Chromatic properties of U-rank one graphs

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1 Motivation

I had long focused on edge-coloring problems in graphs, but only recently did I fully appreciate the significance of vertex-coloring problems. This shift was motivated by the work of Halevi, Kaplan, and Shelah, who investigated vertex colorings of uncountable graphs. Their analysis of stable or simple uncountable graphs uses a deep model-theoretic analysis.

In this note, we investigate a stable graph whose theory has U-rank one. Very roughly, for such a graph, we show the following:

- If the chromatic number of G is infinite, then G has an infinite complete subgraph.

2 Definitions

We first recall fundamental definitions in graph theory.

Definition 1 (Graph)**.** *A structure $G = (G, R)$ is called a graph if R is a binary relation on G that is irreflexive and symmetric. If $R(a, b)$ holds, the nodes a and b are said to be adjacent.*

Definition 2 (Coloring and Chromatic Number)**.** *Let (G, R) be a graph. A function $f : G \to \kappa$ is a valid coloring if $R(a, b) \Rightarrow f(a) \neq f(b)$. The minimum cardinality κ for which such a coloring exists is the chromatic number, denoted $\chi(G)$.*

Definition 3 (Degree)**.** *For $a \in G$, the degree of a in G is defined as $deg_G(a) := |\{g \in G : R(a, g)\}|$.*

3 Examples and Observations

Example 4.

1. *If a graph G contains a clique (complete subgraph) of size κ, then $\chi(G) \geq \kappa$.*
2. *A triangle-free random graph does not have a clique larger than size 3, yet its chromatic number $\chi(G) \geq \omega$.*
3. *Let K_n be a complete graph of size n. The graph $G = \bigoplus_{n\in\omega} K_n$ is a stable graph of U-rank 2. In this case, $\chi(G) = \omega$, yet G contains no infinite clique (though one exists in the monster model).*

4 Facts and Lemmas

The following facts regarding partitions and bounded degrees are essential for the main proof:

Fact 5. *Suppose $\chi(G)$ is infinite.*

1. *If $G = G_0 \sqcup \cdots \sqcup G_{n-1}$ is a finite partition of the vertices, there exists some $i < n$ such that $\chi(G_i)$ is infinite.*

2. *If $R = C_0 \sqcup \cdots \sqcup C_{n-1}$ is a finite partition of the edges, there exists some $i < n$ such that (G, C_i) has an infinite chromatic number.*

Fact 6. *If the degrees in G are uniformly bounded by a finite number d, then $\chi(G) \leq d + 1$, meaning the chromatic number is finite.*

The following lemma, implicit in Halevi et al., links strong types and independence to the existence of cliques:

Lemma 7. *Let G be an infinite graph whose theory T is stable, and let $A \subset G$. If there exist a, b in the monster model $\mathcal{M} \succ G$ such that $R(a, b)$ holds, $stp(a/A) = stp(b/A)$, and $a \downarrow_A b$, then G contains an infinite clique.*

5 Main Result

Theorem 8. *Let G be a graph whose theory has U-rank one (every element in the monster model has U-rank ≤ 1). If $\chi(G) \geq \omega$, then G must contain an infinite clique.*

Proof. Assume for contradiction that G does not contain an infinite clique. Let $\Gamma(x, y)$ be the set of formulas:

$$\{(\exists^{<n} u)\phi(u, y) \to \neg\phi(x, y) : n \in \omega, \phi(x, y) \text{ a formula}\}.$$

The following claim can be proved using Lemma 7 above.

Claim 1. *The set $\Delta(x, y) :=$ '$stp(x) = stp(y)$'$\cup \{R(x, y)\} \cup \Gamma(x, y)$ is inconsistent.*

Since Δ is inconsistent, by compactness there exist a finite equivalence relation $E(x, y)$, an integer $n \in \omega$, and finitely many formulas $\phi_i(x, y)$ for $i < k$ such that $(\forall x \forall y)[E(x, y) \wedge R(x, y) \to \bigvee_{i<k} (\exists^{<n} u)\phi_i(u, y) \wedge \phi_i(x, y)]$ holds. Because there are only finitely many E-classes, there must be an E-class C such that $\chi(C)$ is infinite. This formula implies that for every $a \in C$, its degree is bounded: $\deg(a) \leq kn$. Uniformly finite degrees imply $\chi(C)$ is finite, which is a contradiction. □

References

[1] Yatir Halevi, Itay Kaplan, and Saharon Shelah, Infinite Cliques in Simple and Stable Graphs, Model Theory (Volume 4, Issue 2, 2024, pp. 231-245).

[2] Saharon Shelah, *Classification Theory and the Number of Non-Isomorphic Models*, 2nd edition, North-Holland, 1990ddd.

[3] Anand Pillay, *An Introduction to Stability Theory*, Oxford University Press, 1983.