On coloring of o-minimal graphs

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Abstract

This paper investigates the chromatic numbers of graphs definable in o-minimal structures. We show that an infinite chromatic number necessarily yields arbitrarily large cliques.

1 Introduction

Let $G = (G, R)$ be a graph, where $R \subset G^2$ is a symmetric and irreflexive relation representing the edges of G. For a cardinal κ, a (vertex) coloring of G with κ colors is a function $f : G \to \kappa$ such that $f(u) \neq f(v)$ for every edge $(u, v) \in R$. The chromatic number $\chi(G)$ is the least cardinal κ for which such a coloring exists.

If G has a complete induced subgraph of size κ, then clearly $\chi(G) \geq \kappa$. However, the converse is not true in general. In [3], Halevi, Kaplan and Shelah studied uncountable graphs with stability and showed that under additional conditions, the converse holds for these graphs. They also studied uncountable graphs with simple theories and demonstrated a weak converse.

The goal of this paper is to demonstrate that other classes of graphs with relatively simple structures exhibit similar properties.

The structure of this paper is as follows. In Section 2, we review basic definitions and facts concerning chromatic numbers. In Section 3, we show that any graph definable in an o-minimal structure with infinite chromatic number contains arbitrarily large finite cliques.

2 Preliminaries

We view a graph as a model-theoretic structure $G = (G, R)$, where $R \subset G^2$ is a symmetric and irreflexive relation representing the edge set. We refer to induced subgraphs simply as subgraphs, so that subgraphs coincide with substructures. When $R = \{(a, b) \in G^2 : a \neq b\}$, the graph G is called complete. A complete subgraph $H \subset G$ is called a clique of G.

Definition 1. Let G be a graph. The chromatic number $\chi(G)$ of G is the smallest cardinal κ such that there exists a function (called a coloring) $f : G \to \kappa$ such that all adjacent vertices have a different color.

Remark 2.

1. If G has a clique of size κ, then $\chi(G) \geq \kappa$. However, the converse does not hold in general, as illustrated in Remark 4.
2. Let G be an infinite graph with $\chi(G) = \omega$, and suppose $G = H_0 \cup \cdots \cup H_{n-1}$ is a finite partition. Then there exists $i < n$ such that $\chi(H_i) = \omega$.
3. Let (G, R) be a graph, and suppose $R = C_0 \cup \cdots \cup C_{n-1}$ is a finite partition of the edge set, where each $C_i \subset R$ is symmetric. If the chromatic number $\chi(G, R)$ is infinite, then there exists some $i < n$ such that $\chi(G, C_i)$ is infinite.

3 Main Results

We consider graphs definable in o-minimal structures.

Theorem 3. *Let $M = (M, <, \dots)$ be an o-minimal structure. Let $G \subset M$ be a definable set, and let $R \subset G^2$ be a definable relation representing the edge set of a graph on G. Suppose that $\chi(G)$ is infinite. Then G contains arbitrarily large finite cliques. In particular, the monster model of G contains an infinite clique.*

Before proving this theorem, we note that it is optimal in a certain sense. To avoid conflict with the notation for open intervals, the ordered pair of x and y will be denoted by $\langle x, y \rangle$.

Remark 4. In Theorem 3, we have to restrict our consideration to the case where the domain of a graph is a one-dimensional definable set G.

Let M be an infinite structure with a definable total order $<$. Consider the following graph, known as a *shift graph*, which is definable in M^2. Put

$$G = \{\langle u_1, u_2 \rangle \in M^2 : u_1 < u_2\},$$

and let $\varphi(u_1, u_2, v_1, v_2)$ be the formula $u_2 = v_1 \vee u_1 = v_2$. Then $\varphi(u_1, u_2; v_1, v_2)$ defines an edge relation R on G. The graph G is triangle-free, yet it has an infinite chromatic number ([2]).

Proof. The proof is a standard application of Ramsey's theorem.

Suppose that $\langle a, b \rangle$ is adjacent to $\langle b, c \rangle$, where $a < b < c$. If $\langle d, e \rangle \in G$ is adjacent to $\langle a, b \rangle$, then we have $a = e$ or $b = d$, which implies that $b \neq e$ and $c \neq d$. Hence, the graph G is triangle-free.

Suppose, for purposes of contradiction, that $\chi(G) = n$ for some positive integer n. Let f be a valid coloring $f : G \to n$. By Ramsey's theorem, there is an infinite subset $A \subset M$ and $m < n$ such that for any $a, b \in A$ with $a < b$, $f(\langle a, b \rangle) = m$. Let $a, b, c \in A$ such that $a < b < c$. Then, $\langle a, b \rangle$ and $\langle b, c \rangle$ are adjacent but have the same color, which contradicts the validity of the coloring f. □

In order to prove Theorem 3, we use the cell decomposition theorem for o-minimal structures (see [4]).

Proof sketch of Theorem 3. Since o-minimality is preserved under elementary equivalence, we may assume that M, and hence G, is sufficiently saturated. By symmetry, we consider $D = \{\langle x, y \rangle \in R \mid y < x\}$ as the edge set.

We decompose D into finitely many cells. By Remark 2, there must exist a $\langle 1, 1 \rangle$-cell C such that the chromatic number of (G, C) is infinite. We choose definable functions $f, g : (d, e) \to G$ such that

$$C = (f, g) = \{\langle x, y \rangle : d < x < e, f(x) < y < g(x)\},$$

where $d, e \in M \cup \{\pm\infty\}$, $f < g$ holds, and each of f and g is either a strictly monotone definable continuous function or a constant function. Notice also that $g(x) \leq x$ holds. For $g(x) = x$, we can easily obtain an infinite clique. For that reason, we may assume that $g(x) < x$.

Consider $C_+ = \{\langle x, y\rangle \in C : y > d\}$ and $C_- = \{\langle x, y\rangle \in C : y \leq d\}$. (G, C_-) cannot have an infinite chromatic number, since it is bipartite. Hence, $\chi(G, C_+) \geq \omega$. In the following, we restrict our attention to the graph (d, e) with the edge set C_+. Put

$$f^*(x) = \max\{f(x), d\}, \qquad g^*(x) = \max\{g(x), d\}.$$

By stipulating that $f^*(d) = g^*(d) = d$, both f^* and g^* may be regarded as definable functions from $[d, e)$ into $[d, e)$. We can write $C_+ = (f^*, g^*)$. By the choice of f and g, f^* and g^* are non-decreasing.

Claim A. *Let $c \in (d,\, e)$ be arbitrary.*

1. *The subgraph induced on the interval $[g^*(c),\, c)$ contains no edge.*
2. *Let $m, n \in \omega$ with $|n-m| \geq 2$. Then there is no edge directly connecting the intervals*

$$\big[(f^*)^{m+1}(c), (f^*)^{m}(c)\big) \quad \textit{and} \quad \big[(f^*)^{n+1}(c), (f^*)^{n}(c)\big)\,.$$

Claim A directly follows from the definition of f^* and g^*. The subsequent claim is essential for the proof of Theorem 3.

Claim B. *Suppose that there exists an interval $(f^*(c), c]$ which contains an infinite descending sequence $\{(g^*)^n(c)\}_{n\in\omega}$. Then the graph G contains an infinite clique.*

By the property of f^* and g^*, every point $u \in ((g^*)^n(c), c]$ is adjacent to all points $v \in (f^*(c), (g^*)^{n+1}(c))$. Using this observation, we can easily find an infinite clique. (End of the Proof of Claim B)

Owing to Claim B, henceforth we assume that no interval of the form $(f^*(c),\, c]$ contains an infinite descending sequence under g^*. Moreover, by saturation, there must exist a number $N \in \omega$ such that, for all $c \in (d,\, e)$,

$$(g^*)^N(c) \leq f^*(c). \tag{$**$}$$

Our goal is to derive a contradiction from this property. It is sufficient to show the following claim.

Claim C. *Let H be a finite connected subgraph of G. H can be colored by using at most $2N$ colors, where N is a number with the property $(**)$.*

By the connectedness of H, there is some $c \in (d, e)$ and $n \in \omega$ such that $H \subseteq [(f^*)^n(c), c)$.

Let $I_i = [\,(f^*)^{i+1}(c),\,(f^*)^i(c)\,)$ for $i < n$. By $(**)$, for some $k < N$, I_i is partitioned into $k+1$ subintervals:

$$I_i = \left[f^*(c^*), (g^*)^k(c^*)\right) \cup \bigcup_{0 \leq j < k} \left[(g^*)^{j+1}(c^*), (g^*)^j(c^*)\right) ,$$

where $c^* = (f^*)^i(c)$. By Claim A, each subinterval above has no edges. So, I_i is validly colored by using at most N colors. Recall that for all $i < n$, I_i, and I_{i-k} with $k \geq 2$ there are no edges that directly connect them. Therefore, $[(f^*)^n(c), c)$ is $2N$-colored and so is H.

(End of the Proof of Claim C) □

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References

[1] R. Diestel, *Graph Theory*, Graduate Texts in Mathematics, vol. 173, Second Edition, Springer, 2000.

[2] P. Erdős, A. Hajnal, Some remarks on set theory. IX. Combinatorial problems in measure theory and set theory, Michigan Math. J. 11(2): 107–127, 1964. DOI: 10.1307/mmj/1028999083

[3] Y. Halevi, I. Kaplan, and S. Shelah, Infinite Cliques in Simple and Stable Graphs, Model Theory, Vol. 4 (2025), No. 3, 231–249. DOI: 10.2140/mt.2025.4.231

[4] L. van den Dries, *Tame Topology and O-Minimal Structures*, London Mathematical Society Lecture Note Series, vol. 248, Cambridge University Press, 1998.