Locally o-minimal structures of finite dp-rank

Hisatomo Maesono
Global Education Center, Waseda University
前園久智
早稲田大学グローバルエデュケーションセンター

概要

abstract Recently I tried to characterize dp-minimal locally o-minimal structures. In this note, we continue the characterization and try to generalize them to strongly dependent, dependent cases.

1. Introduction

At first we recall some definitions.

Definition 1. Let M be a densely linearly ordered structure without endpoints.

M is *o-minimal* if every definable subset of M^1 is a finite union of points and intervals.

M is *locally o-minimal* if for any element $a \in M$ and any definable subset $X \subset M^1$, there is an open interval $I \subset M$ such that $I \ni a$ and $I \cap X$ is a finite union of points and intervals.

M is *uniformly locally o-minimal* if for any formula $\varphi(x, \bar{y})$ over $\emptyset$ and any $a \in M$, there is an open interval $I \ni a$ such that $I \cap \varphi(M, \bar{b})$ is a finite union of points and intervals for any $\bar{b} \in M^n$ where $\varphi(M, \bar{b})$ is the realization set of $\varphi(x, \bar{b})$ in M.

M is *strongly locally o-minimal* if for any $a \in M$, there is an open interval $I \ni a$ such that whenever X is a definable subset of M^1, then $I \cap X$ is a finite union of points and intervals.

M is *definably complete* if any definable subset X of M^1 has the supremum and infimum in $M \cup \{\pm\infty\}$.

Example 2. [1], [2] *etc.*

$(\mathbb{R}, +, <, \mathbb{Z})$ where $\mathbb{Z}$ is the interpretation of a unary predicate, and $(\mathbb{R}, +, <, \sin)$ and ultraproducts of o-minimal structures are definably complete locally o-minimal structures.

We recall some fundamental facts.

Fact 3. [1], [2]

Definably complete local o-minimality is preserved under elementary equivalence.

Proposition 4. [2]

Let M be a uniformly locally o-minimal structure. Suppose that M is ω-saturated. Then M is strongly locally o-minimal.

Here we recall 1-types of locally o-minimal structures.

Definition 5. Let M be a densely linearly ordered structure and $p(x) \in S_1(M)$.

We say that $p(x)$ is *cut over M* if for any $a \in M$, if $a < x \in p(x)$, then there is $b \in M$ such that $a < b < x \in p(x)$, and similarly if $x < a \in p(x)$, then there is $c \in M$ such that $x < c < a \in p(x)$.

We say that $q(x) \in S_1(M)$ is *noncut over M* if $q(x)$ is not a cut type.

Here we consider nonisolated types only.

It is known that o-minimal structures have a small amount of definable sets. However, locally o-minimal structures have definable complexity to some extent.

Definition 6. An *independence pattern* of length κ is a sequence of pairs $(\phi^{\alpha}(\bar{x}, \bar{y}), k^{\alpha})_{\alpha<\kappa}$ of formulas such that there exists an array $\langle \bar{b}_i^{\alpha} : \alpha < \kappa, i < \lambda \rangle$ for some $\lambda \geq \omega$ such that :

· for each $\alpha < \kappa$, the set $\{\phi^{\alpha}(\bar{x}, \bar{b}_i^{\alpha}) : i < \lambda\}$ is k^{α}-inconsistent, and

· for all $\eta \in \lambda^{\kappa}$, the set $\{\phi^{\alpha}(\bar{x}, \bar{b}_{\eta(\alpha)}^{\alpha}) : \alpha < \kappa\}$ is consistent.

A thoery T is *inp-minimal* if there is no inp-pattern of length two in a single variable x.

A theory T has *the tree property of the second kind* (TP_2) if there is an inp-pattern of size ω for which the formulas $\phi^{\alpha}(\bar{x}, \bar{y})$ in the definition above are all equal to some $\phi(\bar{x}, \bar{y})$.

Fact 7. *There are locally o-minimal structures whose theories have TP_2.*

For example, some modified simple product by M.Fujita, and some ultraproduct of o-minimal structures by A.Tsuboi and the author.

Thus we try to characterize locally o-minimal structures by stability theoretic properties under some additional conditions.

2. Locally o-minimal structures of finite dp-rank

In this section, we investigate locally o-minimal structures whose theories are dp-minimal at first.

We recall the definition of dp-rank.

Definition 8. Let $p(\bar{x})$ be a partial type over a set $A \subset \mathcal{U}$ where $\mathcal{U}$ is the monster model.

We define *dp-rank* of $p(\bar{x})$ as follows :

Let μ be a cardinal.

We say that $p(\bar{x})$ has dp-rank $< \mu$ if given any realization $\bar{a}$ of p and any family $(I_t : t < \mu)$ of mutually A-indiscernible sequences, at least one of them is indiscernible over $A\bar{a}$.

Dp-minimal theories are theories in which all 1-types have dp-rank 1 (dp-rk$(x = x) = 1$).

Fact 9. *A theory T is dp-minimal*
if and only if
T is NIP and inp-minimal.

I showed the next theorem before. After that, I tried to generalize this fact to locally o-minimal structures of finite dp-rank.

Theorem 10. *Let M be locally o-minimal and $Th(M)$ be dp-minimal.*
Then M is uniformly locally o-minimal.

For the proof in finite dp-rank case, I tried to modify the next result.

Theorem 11. [12] *Assume that p is a dependent type over A with dp-rank$(p) \geq \kappa$.*
Then there are some $A' \supset A$, some $a \models p$, and A'-mutually indiscernible sequences $\{ I_i \mid i < \kappa \}$ such that each of them is not indiscernible over $A'a$ and all tuples in each I_i satisfy p.

Problem 12. *Are locally o-minimal structures of finite dp-rank uniformly local ones ?*
Are there counterexamples ?

There are some results about types in dp-minimal locally o-minimal structures by means of the local monotonicity theorem for strongly locally o-minimal structures proved in [2].

Proposition 13. *Let M be a dp-minimal locally o-minimal structure and $p(x), q(y) \in S_1(M)$. And let $p(x)$ be noncut and $q(y)$ be cut such that $q(y)$ is not order-complete, that is, $q(y) \restriction <$ is not complete.*
Then there are no realizations $a \models p(x)$ and $b \models q(y)$ such that a and b have a common interval with OM-property in any strongly locally o-minimal structure $N \succ M$ (where OM-property means that the intersection of every definable subset of M^1 and the interval is a finite union of points and intervals).

In general, unbounded noncut types are not definable in locally o-minimal structures. I showed the next result in [13]. But there is some incorrect part in it which was pointed out by A.Tsuboi. Thus I show it again.

Fact 14. *Let M be locally o-minimal and $Th(M)$ be dp-minimal.*
And let $p(x) \in S_1(M)$ be unbounded noncut.
Then $p(x)$ is definable.

Proof ;

We may assume that $p(x) \vdash \{m < x \,:\, m \in M\}$*. Let* $\phi(x, \bar{y})$ *be an* L*-formula. We consider the equivalence relation* $E_\phi(\bar{y}, \bar{z})$ *defined on tuples by* $E(\bar{b}, \bar{b}')$ *if and only if* $\exists z \forall x\,(\,x > z \longrightarrow (\,\phi(x, \bar{b}) \longleftrightarrow \phi(x, \bar{b}')\,)\,)$*.*

We denote the formula $\phi(x, \bar{y}_0) \triangle \phi(x, \bar{y}_1) := (\phi(\bar{x}, \bar{y}_0) \wedge \neg\phi(\bar{x}, \bar{y}_1)\,) \vee (\,\neg\phi(\bar{x}, \bar{y}_0) \wedge \phi(\bar{x}, \bar{y}_1))$*. We take any indiscernible sequence* $I := \{\,\bar{b}_i \,:\, i < \omega\,\}$ *over* $\emptyset$ *and consider the set of formulas* $\Phi(x) := \{\,\phi(x, \bar{b}_{2i}) \triangle \phi(x, \bar{b}_{2i+1}) \,:\, i < \omega\,\}$*. We may assume that* $I \subset N$ *for some sufficiently saturated model* $N \succ M$*. If* $\Phi(x)$ *is consistent, then the alternation rank of* $\phi(x, \bar{y})$ *is infinite. Thus it contradicts that* $Th(M)$ *is* NIP*. So the set of formulas* $\Phi(x)$ *is* k*-inconsistent for some* $k < \omega$*.*

If the realization set $\phi(N, \bar{b}_{2i}) \triangle \phi(N, \bar{b}_{2i+1})$ *is cofinal in* N *for some* $i < \omega$*, then for any* $i < \omega$*, the set* $\phi_i(N) := \phi(N, \bar{b}_{2i}) \triangle \phi(N, \bar{b}_{2i+1})$ *is cofinal. Thus we can take infinitely many intervals which have realizations of* ϕ_i *for any* $i < \omega$*. Then this contradicts the inp-minimality.*

Thus this set $\phi(N, \bar{b}_{2i}) \triangle \phi(N, \bar{b}_{2i+1})$ *cannot be cofinal, so* $\bar{b}_{2i}$ *and* $\bar{b}_{2i+1}$ *are* E_ϕ*-equivalent. Then* $E_\phi(\bar{y}, \bar{z})$ *is a bounded equivalence relation. As* E_ϕ *is definable,* E_ϕ *has finitely many classes. By compactness theorem,* E_ϕ *has* n_ϕ *classes for some* $n_\phi < \omega$*.*

Let $\psi(x, \bar{y}) \in L$*-formula. And let* $\{\,\bar{m}_i \,:\, i < n_\psi\,\}$ *be the set of representatives of the* E_ψ*-classes in* M*. We can take* $d\psi(\bar{y}) := \exists z \forall x(\,x > z \longrightarrow \bigvee_{i < n_\psi}(\psi(x, \bar{y}) \longleftrightarrow \psi(x, \bar{m}_i))$*.* ∎

3. Some characterization about forking of types

There is a conjecture about nonforking extensions of types in NIP theories by A.Chernikov and P.Simon in [9].

Conjecture 15. *Let* T *be* NIP *and* $M \models T$*. And let* $\phi(x; d) \in L(\mathcal{U})$ *be a formula nonforking over* M*.*

Then there is $\theta(y) \in \mathrm{tp}(d/M)$ *such that the partial type* $\{\phi(x; d') : d' \in \theta(\mathcal{U})\}$ *is consistent.*

P.Simon argued about this conjecture under the assumption that T is dp-minimal in [5]. We can confirm that this conjecture holds for dp-minimal locally o-minimal structures by another result by Simon and S.Starchenko.

Proposition 16. [7] *Assume that* T *is dp-minimal locally o-minimal structure. Let* $M \models T$ *and* $\phi(x; d) \in L(\mathcal{U})$ *be nonfroking over* M*.*

Then $\phi(x; d)$ *extends to a complete* M*-definable type.*

So by the proposition above,

Theorem 17. *Let T be dp-minimal locally o-minimal and $M \models T$.*
Then Conjecture 15 holds.

4. Honest definition of locally o-minimal structures

It is known that definability of types holds in NIP theories to some extent. We apply it to characterization of locally o-minimal structures.

In this section, we consider pairs of structures. The base language is L and we define the language L_P where we add L a new unary predicate $P(x)$. And by the pair (M, A), we mean the L_P-expansion of M obtained by $\models P(a) \longleftrightarrow a \in A$.

Definition 18. Let $X \subseteq M^k$ be externally definable (that is, definable using parameters outside M).

Then an *honest definition* of X is a definition $\theta(x, d)$ of X, $d \in \mathcal{U}$ such that :

$\mathcal{U} \models \theta(x, d) \longrightarrow \psi(x)$ for every $\psi(x) \in L(M)$ such that $X \subseteq \psi(M)$.

Let T be NIP and (M, A) a pair with $M \models T$. And let $\phi(x, a) \in L_P(M)$.

An *honest definition of* $\phi(x, a)$ *over* A is a formula $\theta(x, c) \in L_P$, $c \in P(\mathcal{U})$ such that $\theta(A, c) = \phi(A, a)$ and $\models (\forall x \in P)\,(\theta(x, c) \longrightarrow \phi(x, a))$.

Proposition 19. [8] *Let $\phi(x, y)$ be NIP.*
Given (M, A) and $c \in M$, there are $(M', A') \succeq (M, A)$ and $\theta(x) \in L(A')$ such that $\phi(A, c) = \theta(A)$ and $\theta(A') \subseteq \phi(A', c)$.

Proposition 20. [8] *Let T be NIP.*
Then every externally definable set $X \subset M^k$ has an honest definition.

It is shown that strong form of honest definition holds in distal theories.

Definition 21. A theory T is *distal* if it satisfies the following property :

Let $I + b + J$ be an indiscernible sequence with I and J infinite.

For any A, if $I + J$ is indiscernible over A, then $I + b + J$ is indiscernible over A.

In NIP theories, distal theory is the farthest one from stability. It is known that linearly ordered dp-minimal structures are distal.

So we consider a problem.

Problem 22. *Are there distal and non dp-minimal locally o-minimal structures ?*

Proposition 23. [9] *Let T be distal, $A \subset M$ and $a \in M$ arbitrary.*
And let $(M', A') \succ (M, A)$ be $|\, M \,|^{+}$-saturated.

Then for any $\phi(x,y)$, there are $\theta(x,z)$ and $b' \in A'$ such that $\models \theta(a,b)$ and $\theta(x,b) \vdash \mathrm{tp}_\phi(a/A)$.

A.Chernikov and P.Simon showed about the dependence of theories of pairs.

Definition 24. We say that an L_P-formula is *bounded* if it is of the form $Q_0 y_0 \in P \cdots Q_n y_n \in P\, \phi(x, y_0, \cdots, y_n)$ where $Q_i \in \{\exists, \forall\}$ and $\phi(x, \bar{y})$ is an L-formula.

We say that T_P is *bounded* if every formula is equivalent to a bounded formula where T_P is a theory of some expanded structure.

Let $A \subset \mathcal{U}$ and Ω be a set of formulas.

We let $A_{ind(\Omega)}$ be the structure with domain A and a relation added for every set of the form $A^n \cap \phi(\bar{x})$ where $\phi(\bar{x}) \in \Omega$.

Theorem 25. [8] *Let $M \models T$ and for some $A \subset M$, $(M, A) \models T_P$.*

Assume that T is NIP, and $A_{ind(L)}$ is NIP and T_P bounded, then T_P is NIP.

And let $M \prec N$ and $(N, M) \models T_P$.

Assume that T is NIP and T_P is bounded, then T_P is NIP.

There are many examples of locally o-minimal structures which are expansions by a unary predicate. So we try to characterize them by the stability theoretic properties. In particular, they constructs many locally o-minimal strucutres by means of simple product construction in [2].

Definition 26. [2] Let L and L_i be relational languages $i = 1, 2$. And let M_i be an L_i-structure and $N = M_1 \times M_2$.

For $A \subset M_1^n$ and $B \subset M_2^n$, we define :

$A * B = \{\langle (a_1, b_1), \cdots, (a_n, b_n) \rangle \in N^n : \langle a_1, \cdots, a_n \rangle \in A,\ \langle b_1, \cdots, b_n \rangle \in B\}$.

For L-structure N, we say that N is a *simple product* of M_1 and M_2 if for any $R(x_1, \cdots, x_n) \in L$, there are M_1-definable sets $A_1, \cdots, A_k \subset M_1^n$ and $B_1, \cdots, B_l \subset M_2^n$ such that R^N is a (positive) boolean combination of the following sets :

$A_i * M_2^n$ and $M_1^n * B_j$ where $i = 1, \cdots, k$ and $j = 1, \cdots, l$.

Theorem 27. [2] *For $i = 1, 2$, let $M_i = (M_i, <^{M_i}, \cdots)$ be an expansion of a linear order. Let $N = (N, <^N, \cdots)$ be a simple product of M_1 and M_2 where $<^N$ is given by the lexicographic ordering.*

1. *Suppose that M_2 is a (strongly) locally o-minimal structure without endpoints.*
   Then N is (strongly) locally o-minimal.
2. *Suppose that M_2 is an o-minimal structure possibly with endpoints. And suppose also that M_1 is a discrete order.*
   Then N is strongly locally o-minimal.

There are some facts which are deduced from Theorem 25. But we set further problem.

Problem 28. *Characterize the dependence of locally o-minimal structures which are expansions by a unary predicate.*

In pariticular, characterize properties about dependence of locally o-minimal structures which are constructed by simple product.

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