On theories with the strong tree property

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A stable theory T is said to have the tree property, if there are a non-isolated type $p \in S(T)$ and $a, b, c \models p$ with $b \downarrow c$ and $\mathrm{tp}(bc/a)$ isolated. Herwig proved that any stable Ehrenfeucht theory has a type with the tree property. In this shot note, we introduce the notion called strong tree property, and prove that any stable Ehrenfeucht theory has the strong tree property. This is a generalization of Herwig's result.

1 Ehrenfeucht theories

Let T be a complete first-order theory in a countable language. Let $I(\omega, T)$ denote the number of non-isomorphic countable models of T. Then T is said to be Ehrenfeucht, if $1 < I(\omega, T) < \aleph_0$. It is seen that if T is Ehrenfeucht, then it is small, i.e, $S(T)$ is countable. It is also known that if T is Ehrenfeucht, then there is a powerful type $p \in S(T)$, i.e, every model realizing p realizes any type over $\emptyset$. A well-known example of Ehrenfeucht theories is $\mathrm{Th}(\mathbb{Q}, <, c_1, c_2, \cdots)$, where $\mathbb{Q}$ is the set of rational numbers, and $(c_i)_{i \in \omega}$ is an increasing sequence. In fact, it can seen that $I(\omega, T) = 3$. The Lachlan conjecture[4] says that any Ehrenfeucht theory is unstable. Our result is related to this conjecture.

2 Tree property

Definition 2.1 (Herwig[2]) Let T be a stable theory. Then a non-isolated $p \in S(T)$ is said to have the tree property, if there are $a, b, c \models p$ with $b \downarrow c$ and $\mathrm{tp}(bc/a)$ isolated.

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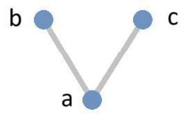


A theory T is said to have the tree property, if it has a type with the tree property.

Fact 2.2 (Herwig[2]) If T is a stable Ehrenfeucht theory, then it has the tree property.

Proof. Since T is Ehrenfeucht, there is a powerful type p. Take any $a \models p$, and let M be a model prime over a. Take any $b, c \models p$ with $b \mathop{\smile\kern-0.65em|}\, c$. Since p is powerful, we can assume that $b, c \in M$. Then $\mathrm{tp}(bc/a)$ is isolated. Hence p has the tree property.

Example 2.3 (Bouscaren-Poizat[1]) The theory of the free pairing function is stable and has the tree property: Take a set M with a 2-ary function f such that $f : M^2 \to M$ is a bijection, and for any term $t(\bar{x})$, $M \models t(x_1, \cdots, x_n) \neq x_i$ for all i. Let f_1, f_2 be definable functions such that the function $x \mapsto (f_1(x), f_2(x))$ is the inverse of f. By QE, $\mathrm{Th}(M)$ is stable. Take the free type p, that is, for $a \models p$ we have $f_{i_1} \cdots f_{i_n}(a) = f_{j_1} \cdots f_{j_m}(a)$ iff $i_1 \cdots i_n = j_1 \cdots j_m$. Then p has the tree property. In fact, for $a \models p$, let $b = f_1(a), c = f_2(a)$. Then $b \mathop{\smile\kern-0.65em|}\, c$, and $\mathrm{tp}(bc/a)$ is isolated (in particular, algebraic). Note that the theory is not small.

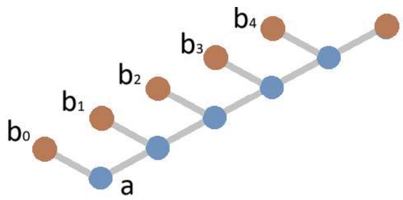
Question 2.4 Is there a small stable theory with the tree property?

3 Strong tree property

Definition 3.1 Let T be a stable theory. Then a non-isolated $p \in S(T)$ is said to have the strong tree property, if there are $a, b, c \models p$ with $b \mathop{\smile\kern-0.65em|}\, c$ and $\mathrm{tp}(bc/a)$ non-algebraic isolated. A theory T is said to have the strong tree property, if it has a type with the tree property.

Proposition 3.2 If T is a stable Ehrenfeucht theory, then it has the strong tree property.

Proof. By Fact 2.2, there is a type $p \in S(T)$ with the tree property. Suppose, by way of contradiction, that p does not have the strong tree property. Then there are $a, b, c \models p$ with $b \mathop{\smile\!\!\!\!\!\!|}\; c$ and $\mathrm{tp}(bc/a)$ algebraic. Using a, b, c, we have an independent set $B = \{b_i\}_{i\in\omega}$ with $b_i \in \mathrm{acl}(a)$ for each i. (See the picture below.)



Then it can be seen that $S(T)$ is uncoutable. (This argument can be found in [3], [2].) The proof is as follows: By reenumerating B, we can assume that $B = \{b_i\}_{i\in\mathbb{Q}}$. For $r \in \mathbb{R}$, let B_r denote $\{b_i\}_{i<r}$. Take a_r with $a_r \mathop{\smile\!\!\!\!\!\!|}_{B_r} a$ and $\mathrm{tp}(a_r/B_r) = \mathrm{tp}(a/B_r)$. Then $\mathrm{tp}(aa_r) \neq \mathrm{tp}(aa_s)$ for any $r, s \in \mathbb{R}$ with $r < s$. In fact, we have $R_\Delta(a/a_r) > R_\Delta(a/a_s)$ for some Δ. Hence $S(T)$ is uncoutable. This contradicts the assumption that T is Ehrenfeucht.

Example 3.3 There is a stable theory with the strong tree property: This can be obtained by an easy modification of Example 2.1: Let (M, f, f_1, f_2) be a structure with the free pairing function. Let X be an infinite set, and let $N = M \times X$. Let $\pi : N \to M$ be a projection with $(a, x) \mapsto a$. For $i = 1, 2$, let $P_i(*, *)$ be a binary relation with $N \models P_i(a^*, b^*)$ iff $M \models f_i(\pi(a^*)) = \pi(b^*)$. Then $\mathrm{Th}(N, P_1, P_2, \pi)$ is stable and has the strong tree property. Note that the theory is not small.

Question 3.4 Is there a small stable theory with the strong tree property?

References

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