MODULAR LAW AND THE GEOMETRY OF THORN FORKING

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ABSTRACT. This is a brief note of [Y1]. We mention that modular law for algebraic closure$^{\mathrm{eq}}$ is equivalent to rosiness and one-basedness with respect to thorn forking. It is known that modular law for algebraic closure$^{\mathrm{eq}}$ is equivalent to base monotonicity for algebraic independence. Joonhee Kim shows that if Kim-forking has base monotonicity, then Kim-forking coincides with forking. We pose questions about Kim-forking without base monotonicity. We are writing [Y1], so we omit proofs of facts which will appear there.

1. AXIOMS OF STRICT INDEPENDENCE RELATIONS.

Let $\mathcal{M}$ be a sufficiently saturated model of a complete L-theory T. Let $\bar{a}$ be a finite tuple of $\mathcal{M}$. $e \in \mathcal{M}^{\mathrm{eq}}$ iff $e = (\bar{a})_E$, where $E(\bar{x}, \bar{y})$ is an $\emptyset$-definible equivalence relation with $\mathrm{lh}(\bar{a}) = \mathrm{lh}(\bar{x}) = \mathrm{lh}(\bar{y})$ and some $\bar{a} \subset \mathcal{M}$. For $e \in \mathcal{M}^{\mathrm{eq}}$ and $A \subset \mathcal{M}^{\mathrm{eq}}$ we write $e \in \mathrm{acl}^{\mathrm{eq}}(A)$ if $|\{\sigma(e) : \sigma \in \mathrm{Aut}(\mathcal{M}^{\mathrm{eq}}/A)\}|$ is finite. We write $e \in \mathrm{dcl}^{\mathrm{eq}}(A)$ if $|\{\sigma(e) : \sigma \in \mathrm{Aut}(\mathcal{M}^{\mathrm{eq}}/A)\}| = 1$. For $B' := \sigma(B)$ with $\sigma \in \mathrm{Aut}(\mathcal{M}^{\mathrm{eq}}/A)$ we write that $B' \equiv_A B$ or $B' \models \mathrm{tp}(B/A)$.
H.Adler [A] introduces a strict independence relation $* \mathop{\smile\hskip-0.9em ^{|}\ }_* *$ for triplet of small subsets of $\mathcal{M}^{\mathrm{eq}}$ satisfying (1)-(9):

(1) invariance: If $A \mathop{\smile\hskip-0.9em ^{|}\ }_B C$ and $\mathrm{tp}(ABC) = \mathrm{tp}(A'B'C')$, then $A' \mathop{\smile\hskip-0.9em ^{|}\ }_{B'} C'$.
(2) monotonicity: If $A \mathop{\smile\hskip-0.9em ^{|}\ }_B C, A' \subseteq A$ and $C' \subseteq C$, then $A' \mathop{\smile\hskip-0.9em ^{|}\ }_B C'$.
(3) RIGHT BASE MONOTONICITY: If $A \mathop{\smile\hskip-0.9em ^{|}\ }_B D$ and $B \subseteq C \subseteq D$, then $A \mathop{\smile\hskip-0.9em ^{|}\ }_C D$. This property is very important for this paper.
(4) left transitivity: If $B \subseteq C \subseteq D, D \mathop{\smile\hskip-0.9em ^{|}\ }_C A$ and $C \mathop{\smile\hskip-0.9em ^{|}\ }_B A$, then $D \mathop{\smile\hskip-0.9em ^{|}\ }_B A$.
(5) left normality: $A \mathop{\smile\hskip-0.9em ^{|}\ }_B C$ implies $AB \mathop{\smile\hskip-0.9em ^{|}\ }_B C$.
(6) left extension: If $A \mathop{\smile\hskip-0.9em ^{|}\ }_B C$ and $C \subseteq D$, then there exists $A' \models \mathrm{tp}(A/BC)$ such that $A' \mathop{\smile\hskip-0.9em ^{|}\ }_B D$.
(7) left finite character: If $\bar{a} \mathop{\smile\hskip-0.9em ^{|}\ }_B C$ for any finite tuple $\bar{a} \subset A$, then $A \mathop{\smile\hskip-0.9em ^{|}\ }_B C$.
(8) local character: For any A there exists a cardinal $\kappa(A)$ such that, for any B, there exists $B_0 \subseteq B$ with $|B_0| < \kappa(A)$ and $A \mathop{\smile\hskip-0.9em ^{|}\ }_{B_0} B$.
(9) anti-reflexivity: $A \mathop{\smile\hskip-0.9em ^{|}\ }_B A$ implies $A \subseteq \mathrm{acl}^{\mathrm{eq}}(B)$.

T is said to be *rosy* if there exists a strict independence relation on $\mathcal{M}^{\mathrm{eq}}$. Theorem 1.14 in [A] shows that (1)-(8) imply symmetry: $A \mathop{\smile\hskip-0.9em ^{|}\ }_B C \Leftrightarrow C \mathop{\smile\hskip-0.9em ^{|}\ }_B A$.

Date: March 31, 2026.
1991 *Mathematics Subject Classification.* 03C45, 03C64.
Key words and phrases. modular law, thorn forking, CM-triviality, weak canonical base.

From now on we work in a saturated rosy eq-structure $(\mathcal{M}^{\mathrm{eq}}, \perp)$. We use the following fact (Remark 2.1 in [Y]) by symmetry and (1)-(8) except (7). But symmetry needs (7).

Fact 1.1. *For any $A, B, C \subset \mathcal{M}^{\mathrm{eq}}$ we have $A \perp_B C$ iff $\mathrm{acl}^{\mathrm{eq}}(A) \perp_{\mathrm{acl}^{\mathrm{eq}}(B)} \mathrm{acl}^{\mathrm{eq}}(C)$.*

Question 1.2. *Is this fact true for any NSOP_1 theory $(\mathcal{M}^{\mathrm{eq}}, \perp^{K,f})$ having existence property for $\perp^f$?*

2. Algebraic independence relation

Here we introduce three conditions on any ternary relation $* \perp_* *$ for the later argument.

Definition 2.1. (1) *We say that $\perp$ has the existence if $\bar{a} \perp_A A$ for any $\bar{a}, A \subset \mathcal{M}^{\mathrm{eq}}$.*

(2) *We say that $\perp$ has the full existence if for any A and $B \subseteq C$ in $\mathcal{M}^{\mathrm{eq}}$ there exists $A' \models \mathrm{tp}(A/B)$ such that $A' \perp_B C$.*
(3) *We say that $\perp$ has the right extensibility if $\bar{a} \perp_B C$ and $B \subseteq C \subseteq D$ implies there exists $A' \models \mathrm{tp}(A/C)$ such that $A' \perp_B D$ in $\mathcal{M}^{\mathrm{eq}}$.*

Remark 2.2. *Suppose that invariance and right transitivity of $\perp$. Then the following are equivalent.*

(1) *the full existence of $\perp$.*
(2) *the existence of $\perp$ and right existensibility of $\perp$*

Proof. ((1) $\Leftarrow$ (2)) is clear. ((1) $\Rightarrow$ (2)) : Suppose that $B \subseteq C \subseteq D$ and $A \perp_B C$. The full existence implies there exists $A' \models \mathrm{tp}(A/C)$ such that $A' \perp_C D$. As $A' \perp_B C$ by invariance, the right transitivity implies $A' \perp_B D$. □

The full existence property for algebraic independence relation $\perp^a$ is firstly shown by H.Adler [A1].

The definiton of algebraic independence relation:
$A \perp^a_C B \Leftrightarrow_{\mathrm{def}} \mathrm{acl}^{\mathrm{eq}}(AC) \cap \mathrm{acl}^{\mathrm{eq}}(BC) = \mathrm{acl}^{\mathrm{eq}}(C)$.

W.Johnson gives a different proof of the full existence condition for $\perp^a$ from H.Adler's one by using B.H.Neumann lemma (Lemma 4.2.1 on pp.140 in [H]). A.Tsuboi gives a transparent proof of the full existence property for algebraic independence relation.

3. Facts on modular law

Fact 3.1. (1) *Put $A' = \mathrm{acl}^{\mathrm{eq}}(A), B' = \mathrm{acl}^{\mathrm{eq}}(B)$. Then $\mathrm{acl}^{\mathrm{eq}}(AB) = \mathrm{acl}^{\mathrm{eq}}(A'B')$. On the other hand, we have $\mathrm{acl}^{\mathrm{eq}}(A' \cap B') = A' \cap B'$ and $\mathrm{acl}^{\mathrm{eq}}(A \cap B) \subseteq A' \cap B'$. We do not know $\mathrm{acl}^{\mathrm{eq}}(A \cap B) \supseteq A' \cap B'$.*

(2) *The algebraic independence relation $\perp^a$ is a strict independece relation except RIGHT BASE MONITONICITY. See* [A1].
(3) *$A \perp^a_B C$ iff $\mathrm{acl}^{\mathrm{eq}}(A) \perp^a_{\mathrm{acl}^{\mathrm{eq}}(B)} \mathrm{acl}^{\mathrm{eq}}(C)$ for any $A, B, C \subset \mathcal{M}^{\mathrm{eq}}$.*
(4) *BASE MONOTONICITY of $\perp^a$: $A \perp^a_B D$ and $B \subseteq C \subseteq D$ implies $A \perp^a_C D$ is equivalent to MODULAR LAW for $\mathrm{acl}^{\mathrm{eq}}$: $D \cap \mathrm{acl}^{\mathrm{eq}}(AC) = \mathrm{acl}^{\mathrm{eq}}((A \cap D)C) = \mathrm{acl}^{\mathrm{eq}}(D \cap AC)$ holds for any algebraically closed $A, C, D \subset \mathcal{M}^{\mathrm{eq}}$ with $C \subseteq D$. And we may assume that A, B, C, D are algebraically closed in $\mathcal{M}^{\mathrm{eq}}$ by (3).*

(5) *If* $(\mathcal{M}^{\mathrm{eq}}, \downarrow)$ *is one-based and has base monotonicity and anti-reflexivity, then modular law follows.*
(6) $A \downarrow^m_B D$ *denotes* $A \downarrow^a_C D$ *for any* $C \subset \mathcal{M}^{\mathrm{eq}}$ *such that* $B \subseteq C \subseteq \mathrm{acl}^{\mathrm{eq}}(BD)$ *i.e. Modular law holds. Then* $\downarrow^m \Rightarrow \downarrow^a$*, and* $\downarrow^a$ *has BASE MONOTONICITY iff* $\downarrow^a = \downarrow^m$*.*
(7) *For any ternary relation* $* \downarrow_* *$*, we define* $\downarrow^*$ *as follows:* $A \downarrow^*_B C$ *denotes for any* $D \supseteq C$ *there exists* A' *such that* $A' \equiv_{BC} A$ *such that* $A' \downarrow_B D$*. i.e.* $\downarrow^*$ *is an extensible independence relation for* $\downarrow$*. We have that* $\downarrow^* \Rightarrow \downarrow$*,* $\downarrow = \downarrow^*$ *iff* $\downarrow$ *satisfies right extensiblit . And* $\downarrow^{d^*} = \downarrow^f$ *and* $\downarrow^{m^*}$ *is the thorn forking relation* $\downarrow^{\text{þ}}$*.*
(8) *We have* $\downarrow \Rightarrow \downarrow^{\text{þ}} (:= \downarrow^{m^*}) \Rightarrow \downarrow^m \Rightarrow \downarrow^a$ *for a strict independence relation* $\downarrow$*.*
(9) $\downarrow^a = \downarrow^{a^*}$ *by Remark 2.2 and the full existence of algebraic independence.*
(10) *Suppose that* $(\mathcal{M}^{\mathrm{eq}}, \downarrow)$ *is rosy and* $\downarrow$*-weak canonical base exists for any strong type. Then* $\downarrow = \downarrow^{\text{þ}}$ *by Theorem 3.3 in* [A].

Proof. (1) Note that $A'B' \subseteq \mathrm{acl}^{\mathrm{eq}}(A'B') \subseteq \mathrm{acl}^{\mathrm{eq}}(AB)$ since $A'B' \subseteq \mathrm{acl}^{\mathrm{eq}}(AB)$. On the other hand, as $AB \subseteq A'B'$, we have $\mathrm{acl}^{\mathrm{eq}}(AB) \subseteq \mathrm{acl}^{\mathrm{eq}}(A'B')$.

(2): See [A1]. (3): Put $A' := \mathrm{acl}^{\mathrm{eq}}(A), B' = \mathrm{acl}^{\mathrm{eq}}(B), C' = \mathrm{acl}^{\mathrm{eq}}(C)$. By (1) we have $\mathrm{acl}^{\mathrm{eq}}(AC) = \mathrm{acl}^{\mathrm{eq}}(A'C')$, so we see that $A \downarrow^a_C B$ iff $A' \downarrow^a_{C'} B'$.
(4): It is shown in [A], but we give the proof. We only consider algebraically closed subsets in $\mathcal{M}^{\mathrm{eq}}$ by (3).
Base monotonicity implies modular law : We have $A \downarrow^a_{A\cap D} D$. By base monotonicity and $C \subseteq D$, we have $A \downarrow^a_{(A\cap D)C} D$, which implies $D \cap \mathrm{acl}^{\mathrm{eq}}(AC) = \mathrm{acl}^{\mathrm{eq}}((A\cap D)C)$.
Modular law implies base monotonicity: Suppose that $A \downarrow_B D$ and $B \subseteq C \subseteq D$. $C \subseteq D \cap \mathrm{acl}^{\mathrm{eq}}(AC) = \mathrm{acl}^{\mathrm{eq}}((D\cap A)C)$ by modular law. As $\mathrm{acl}^{\mathrm{eq}}(AB) \cap D = B$, we have $C \subseteq \mathrm{acl}^{\mathrm{eq}}(AC \cap D) = \mathrm{acl}^{\mathrm{eq}}((D\cap A)C) \subseteq \mathrm{acl}^{\mathrm{eq}}(BC) = C$. So we see $A \downarrow^a_C D$ as desired. (5): Clear. (6) (7) (8) (9) (10) : See [A]. □

4. A new characterization of modular law

The definition of one-basedness for $\downarrow$: $D \downarrow_{D\cap A} A$ for any $A = \mathrm{acl}^{\mathrm{eq}}(A), D = \mathrm{acl}^{\mathrm{eq}}(D) \subset \mathcal{M}^{\mathrm{eq}}$. Note that $(\mathcal{M}^{\mathrm{eq}}, \downarrow^a)$ is always one-based.
We show the following fact in [Y1], so we omit its proof.

Fact 4.1. (1) *Modular law holds (iff* $\downarrow^a = \downarrow^m$*) iff* $\downarrow^a = \downarrow^{\text{þ}}$ *with rosiness iff* $(\mathcal{M}^{\mathrm{eq}}, \downarrow^{\text{þ}})$ *is rosy one-based.*
(2) *Let* $(\mathcal{M}^{\mathrm{eq}}, \downarrow)$ *be rosy. The following are equivalent.*
  (a) $(\mathcal{M}^{\mathrm{eq}}, \downarrow)$ *is one-based.*
  (b) $\downarrow = \downarrow^a = \downarrow^{\text{þ}}$*.*
  (c) $\downarrow^a = \downarrow^m$ *and the existence of* $\downarrow$*-weak canonical base for any strong type.*
(3) *Question : Find a rosy non one-based* $(\mathcal{M}^{\mathrm{eq}}, \downarrow)$ *such that* $(\mathcal{M}^{\mathrm{eq}}, \downarrow^{\text{þ}})$ *is one-based. Can we find it as a non one-based treeless* $(\mathcal{M}^{\mathrm{eq}}, \downarrow^{G,S})$*? Here* $\downarrow^{G,S} (\Rightarrow \downarrow^{\text{þ}})$ *is the generic stability independence in treeless theories. See* [KRS].

(4) $(\mathcal{M}^{\mathrm{eq}}, \mathop{\smile\!\!\!\!\!\!\mid}^{a})$ *is always CM-trivial.*
(5) *If* $(\mathcal{M}^{\mathrm{eq}}, \mathop{\smile\!\!\!\!\!\!\mid})$ *is rosy and CM-trivial, then modular law holds iff* $\mathop{\smile\!\!\!\!\!\!\mid} = \mathop{\smile\!\!\!\!\!\!\mid}^{\mathfrak{p}} = \mathop{\smile\!\!\!\!\!\!\mid}^{a}$.

5. Topics related to Geometric Elimination of Imaginary

We say that T has GEI if for any $e = (\bar{a})_E \in \mathcal{M}^{\mathrm{eq}}$ there exists a finite real tuple $\bar{b} \subset \mathcal{M}$ such that $\mathrm{acl}^{\mathrm{eq}}(e) = \mathrm{acl}^{\mathrm{eq}}(\bar{b})$.

Definition 5.1. We say that T has *quasi*-GEI if we have

$$\mathrm{acl}^{\mathrm{eq}}(\mathrm{acl}(A) \cap \mathrm{acl}(B)) = \mathrm{acl}^{\mathrm{eq}}(A) \cap \mathrm{acl}^{\mathrm{eq}}(B)$$

for any $A, B \subset \mathcal{M}$. "$\subseteq$" always holds.

Put $X = \varphi(\bar{x}, \bar{a})^{\mathcal{M}}$, a definable set over $\bar{a}$. Let

$$E_\varphi(\bar{y}, \bar{z}) \equiv \forall \bar{x}(\varphi(\bar{x}, \bar{y}) \leftrightarrow \varphi(\bar{x}, \bar{z})).$$

$\lceil X \rceil := (\bar{a})_{E_\varphi}$, the canonical parameter of X in $\mathcal{M}^{\mathrm{eq}}$.
We say X is almost definable over $A = \mathrm{acl}(A) \subset \mathcal{M}$ if $\lceil X \rceil \in \mathrm{acl}^{\mathrm{eq}}(A)$.

Fact 5.2. *The following are equivalent.*

(1) *GEI*
(2) *Quasi-GEI*
(3) *If* $A = \mathrm{acl}(A), B = \mathrm{acl}(B) \subset \mathcal{M}$, X *is almost definable over both* A *and* B, *then* X *is almost definable over* $A \cap B$.

Definition 5.3. (1) We say that $\mathcal{M}$ has modular law in the real sort if $\mathrm{acl}^{\mathrm{eq}}(D) \cap \mathrm{acl}^{\mathrm{eq}}(AC) = \mathrm{acl}^{\mathrm{eq}}((A \cap D)C) = \mathrm{acl}^{\mathrm{eq}}(D \cap AC)$ holds for any $A = \mathrm{acl}(A), C = \mathrm{acl}(C) \subseteq D = \mathrm{acl}(D) \subset \mathcal{M}$.
(2) We say that $\mathcal{M}^{\mathrm{eq}}$ has quasi-modular law if we have that $\mathrm{acl}^{\mathrm{eq}}(\mathrm{acl}(D) \cap \mathrm{acl}(AC)) \subseteq \mathrm{acl}^{\mathrm{eq}}(\mathrm{acl}^{\mathrm{eq}}(D) \cap (\mathrm{acl}^{\mathrm{eq}}(A)\mathrm{acl}^{\mathrm{eq}}(C)))$ for any $A, C, D \subset \mathcal{M}$ with $C \subseteq D$.

We show the following fact in [Y1].

Fact 5.4. (1) *Modular law and GEI iff modular law in the real sort.*
(2) *Modular law implies quasi-modular law.*
(3) *Quasi-modular law and GEI implies modular law.*

Remark 5.5. (1) $(\mathcal{M}, \mathop{\smile\!\!\!\!\!\!\mid}^{\mathfrak{p}})$ *is modular iff modular law in the real sort by Proposition 4.1.1 and Fact 5.4.1.*
(2) *Weak one-basedness does not imply quasi-modular law : Consider Peterzil's o-minimal theory* $T = \mathrm{Th}(\mathbb{R}, <, +, 0, 1, \pi|(-1, 1))$ *with CF-property, (G)EI and a strong type without its weak canonical base. If* T *had quasi-modular law,* T *would be one-based by Fact 5.4.3, a contradiction.*
(3) *Question: Does quasi-modular law imply weak one-basedness?*
(4) *CM-triviality in the real sort and quasi-modular law implies modularity with respect to* $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathfrak{p}}$, *since CM-triviality in the real sort implies GEI. K.Ikeda's* ω*-categorical strictly stable Fraïssé limit hypergraph do not have quasi-modular law.*
(5) *Question: Does D.Evans'* ω*-categorical* SU *rank one CM-trivial structure without GEI have quasi-modular law? If so, modular law strictly implies quasi-modular law.*

6. Questions

Question 6.1. (1) *Find one-based* $(\mathcal{M}^{\mathrm{eq}}, \perp)$ *without base monotonicity, so* $(\mathcal{M}^{\mathrm{eq}}, \perp)$ *is not rosy. Can we find it in* NSOP_1 $(\mathcal{M}^{\mathrm{eq}}, \perp^{K,f})$*?*
We have $\perp^{f} \Rightarrow \perp^{K,f}$.

(2) *We have* $\perp \Rightarrow \perp^{\mathfrak{p}} (:= \perp^{m^*}) \Rightarrow \perp^{m} \Rightarrow \perp^{a}$ *for a strict independence relation* $\perp$. *If* $\perp^{a} = \perp^{m}$*, then* $\perp^{a} = \perp^{a^*} = \perp^{m^*} =: \perp^{\mathfrak{p}}$ *and rosiness follows. Non-rosiness implies* $\perp^{a} \neq \perp^{m}$. *Non one-based rosy theries with the existence of weak canonical base for any strong type are rosy theories with* $\perp^{a} \neq \perp^{m}$ *by Proposition 4.1.(2).*

(3) *Peterzil's non one-based o-minimal theory with CF-property having a strong type without its weak canonical base have* $\perp^{a} \neq \perp^{m}$ *by Proposition 4.1 (1).*

(4) *Find a* $\perp$*-rosy theory with* $\perp^{a} = \perp^{m}$ *but without* $\perp$*-weak canonical base for a strong type. Note that it is not one-based with respect to* $\perp$ *by Proposition 4.1 (2).*

(5) *Any simple theory with elimination of hyperimaginaries is rosy with* NSOP_1. *Any o-minimal theory is rosy with* SOP_1. *Find an* NSOP_1 *theory without rosiness, a candidate is the Džamonja-Shelah's theory* T^*_{feq} *which is* NSOP_1 *not simple by* [DS].

(6) *Conjecture: Simplicity (with elimination of hyperimaginaries)=*NSOP_1*+rosiness.*

(7) *Fact: In treeless theories, the generic stability independence relation* $\perp^{G,S}$ *is considered. See* [KRS].
  (a) *Stability implies treelessness implies rosiness.*
  (b) *Treelessness+*NSOP_1 *implies simplicity. So* T^*_{feq} *is not treeless.*

(8) *Joonhee Kim shows that if* $\perp^{f}$ *has existence, then* $\perp^{K,f} \Rightarrow \perp^{a}$ *by* [K].
S.Mutchnik's question : Suppose that $\perp^{K,f} = \perp^{a}$ *in an* NSOP_1 *theory* T. *Does* $\perp^{f}$ *have the existence property? In general, if* $\perp^{K.f}$ *has base monotonicity,* $\perp^{K,f} = \perp^{f}$ *follows by* [K]. *As* $\perp^{f}$ *has base monotonicity by Lemma 1.17 and Lemma 1.22 in* [A], $\perp^{K,f}$ *has base monotonicity iff* $\perp^{K,f} = \perp^{f}$. *If* $\perp^{K,f} = \perp^{a}$ *has base monotonicity, then* $\perp^{K,f} = \perp^{f} = \perp^{a} = \perp^{\mathfrak{p}}$ *and* $\perp^{f}$ *has the existence property. So we need to consider the case that* $\perp^{K.f}$ *does not have base monotonicity.*

(9) *In* NSOP_1 *theories,* $\perp^{f}$ *has the existence iff* $\perp^{K,f}$ *has base monotonicity?* *"If" is true because* $\perp^{f} = \perp^{K,f}$ *(by* [K]*) has the existence property in* NSOP_1 *theories by* [KKL]. *"Only if" is open.*

(10) *Find* SOP_1 *theories without rosiness. There are definably complete locally o-minimal theories without rosiness : Fujita introduces the discrete closure which has dimesion* [F]. *The independence relation by discrete closure does not have anti-reflexivity because* $\dim(a/A) = 0$ *iff* a *is a realization of a discrete closed formula over* A. *We do not know whether* $a \in \mathrm{acl}(A)$ *or not. Find such a theory with* NSOP. *Recall that* $\mathrm{SOP} \Rightarrow \mathrm{SOP_{n+1}} \Rightarrow \mathrm{SOP_n} \Rightarrow \mathrm{SOP}_3 \Rightarrow \mathrm{SOP}_2 \Leftrightarrow \mathrm{SOP}_1$ *for any* $n \geq 4$.

7. Appendix between imaginary sets and real sets

We also show the following fact in [Y1].

Fact 7.1. *(1)'s are Poizat's style. (2)'s are Pillay's style. (3)'s are new.*
The following are characterizations of E.I.

(1) *EI : Any definable set X has the smallest definably cloesd set $B \subset \mathcal{M}$ such that X is definable over B.*
(2) *For any $e \in \mathcal{M}^{\mathrm{eq}}$ there exists real tuple $\bar{b} \subset \mathcal{M}$ such that $\mathrm{dcl}^{\mathrm{eq}}(e) = \mathrm{dcl}^{\mathrm{eq}}(\bar{b})$.*
(3) *For any $A, B \subset \mathcal{M}$ we have $\mathrm{dcl}^{\mathrm{eq}}(A) \cap \mathrm{dcl}^{\mathrm{eq}}(B) = \mathrm{dcl}^{\mathrm{eq}}(\mathrm{dcl}(A) \cap \mathrm{dcl}(B))$.*

The following are characterizations of WEI.

(1) *WEI : Any definable set X has the smallest algebraically closed set $B \subset \mathcal{M}$ such that X is definable over B.*
(2) *For any $e \in \mathcal{M}^{\mathrm{eq}}$ there exists real tuple $\bar{b} \subset \mathcal{M}$ such that $e \in \mathrm{dcl}^{\mathrm{eq}}(\bar{b})$ and $\bar{b} \in \mathrm{acl}^{\mathrm{eq}}(e)$.*
(3) *For any $A, B \subset \mathcal{M}$ we have $\mathrm{dcl}^{\mathrm{eq}}(\mathrm{acl}(A)) \cap \mathrm{dcl}^{\mathrm{eq}}(\mathrm{acl}(B)) = \mathrm{dcl}^{\mathrm{eq}}(\mathrm{acl}(A) \cap \mathrm{acl}(B))$.*

The following are characterizations of GEI.

(1) *GEI : Any definable set X has the smallest algebraically closed set $B \subset \mathcal{M}$ such that X is almost definable over B.*
(2) *For any $e \in \mathcal{M}^{\mathrm{eq}}$ there exists real tuple $\bar{b} \subset \mathcal{M}$ such that $\mathrm{acl}^{\mathrm{eq}}(e) = \mathrm{acl}^{\mathrm{eq}}(\bar{b})$.*
(3) *For any $A, B \subset \mathcal{M}$ we have $\mathrm{acl}^{\mathrm{eq}}(A) \cap \mathrm{acl}^{\mathrm{eq}}(B) = \mathrm{acl}^{\mathrm{eq}}(\mathrm{acl}(A) \cap \mathrm{acl}(B))$.*

Acknowledement: This work was suppoted by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University. The author would like to thank Joonhee Kim's kind suggestions.

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