Boundedness in a quasilinear attraction-repulsion chemotaxis system with flux limitation

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1 Introduction

In this paper we consider the quasilinear attraction-repulsion chemotaxis system

$$\begin{cases} u_t = \nabla \cdot \big((u+1)^{m-1}\nabla u - \chi u(u+1)^{p-2}\nabla v + \xi u(u+1)^{q-2}\nabla w\big), \\ \tau v_t = \Delta v + \alpha u - \beta v, \\ \tau w_t = \Delta w + \gamma u - \delta w \end{cases} \tag{QAR}$$

in a smoothly bounded domain $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ under Neumann boundary conditions, where

$$m, p, q \in \mathbb{R}, \qquad \chi, \xi, \alpha, \beta, \gamma, \delta > 0 \quad \text{and} \quad \tau \in \{0, 1\}$$

are constants. When $m = 1$ and $p = q = 2$, boundedness and finite-time blow-up in (QAR) were classified the sizes of χ and ξ (see e.g., Tao–Wang [**21**], Fujie–Suzuki [**14**]). Thus the following question arises:

Can we classify boundedness and finite-time blow-up in (QAR)
when $m \neq 1$, $p \neq 2$ and $q \neq 2$ by the sizes of p, q and χ, ξ?

The purpose of this paper is to give an answer to this question.

2 Quasilinear attraction-chemotaxis system

Before considering (QAR), we precisely review the case $m = 1$ and $p = q = 2$. In this case, Tao and Wang [**21**] showed boundedness when $n \geq 2$ and $\chi\alpha - \xi\gamma < 0$, and finite-time blow-up under the conditions that $n = 2$, $\chi\alpha - \xi\gamma > 0$, $\|u_0\|_{L^1(\Omega)} > \frac{8\pi}{\chi\alpha - \xi\gamma}$ and that $\int_\Omega u_0(x)|x - x_0|^2\,dx$ is small (x_0 is a fixed point). In the literature, finite-time blow-up was obtained by using the transformation $z := \chi v - \xi w$ and deducing (QAR) to a Keller–Segel system. However, this transformation does not work well for the reduction to Keller–Segel systems when $m \neq 1$, $p \neq 2$ and $q \neq 2$. Thus we shift our method to an energy estimate toward boundedness, and a moment-type functional (see e.g., Winkler [**23**] and Tanaka–Yokota [**20**]) for finite-time blow-up. With these strategies we obtained the following results.

Theorem 2.1 (C.–Yokota [**10**]). Let $n \in \mathbb{N}$. Assume that $p < q$ or $[p = q$ and $\chi\alpha - \xi\gamma < 0]$. Then for all nonnegative initial data $u_0 \in C(\overline{\Omega}) \setminus \{0\}$ there exists a unique global classical solution (u, v, w) of (QAR) with $\tau = 0$ such that the solution is bounded i.e. $\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C$ for all $t > 0$ with some $C > 0$.

Theorem 2.2 (C.–Yokota [**10**]). Let $\Omega = B_R(0) \subset \mathbb{R}^n$ $(n \geq 3,\ R > 0)$. Assume that $p > q$ or $[p = q$ and $\chi\alpha - \xi\gamma > 0]$. Then there exists $u_0 \in C(\overline{\Omega}) \setminus \{0\}$ such that the corresponding solution of (QAR) with $\tau = 0$ blows up in a finite time T in the sense that $\limsup_{t \nearrow T} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty$.

Remark. Boundedness was obtained when χ, ξ are functions (C.–Mizukami–Yokota [**6, 7**], C.–Yokota [**11**], C.–Mizukami [**5**]), and when the diffusion is degenerate (C. [**1**]). Also, in the case $m = 1$ and $p = q = 2$, global solvability can be obtained even if $u_0 \in L^2(\Omega)$ (C.–Saga–Yokota [**8**]).

The strategy for the proof of Theorem 2.1 is to establish the differential inequality

$$\frac{d}{dt} \int_\Omega (u+1)^\sigma \leq -c_1 \Big(\int_\Omega (u+1)^\sigma \Big)^{1+\theta_1} + c_2 \tag{1}$$

with some $\sigma > n$, $c_1, c_2, \theta_1 > 0$. The key to the derivation of (1) is to take advantage of the effect of repulsion. More precisely, we will estimate positive terms like $\chi\alpha \int_\Omega u^{\sigma+p-2}$ by the negative term $-\xi\gamma \int_\Omega u^{\sigma+q-2}$. On the other hand, the cornerstone of the proof of Theorem 2.2 is the derivation of the differential inequality

$$\phi'(s_0, t) \geq c_3 s_0^{-\theta_2} \phi^2(s_0, t) - c_4 s_0^{\theta_3}, \tag{2}$$

where $c_3, c_4, \theta_2, \theta_3 > 0$ are constants. Here the moment-type functional ϕ is defined as

$$\phi(s_0, t) := \int_0^{s_0} s^{-b}(s_0 - s) U(s, t)\, ds,$$

where U is the mass accumulation function given by

$$U(s, t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u(\rho, t)\, d\rho$$

for $s > 0$, $t > 0$ and $b \in (0, 1)$. To derive the inequality (2) we utilize the attraction term. More precisely, the key is to handle a term derived from the repulsion term by exploiting the effect of attraction.

3 Recent works

We next introduce recent works on related problems to (QAR).

3.1 Flux limited case

We first consider the system (QAR) with $\tau = 0$ involving flux limitation, that is,

$$\begin{cases} u_t = \nabla \cdot \Big((u+1)^{m-1}\nabla u - \dfrac{\chi u(u+1)^{p-2}}{(1+|\nabla v|^2)^k}\nabla v + \dfrac{\xi u(u+1)^{q-2}}{(1+|\nabla w|^2)^\ell}\nabla w \Big), \\ 0 = \Delta v + \alpha u - \beta v, \\ 0 = \Delta w + \gamma u - \delta w \end{cases} \tag{ARFL}$$

in a smoothly bounded domain $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ under Neumann boundary conditions, where $m, p, q \in \mathbb{R}$, $\chi, \xi, \alpha, \beta, \gamma, \delta > 0$ and $k, \ell \geq 0$ are constants. When $w = 0$, properties of solutions to this system were obtained (see e.g., Kohatsu [**16, 17**], Kohatsu–Senba [**18**]). Concerning (ARFL) with $w \neq 0$, in the one-dimensional case, we can obtain the following boundedness results.

Theorem 3.1 (C.–Hasegawa–Kohatsu–Yokota [**2**])**.** Let $n = 1$. Assume that $p, q \in \mathbb{R}$ fulfill $p < q$ and that $k \geq 0$ as well as $0 \leq \ell < \frac{1}{2}$. Suppose that nonnegative initial data $u_0 \in C(\overline{\Omega}) \setminus \{0\}$ satisfies

$$\begin{cases} \|u_0\|_{L^1(\Omega)} < \infty & \text{if } \ell = 0, \\ \|u_0\|_{L^1(\Omega)} < \dfrac{1}{\gamma}\sqrt{\left(\dfrac{1}{2\ell}\right)^{\frac{1}{\ell}} - 1} & \text{if } 0 < \ell < \dfrac{1}{2}. \end{cases} \tag{3}$$

Then there exists a unique classical solution (u, v, w) of (ARFL) such that the solution is bounded i.e. $\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C$ for all $t > 0$ with some $C > 0$.

Theorem 3.2 (C.–Hasegawa–Kohatsu–Yokota [**2**])**.** Let $n = 1$. Assume that $p, q \in \mathbb{R}$ fulfill $p = q$ and that $k \geq 0$ as well as $0 \leq \ell < \frac{1}{2}$. Suppose that nonnegative initial data $u_0 \in C(\overline{\Omega}) \setminus \{0\}$ satisfies (3) and

$$\chi\alpha - \xi\gamma \left\{ \left(1 + \gamma^2 \|u_0\|^2_{L^1(\Omega)}\right)^{-\ell} - 2\ell \right\} < 0.$$

Then the conclusion of Theorem 3.1 holds.

Remark. In the case $n \geq 2$, boundedness in (ARFL) in the radial setting was recently obtained (C.–Hasegawa–Yokota [**3**]).

The strategy for the proofs of Theorems 3.1 and 3.2 is to derive the differential inequality

$$\frac{d}{dt}\int_\Omega (u+1)^\sigma \le -c_5\left(\int_\Omega (u+1)^\sigma\right)^\kappa + c_6$$

with some $\sigma > 1$ and $c_5, c_6, \kappa > 0$. In order to explain the key to deriving this we replace $u+1$ with u in (ARFL). For large σ we have

$$\begin{aligned}\frac{1}{\sigma}\cdot\frac{d}{dt}\int_\Omega u^\sigma &= -(\sigma-1)\int_\Omega u^{\sigma+m-3}|\nabla u|^2 + \chi(\sigma-1)\int_\Omega u^{\sigma+p-3}(1+|\nabla v|^2)^{-k}\nabla u\cdot\nabla v \\ &\quad - \xi(\sigma-1)\int_\Omega u^{\sigma+q-3}(1+|\nabla w|^2)^{-\ell}\nabla u\cdot\nabla w.\end{aligned}$$

Using integration by parts, we furthermore see that

$$\begin{aligned}&\frac{1}{\sigma}\cdot\frac{d}{dt}\int_\Omega u^\sigma \\ &= -\frac{4(\sigma-1)}{(\sigma+m-1)^2}\int_\Omega \left|\nabla u^{\frac{\sigma+m-1}{2}}\right|^2 \\ &\quad - \frac{\chi(\sigma-1)}{\sigma+p-2}\int_\Omega u^{\sigma+p-2}\left\{(1+|\nabla v|^2)^{-k}\Delta v - k(1+|\nabla v|^2)^{-k-1}\nabla(|\nabla v|^2)\cdot\nabla v\right\} \\ &\quad + \frac{\xi(\sigma-1)}{\sigma+q-2}\int_\Omega u^{\sigma+q-2}\left\{(1+|\nabla w|^2)^{-\ell}\Delta w - \ell(1+|\nabla w|^2)^{-\ell-1}\nabla(|\nabla w|^2)\cdot\nabla w\right\} \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{4}$$

In particular, if $k = \ell = 0$, then we infer from the second and third equations in (ARFL) that

$$\begin{aligned} I_2 &\le \frac{\chi\alpha(\sigma-1)}{\sigma+p-2}\int_\Omega u^{\sigma+p-1}, \\ I_3 &= -\frac{\xi\gamma(\sigma-1)}{\sigma+q-2}\int_\Omega u^{\sigma+q-1} + \frac{\xi\delta(\sigma-1)}{\sigma+q-2}\int_\Omega u^{\sigma+q-2}w. \end{aligned} \tag{5}$$

When $p < q$, thanks to the Young inequality we can control the integral $\int_\Omega u^{\sigma+p-1}$ by the first term on the right-hand side of (5), and in the case $p = q$ we can still control the integral provided that $\chi\alpha - \xi\gamma < 0$. Therefore, deriving the first term on the right-hand side of (5) will be crucial. However, when $\ell > 0$, the third term I_3 in (4) contains the problematic quantity $\nabla(|\nabla w|^2)\cdot\nabla w = 2(D^2 w\nabla w)\cdot\nabla w$. This is where a mathematical difficulty arises. In the one-dimensional case, this quantity can be transformed into $\nabla(|\nabla w|^2)\cdot\nabla w = (w_x^2)_x w_x = 2w_x^2 w_{xx}$. Consequently, the third equation in (ARFL) with $n = 1$ can be utilized, but estimating w_x is now required. To this end, we take advantage of the one-dimensional setting and apply the fundamental theorem of calculus to estimate w_x.

3.2 Balanced case

As shown in [**10**], boundedness and finite-time blow-up in (QAR) were classified by the sizes of p, q and χ, ξ as in Table 1.

<table>
<tr><td rowspan="2">$p < q$</td><td colspan="2">$p = q$</td><td rowspan="2">$p > q$</td></tr>
<tr><td>$\chi\alpha - \xi\gamma < 0$</td><td>$\chi\alpha - \xi\gamma > 0$</td></tr>
<tr><td colspan="2">Boundedness</td><td colspan="2">Blow-up</td></tr>
</table>

Table 1: Boundedness/Blow-up in (QAR).

This means that the case $p = q$ is critical. Then the following question arises:

What happen in the fully parabolic case?

However, in the fully parabolic case, it is difficult to classify boundedness and finite-time blow-up by the sizes of p, q and χ, ξ. On the other hand, when $w = 0$, this classification was carried out by the sizes of m and p (see e.g., Tao–Winkler [**22**], Ishida–Seki–Yokota [**15**] for boundendess, and Cieślak–Stinner [**12, 13**] for blow-up), which hints on the fully parabolic case. Thus we focus on the system (QAR) with $\tau = 1$ and $p = q$, that is,

$$\begin{cases} u_t = \nabla \cdot \big((u+1)^{m-1}\nabla u - \chi u(u+1)^{p-2}\nabla v + \xi u(u+1)^{p-2}\nabla w\big), \\ v_t = \Delta v + \alpha u - \beta v, \\ w_t = \Delta w + \gamma u - \delta w \end{cases} \tag{BAR}$$

in a smoothly bounded domain $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ under Neumann boundary conditions, where $m, p \in \mathbb{R}$ and $\chi, \xi, \alpha, \beta, \gamma, \delta > 0$ are constants. In order to analyze this system we introduce the transformation $z := \chi v - \xi w$ to deduce this to the system

$$\begin{cases} u_t = \nabla \cdot \big((u+1)^{m-1}\nabla u - \chi u(u+1)^{p-2}\nabla z\big), \\ z_t = \Delta z - \delta z + \theta u + \chi(\delta - \beta)v, \\ v_t = \Delta v + \alpha u - \beta v, \end{cases} \tag{BAR$'$}$$

where $\theta := \chi\alpha - \xi\gamma$. By extending a method in Tao–Winkler [**22**] and applying it to (BAR)$'$, we can prove boundedness in (BAR).

Theorem 3.3 (C.–Hasegawa–Yokota [**3**])**.** Let $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ be a bounded convex domain with smooth boundary. Assume that

$$\chi\alpha - \xi\gamma > 0$$

and

$$p - m < \frac{2}{n}.$$

Then there is a unique classical solution (u, v, w) of (BAR) such that the solution is bounded i.e.

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} \le C$$

for all $t > 0$ with some $C > 0$.

To state a result on finite-time blow-up we put

$$\widetilde{\mathcal{F}}(u,v,w) := \frac{1}{2}\int_\Omega |\nabla(\chi v - \xi w)|^2 + \frac{1}{2}\int_\Omega (\chi v - \xi w)^2 - \theta \int_\Omega u(\chi v - \xi w) + \theta \int_\Omega G(u),$$
$$G(s) := \int_{s_0}^{s}\int_{s_0}^{\sigma} \frac{(\tau+1)^{m-1}}{\tau(\tau+1)^{p-2}}\, d\tau d\sigma \quad (s, s_0 > 1).$$

By constructing the Lyapunov like functional based on Lankeit [**19**], we obtain finite-time blow-up in (BAR).

Theorem 3.4 (C.–Uemura–Yokota [**9**])**.** Let $\Omega = B_R \subset \mathbb{R}^n$ $(n \in \{2,3\},\ R > 0)$. Assume that

$$\chi\alpha - \xi\gamma > 0$$

and

$$p \ge 2 \quad \text{and} \quad p - m > \frac{2}{n}.$$

Let $M > 0$ and $A > 0$. Then there exists a constant $K(M,A) > 0$ such that if (u_0, v_0, w_0) belongs to the set

$$\mathcal{B}(M,A) := \Big\{ (u_0,v_0,w_0) \in C^0(\overline{\Omega}) \times W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega) \,\Big|\, u_0, v_0 \text{ and } w_0 \text{ are radially symmetric and positive in } \overline{\Omega},\ \int_\Omega u_0 = M,\ \|\chi v_0 - \xi w_0\|_{W^{1,2}(\Omega)} \le A,\ \widetilde{\mathcal{F}}(u_0,v_0,w_0) < -K(M,A) \Big\},$$

then the solution (u, v, w) of (BAR) blows up in a finite time T in the sense that

$$\limsup_{t \nearrow T} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$$

holds.

We briefly explain the main idea behind the proofs of Theorems 3.3 and 3.4. We first focus on Theorem 3.3. When $\beta = \delta$, the term $\chi(\beta - \delta)v$ in (BAR)$'$ vanishes. Then this system has the same structure as a quasilinear Keller–Segel system, which allows us to utilize the energy functional introduced by Tao–Winkler [**22**] to obtain boundedness. However, in the case $\beta \neq \delta$, this structure breaks down because of the additional term $\chi(\beta - \delta)v$, and boundedness can no longer be derived by the same energy functional. To overcome this issue we construct the new energy functional

$$y(t) := \frac{1}{\sigma}\int_\Omega (u(\cdot,t)+1)^\sigma + \frac{1}{q}\int_\Omega |\nabla z(\cdot,t)|^{2q} + \frac{1}{\sigma+m-1}\int_\Omega v^{\sigma+m-1}(\cdot,t),$$

where $\sigma > n$, $q > 1$. This functional allows us to derive the differential inequality

$$y'(t) \le -c_7 y^\kappa(t) + c_8,$$

where $\kappa > 1$, $c_7 > 0$, $c_8 > 0$. As a consequence, we can arrive at the conclusion of Theorem 3.3.

On the other hand, the proof of Theorem 3.4 relies on two key ingredients. The first one is to establish the following inequality

$$\mathcal{F}(u,z) \ge -c_9 \cdot (\mathcal{D}^{\overline{\theta}}(u,z) + 1) \quad \left(\exists\, \overline{\theta} \in \left(\frac{1}{2}, 1\right)\right)$$

with $c_9 > 0$, where

$$\mathcal{F}(u,z) := \frac{1}{2}\int_\Omega |\nabla z|^2 + \frac{1}{2}\int_\Omega z^2 - \theta\int_\Omega uz + \theta\int_\Omega G(u)$$

and

$$\mathcal{D}(u,z) := \int_\Omega (\Delta z - \delta z + \theta u)^2 + \theta\int_\Omega u(u+1)^{p-2}\left|\frac{(u+1)^{m-1}}{u(u+1)^{p-2}}\nabla u - \nabla z\right|^2.$$

The second one is to establish the following inequality

$$\frac{d}{dt}\mathcal{F}(u,z) + \frac{1}{2}\mathcal{D}(u,z) \le c_{10},$$

where $c_{10} > 0$. By combining the above two inequalities, we obtain

$$\frac{d}{dt}\Big(-\frac{1}{c_9}\mathcal{F}(u,z) - 1\Big) \ge -\frac{c_{10}}{c_9} + \frac{1}{2c_9}\Big(-\frac{1}{c_9}\mathcal{F}(u,z) - 1\Big)_+^{\frac{1}{\overline{\theta}}},$$

which leads to the conclusion of Theorem 3.4.

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