RAMSEY TRANSFER PRINCIPLES FROM THE PARTITE CONSTRUCTION

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Abstract. We review the partite construction, a powerful method for proving the Ramsey property of classes of finite structures. In particular, we focus on the notion of locally finite subclasses.

1. Introduction

The purpose of this article is to introduce the recent developments of method of partite construction to model theorists. Most of the contents in this article can be found on [HK26].

First, we recall some basic definitions.

Definition 1.1. Let λ, μ be cardinals and n, k be natural numbers.

1. $[\lambda]^n := \{A \subseteq \lambda : |A| = n\}$.
2. $\lambda \to (\mu)^n_k$ means that for any $c : [\lambda]^n \to k$, there is a subset μ' of λ such that $|\mu'| = \mu$ and c restricted to $[\mu']^n$ is constant (i.e., monochromatic with respect to c).

Using above notation, we can restate Ramsey's theorem (on linear orders).

Theorem 1.2 (Ramsey's theorem)**.**

(1) (Infinite) For any $a, r \in \omega \setminus \{0\}$, the partition arrow $\omega \to (\omega)^a_r$ holds.
(2) (Finite) For any $a, b, r \in \omega \setminus \{0\}$, there is $c \in \omega$ satisfying the partition arrow $c \to (b)^a_r$.

Note that $[\omega]^a$ can also be seen as a collection of embeddings of $(a, <)$ in $(\lambda, <)$, and any subset of ω of cardinal ω as an embedding of ω in ω. Structural Ramsey theory aims to generalize the finite Ramsey's theorem to general structures.

Definition 1.3. Given $\mathcal{L}$-structures A and B, $\binom{B}{A}$ is the collection of all embeddings from A to B. (Imagine labeled copies of A in B.)

Definition 1.4 (Ramsey property)**.** Let $\mathcal{K}$ be a class of finite $\mathcal{L}$-structures. We say $\mathcal{K}$ has the **Ramsey property** (for embeddings) if for any $A, B \in \mathcal{K}$ and $r \geq 1$, there is $C \in \mathcal{K}$ such that for any r-coloring of $\binom{C}{A}$, there is $f \in \binom{C}{B}$ such that $\{f \circ g : g \in \binom{B}{A}\}$ is monochromatic. Using the partition arrow notation, $\mathcal{K}$ has the Ramsey property if and only if

$$(\forall A, B \in \mathcal{K})(\forall r \in \omega \setminus \{0\})(\exists C \in \mathcal{K})(C \to (B)^A_r).$$

We will also say that $\mathcal{K}$ is a **Ramsey class**, or simply $\mathcal{K}$ is **Ramsey**.

Partition arrows may be expressed using chromatic numbers. For graphs, it is very easy to point out the connection between the partition arrow and chromatic numbers:

Observation 1.5. *Let G be a graph. Then $G \to (edge)^{vertex}_r$ if and only if $\chi(G) > r$.*

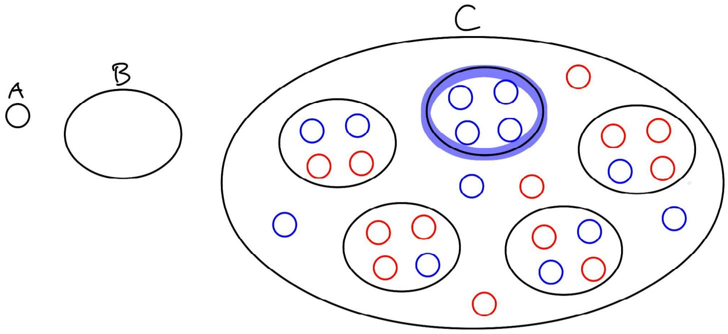
2. Partite construction

We now start to introduce basic definitions of the main topic.

Definition 2.1. Let $\mathcal{L}_P := \mathcal{L} \cup P$ be a relational language extending $\mathcal{L}$ by finitely many predicates. An **$\mathcal{L}_P$-partite system** is any finite $\mathcal{L}_P$-structure A satisfying the following two conditions:

1. For every $v \in A$, there is precisely one predicate $p \in \mathcal{L}_P \setminus \mathcal{L}$ containing v.

The author is supported by the G-LAMP Program of the NRF of Korea grant funded by the Ministry of Education (Grant No. RS-2024-00441954), the National Research Foundation of Korea(NRF) grant funded by the Ministry of Education (No. RS-2024-00441954), and the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.



(2) For every $p \in \mathcal{L}_P \setminus \mathcal{L}$ and every relation $R \in \mathcal{L}$, each $\bar{x} \in R^A$ contains at most one element from p^A.

An $\mathcal{L}_P$-partite system A is **transversal** if $A = \mathcal{L}_P \setminus \mathcal{L}$.

Theorem 2.2 (Partite lemma)**.** *Let $\mathcal{L}_P$ be as above, and let A, B be $\mathcal{L}_P$-partite systems such that A is transversal. Then there is $\mathcal{L}_P$-partite system C such that $C \to (B)^A_2$.*

By repeatedly applying the partite lemma with appropriate variations, the Nešetřil-Rödl theorem can be proved.

Definition 2.3 ([EHN21])**.** A structure is **irreducible** if it is not the free amalgam of any two of its proper substructures. (We allow emtpy set as an empty structure.)

Remark 2.4. If the underlying language $\mathcal{L}$ contains a function symbol with arity ≥ 2, then any $\mathcal{L}$-structure M of cardinal ≥ 2 is **NOT irreducible**: Let $f \in \mathcal{L}$ be a function symbol of arity n. For any $a_0, a_1 \in M$, choose any $a_2, \ldots, a_{n-1} \in M$. Then there is some $b \in M$ such that $f(a_0, a_1, \ldots, a_{n-1}) = b$, which means that any pair of elements (a_0, a_1) have ‘some’ relation. Thus M is never a free amalgam of two of its proper substructures.

We remedy this trouble by **setting the values of the function as (possibly empty) subsets of the model**. That is, for $f \in \mathcal{L}$ and $\bar{a}$ in M, $f^M(\bar{a}) \subseteq M$. The cost of this generalization is serious; the satisfaction of a first-order formula by a model is now not well-defined. But as long as we are interested only on the Ramsey property and the partite construction, we do not care much about formulas; we only care about homomorphisms and embeddings, which preserves interpretations of symbols between models.

Theorem 2.5 ((Generalized) Nešetřil-Rödl theorem, [HN19])**.** *Let $\mathcal{L}$ be* **any** *language containing a binary relation $<$, and let $\mathcal{F}$ be a set of finite $\mathcal{L}$-structures. Assume for every $A \in \mathcal{F}$, $A \restriction (\mathcal{L} \setminus \{<\})$ is irreducible. Then the class of all finite linearly ordered $\mathcal{F}$-free $\mathcal{L}$-structures is a Ramsey class.*

Example 2.6. The class of all finite linearly ordered graphs is Ramsey by the Nešetřil-Rödl theorem. It can be easily proven by invoking the axioms of graphs by forbidding appropriate $\{E\}$-irreducible $\{E, <\}$-structures.

3. Locally finite subclasses

We will focus on the Ramsey transfer principle of the following manner: If $\mathcal{R}$ is a Ramsey class (typically given by the Nešetřil-Rödl theorem) and $\mathcal{K} \subseteq \mathcal{R}$ satisfies “some properties”, then $\mathcal{K}$ is also a Ramsey class.

Definition 3.1 ([HN19])**.** Let A, B be $\mathcal{L}$-structures and $f : A \to B$.

(1) f is a **homomorphism** if for every relation $R \in \mathcal{L}$, $R(\bar{a})$ implies $R(f(\bar{a}))$ (but not necessarily conversely), and preserves every function symbol in $\mathcal{L}$.
(2) f is an **embedding** if f is a homomorphism, preserves every relation in $\mathcal{L}$, and injective.
(3) A homomorphism $f : A \to B$ is a **homomorphism-embedding** if $f \restriction C$ is an embedding whenever C is an irreducible substructure of A.
(4) If $f : A \to B$ is a homomorphism-embedding, we say B is a **completion** of A.

Definition 3.2 ([HN19])**.** Let $\mathcal{K} \subseteq \mathcal{R}$ be classes of finite $\mathcal{L}$-structures. $\mathcal{K}$ is a **locally finite subclass of $\mathcal{R}$** if for every $B \in \mathcal{K}$ and every $C_0 \in \mathcal{R}$, there is $n = n(B, C_0) \in \omega$ such that if a finite $\mathcal{L}$-structure C satisfies the following conditions:

(1) If $D \subseteq C$ is irreducible, then there is an embedding $f : B \to C$ with $D \subseteq f[B]$,
(2) C_0 is a completion of C, and
(3) for every substructure D of C on $\leq n$ elements, there is a completion of D in $\mathcal{K}$,

then there is a completion of C in $\mathcal{K}$.

The structure C of the previous definition can be constructed as a result of the repeated use of the induced partite construction. For the details, consult [HK26].

Theorem 3.3 ([HN19])**.** *Let $\mathcal{L}$ be any language, let $\mathcal{R}$ be a Ramsey class of finite irreducible $\mathcal{L}$-structures and let $\mathcal{K}$ be a hereditary locally finite subclass of $\mathcal{R}$ with the amalgamation property. Then $\mathcal{K}$ is a Ramsey class.*

For a canonical example, we will prove that the class of finite ordered metric spaces is a Ramsey class.

Definition 3.4.

(1) Given $S \subseteq \mathbb{R}^{>0}$, an **$S$-edge-labeled graph** is a relational structure with $|S|$-binary relations such that all of them are symmetric and irreflexive, and every pair of vertices is in at most one relation.
(2) An $\mathbb{R}^{>0}$-edge-labeled triangle is **non-metric** if it has labels a, b, c and $a > b + c$.
(3) A complete $R^{>0}$-edge-labeled graph which embeds no non-metric triangles is a **metric space**.

Theorem 3.5 ([Neš07])**.** *The class of finite linearly ordered metric spaces is a Ramsey class.*

An $\mathbb{R}^{>0}$-edge-labeled cycle is **non-metric** if it has labels $a_1, \dots, a_n$ and $a_1 > a_2 + \cdots + a_n$.

Lemma 3.6. *The class $\mathcal{K}$ of all finite metric spaces is a locally finite subclass of the class $\mathcal{R}$ of all finite $\mathbb{R}^{>0}$-edge labeled graphs.*

Sketch of the proof. Given $B \in \mathcal{K}$, let S be the set of distances in B, and put $n = \left\lceil \frac{\max(S)}{\min(S)} \right\rceil$. Let C satisfy the assumptions of the definition of local finiteness of subclasses (we need only (1) and (3)):
(1) If $D \subseteq C$ is irreducible, then there is an embedding $f : B \to C$ with $D \subseteq f[B]$, and (3) for every substructure D of C on $\leq n$ elements, there is a completion of D in $\mathcal{K}$.
By (1), C is a finite S-edge-labeled graph. Note that the largest non-metric cycle using distances from S has at most n vertices, thus (3) ensures that C has a no non-metric cycle. Thus $\mathcal{K}$ is a locally finite subclass of $\mathcal{R}$. □

But the class $\mathcal{K}$ of all finite metric spaces is not Ramsey.
Problem: $\mathcal{K} \subseteq \mathcal{R}$ but the class $\mathcal{R}$ of all finite $\mathbb{R}^{>0}$-edge labeled graphs is not Ramsey.

Definition 3.7. Let $\mathcal{K}$ be a class of finite structures. The **free ordering** of $\mathcal{K}$, denoted by $\mathcal{K}^<$, is the class of all $\mathcal{L} \cup \{<\}$-structures A such that $<$ is a linear order in A and $A \restriction \mathcal{L} \in \mathcal{K}$.

Lemma 3.8. *Let $\mathcal{R}$ be a class of finite $\mathcal{L}$-structures and $\mathcal{K}$ be a locally finite subclass of $\mathcal{R}$ consisting of irreducible substructures. Then $\mathcal{K}^<$ is a locally finite subclass of $\mathcal{R}^<$.*

This lemma completes the proof that $\mathcal{K}^<$ is Ramsey, since the free ordering $\mathcal{R}^<$ of the class $\mathcal{R}$ of all finite $\mathbb{R}^{>0}$-edge labeled graphs is Ramsey by the Nešetřil-Rödl theorem, and metric spaces are irreducible.

4. Concluding Remarks

In general, methods based on the partite construction (including the concept of locally finite subclasses) is NOT appropriate for "tree structures". Ramsey property of the following well-known examples can be proved purely combinatorially, or by using modeling property.

Example 4.1 ([TT12], [KKS14])**.**

(1) The class of trees in the language $\mathcal{L}_s = \{\trianglelefteq, \wedge, <_{lex}, (P_\alpha)_{\alpha \in \omega}\}$.
(2) The class of trees in the language $\mathcal{L}_0 = \{\trianglelefteq, \wedge, <_{lex}\}$.

The partite construction is based on the Ramsey's theorem, but Ramsey property of tree structures needs "something stronger", and specific arguments that applies to particular structures are necessary.

Question: Will it be possible to obtain a (in some sense) stronger general principle for producing Ramsey classes?

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