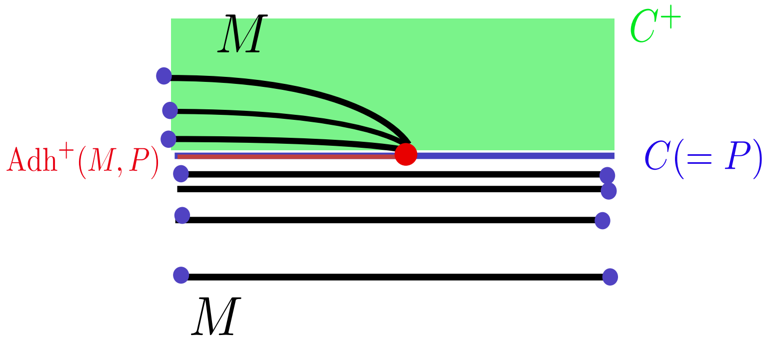


図 3　Non multi-cell case

manner. For every $x \in \pi(M)$, set

$$\mathrm{Adh}^+(M,P,x) := \{x\} \times \{t \in \mathbb{R} \mid (x,t) \in P \cap \mathrm{fr}(M \cap (\{x\} \times (t,\infty)))\}.$$

We define $\mathrm{Adh}^-(M,P,x)$ similarly.

If connected components of $M$ are 'neat', we have

$$\mathrm{Adh}^+(M,P) \cap \pi^{-1}(x) = \mathrm{Adh}^+(M,P,x) \text{ and } \mathrm{Adh}^-(M,P) \cap \pi^{-1}(x) = \mathrm{Adh}^-(M,P,x)$$

for every $x \in \pi(M)$.

In fact, in Figure 3, the equality $\mathrm{Adh}^+(M,P) \cap \pi^{-1}(x) = \mathrm{Adh}^+(M,P,x)$ does not hold at the center of the figure. An well-ordered pair is called *extremely well ordered* if the above two equalities hold for every $x \in \pi(M)$.

We have explained main ingredients of a key lemma. Here is the full description of the key lemma.

**Lemma 3.4** (Key Lemma). *A pseudo-special submanifold $M$ of $\mathbb{R}^n$ satisfying the following conditions is a multi-cell.*

*(1) $M$ enjoys property (†).*
*(2) $\pi(M)$ is a multi-cell.*
*(3) $\mathrm{cl}(M) \cap \pi^{-1}(x) = \mathrm{cl}(M \cap \pi^{-1}(x))$ for every $x \in \pi(M)$;*
*(4) There exists $N \in \mathbb{N}$ with $\mathrm{rank}(\mathrm{cl}(M \cap \pi^{-1}(x))) = N$ for every $x \in \pi(M)$;*
*(5) If $N > 1$, for every $1 \leq i < N$, the definable set*

$$P_i = \bigcup_{x \in \pi(M)} \mathrm{iso}_i(\mathrm{cl}(M) \cap \pi^{-1}(x))$$

*is a multi-cell. Observe that this condition implies that* $(M, P_i)$ *and* $(P_j, P_k)$ *are well-ordered pairs for* $1 \leq i \leq N-1$ *and* $1 \leq j < k \leq N-1$*;*

*(6)* $(M, P_i)$ *and* $(P_j, P_k)$ *are extremely well-ordered pairs for* $1 \leq i \leq N-1$ *and* $1 \leq j < k \leq N-1$.

Several conditions in this lemma are not explained yet. Property (†) is technical, and it is not worth introducing in this conceptional paper. $\mathrm{iso}(S)$ denotes the set of isolated points in $S$. For every nonnegative integer $m$, we define $\mathrm{iso}_m(S)$ as follows:

- $\mathrm{iso}_0(S) = \mathrm{iso}(S)$;
- $\mathrm{iso}_m(S) = \mathrm{iso}(S \setminus \bigcup_{i=0}^{m-1} \mathrm{iso}_i(S))$;

Define $\mathrm{rank}(S)$ as the minimum of positive integers $m$ such that $\mathrm{iso}_m(S) = \emptyset$.

Our proof of Lemma 3.4 in [5] is long and we do not repeat it here.

In the last of this note, we give a comment on the generalization of our theorems to the case where the underlying space is not $\mathbb{R}$. First of all, there is an o-minimal structure in which every connected component of definable sets is singleton. See [1, Example 2.4.1]. So, a straightforward generalization of our theorem to the non-real case is false even if $\mathfrak{R}$ is o-minimal. In o-minimal structures, we consider 'definably connected components' instead of connected components. In almost o-minimal structures, the notion of semi-definable connected components is introduced in [4], and it may be a substitute of that of definable connected components. In d-minimal structures, such substitutes are not found yet. So a reasonable formulation of our theorems in a general context is hard.

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