Well-posedness of entropy solutions to nonhomogeneous conservation laws

Takanori Ebata

JSPS Postdoctoral Fellow, Niigata University, Japan

and

Hiroki Ohwa

Department of Mathematics, Faculty of Science, Niigata University, Japan

1 Introduction

In this paper, we establish the well-posedness of entropy solutions to nonhomogeneous conservation laws

$$u_t + f(u)_x = g(x,t) \quad (x \in \mathbb{R},\ t > 0), \tag{1.1}$$

$$u(x,0) = \bar{u}(x) \quad (x \in \mathbb{R}). \tag{1.2}$$

Here, $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$, $\bar{u} \in L^\infty(\mathbb{R})$, and $g : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ satisfies $g \in L^\infty\big(\mathbb{R} \times [0,T]\big)$ for each $T > 0$.

To study the well-posedness of the Cauchy problem (1.1)–(1.2), it is necessary to consider weak solutions, since classical solutions may not exist globally due to the formation of discontinuities. Among weak solutions, we focus on entropy solutions, which satisfy additional admissibility conditions that ensure both uniqueness and physical relevance. Below, we recall the definition of entropy solutions to the Cauchy problem (1.1)–(1.2) (see Kružkov [6]).

Definition. A function $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ is an entropy solution to the Cauchy problem (1.1)–(1.2), if it satisfies the following conditions (i)–(iii):

(i) $u(x,0) = \bar{u}(x)$ (a.e. $x \in \mathbb{R}$).

(ii) For any $T > 0$, $u \in L^\infty\big(\mathbb{R} \times [0,T]\big) \cap C\big([0,T]; L^1_{\mathrm{loc}}(\mathbb{R})\big)$.

(iii) For any $k \in \mathbb{R}$ and any $\phi \in C_c^\infty(\mathbb{R} \times (0,\infty))$ with $\phi \geq 0$, the following inequality holds:

$$\int_0^\infty \int_{\mathbb{R}} \Big(|u-k|\phi_t + \mathrm{sign}\,(u-k)\big(f(u)-f(k)\big)\phi_x \Big)\, dxdt \geq -\int_0^\infty \int_{\mathbb{R}} \mathrm{sign}\,(u-k) g(x,t)\phi\, dxdt, \tag{1.3}$$

where

$$\mathrm{sign}\,(w) = \begin{cases} -1 & (w<0) \\ 0 & (w=0) \\ 1 & (w>0) \end{cases}.$$

The well-posedness of entropy solutions to the Cauchy problem (1.1)–(1.2) was established by Kružkov [6] under suitable regularity assumptions, such as sufficient smoothness of the flux function f and the source term g. In the case where the source term depends on the unknown function, i.e., $g = g(u)$, the well-posedness was proved by Holden and Risebro [5] and Langseth, Tveito and Winther [7] under the assumptions $f \in C^2(\mathbb{R})$, $g \in \mathrm{Lip}(\mathbb{R})$, and $\bar{u} \in BV(\mathbb{R})$. Here, $BV(\mathbb{R})$ denotes the set of all functions $\psi : \mathbb{R} \to \mathbb{R}$ with bounded total variation, that is,

$$\mathrm{T.V.}\,\{\psi\} := \sup\left\{ \sum_{j=1}^{\Lambda} \big|\psi(x_j) - \psi(x_{j-1})\big| \,\Big|\, \Lambda \geq 1,\, x_0 < x_1 < \cdots < x_\Lambda \right\} < \infty.$$

More recently, Ebata, Ohwa and Tomita [4] extended the proof technique of Holden and Risebro [5] and Langseth, Tveito and Winther [7] to establish the well-posedness of entropy solutions in the case $g = g(u)$, under the weaker assumptions $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$, $g \in \mathrm{Lip}(\mathbb{R})$, and $\bar{u} \in L^\infty(\mathbb{R})$. The aim of this paper is to apply their method to the Cauchy problem (1.1)–(1.2) and to establish its well-posedness under similarly weak assumptions, thus relaxing the original conditions imposed by Kružkov [6].

E-mail addresses: ebata@m.sc.niigata-u.ac.jp (T. Ebata), hiroohwa@math.sc.niigata-u.ac.jp (H. Ohwa).

2 Uniqueness

To prove the uniqueness of entropy solutions for the Cauchy problem (1.1)–(1.2), we establish the continuous dependence of entropy solutions on both the initial data and the nonhomogeneous term.

Theorem 1. *Let $T > 0$, and let $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$, $g, \tilde{g} \in L^\infty\big(\mathbb{R} \times [0,T]\big)$, and $\bar{u}, \bar{v} \in L^\infty(\mathbb{R})$. Denote by u the entropy solution to the Cauchy problem (1.1)–(1.2), and by v the entropy solution to the Cauchy problem*

$$v_t + f(v)_x = \tilde{g}(x,t) \quad (x \in \mathbb{R},\ t > 0),$$
$$v(x,0) = \bar{v}(x) \quad (x \in \mathbb{R}).$$

Set

$$M = \max\big\{\|u\|_{L^\infty(\mathbb{R}\times[0,T])}, \|v\|_{L^\infty(\mathbb{R}\times[0,T])}\big\},$$

and let $L \geq 0$ denote the Lipschitz constant of f on the closed interval $[-M, M]$. Then, for any $R > 0$ and any $t \in [0,T]$, we have

$$\int_{|x|\leq R} \big|u(x,t) - v(x,t)\big|\, dx \leq \int_{|x|\leq R+Lt} \big|\bar{u}(x) - \bar{v}(x)\big|\, dx + \int_0^t \int_{|x|\leq R+L(t-s)} \big|g(x,s) - \tilde{g}(x,s)\big|\, dxds. \tag{2.1}$$

Remark. Theorem 1 has been extended by Ebata, Hiraguri and Ohwa [3] to the case where $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$, $g \in \mathrm{Lip}_{\mathrm{loc}}\big(\mathbb{R} \times [0,\infty)\big)$ and $\bar{u} \in BV(\mathbb{R})$, in the form of a continuous dependence estimate with respect to the flux function.

Proof sketch of Theorem 1. Let $k, k' \in \mathbb{R}$ be arbitrary and $\phi \in C_c^\infty\big(\mathbb{R} \times (0,\infty) \times \mathbb{R} \times (0,\infty)\big)$ with $\phi \geq 0$. Then, by inequality (1.3), we have

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}} \Big(\big|u(x,t) - k\big|\phi_t + \mathrm{sign}\big(u(x,t) - k\big)\Big(f\big(u(x,t)\big) - f(k)\Big)\phi_x \\
&\qquad + \mathrm{sign}\big(u(x,t) - k\big) g(x,t)\phi \Big)\, dxdt \geq 0, \\
&\int_0^\infty \int_{\mathbb{R}} \Big(\big|v(y,s) - k'\big|\phi_s + \mathrm{sign}\big(v(y,s) - k'\big)\Big(f\big(v(y,s)\big) - f(k')\Big)\phi_y \\
&\qquad + \mathrm{sign}\big(v(y,s) - k'\big) \tilde{g}(y,s)\phi \Big)\, dyds \geq 0.
\end{aligned}$$

Taking $k = v(y,s)$ and $k' = u(x,t)$, and then integrating the resulting inequalities with respect to (y,s) and (x,t), respectively, and adding them together, we obtain

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \Big(\big|u(x,t) - v(y,s)\big|(\phi_t + \phi_s) + \mathrm{sign}\,\big(u(x,t) - v(y,s)\big)\Big(f\big(u(x,t)\big) - f\big(v(y,s)\big)\Big)(\phi_x + \phi_y) \\
&\qquad + \mathrm{sign}\big(u(x,t) - v(y,s)\big)\big(g(x,t) - \tilde{g}(y,s)\big)\phi \Big)\, dxdydtds \geq 0.
\end{aligned} \tag{2.2}$$

Let $\delta \in C_c^\infty(\mathbb{R})$ be defined by

$$\delta(x) = \begin{cases} C \exp\left(\dfrac{1}{|x|^2 - 1}\right) & (|x| < 1) \\ 0 & (|x| \geq 1) \end{cases},$$

where C is a constant chosen so that

$$\int_{\mathbb{R}} \delta(x)\, dx = 1.$$

For each $h > 0$, we define

$$\delta_h(x) = h\,\delta(hx), \quad \zeta_h(x) = \int_{-\infty}^{x} \delta_h(s)\,ds \quad (x \in \mathbb{R}).$$

Let $\tau \in (0, T)$ be arbitrary, and let $h, i, j > 0$. Define a function $\phi : \mathbb{R} \times [0, T] \times \mathbb{R} \times [0, T] \to \mathbb{R}$ by

$$\phi(x, t, y, s) = \Big(1 - \zeta_h\big(|x| - R - L(\tau - t)\big)\Big)\Big(\zeta_i\big(t - \frac{2}{i}\big) - \zeta_i(t - \tau)\Big)\delta_j(x - y)\delta_j(t - s).$$

Then, for all sufficiently large $h, i, j > 0$, we have $\phi \in C_c^\infty\big(\mathbb{R}\times(0, T)\times\mathbb{R}\times(0, T)\big)$ with $\phi \geq 0$. Substituting this ϕ into inequality (2.2), we obtain

$$\begin{aligned}\int_0^T\!\!\int_0^T\!\!\int_{\mathbb{R}}\!\int_{\mathbb{R}} \Bigg(&\big|u(x,t) - v(y,s)\big|\Big(1 - \zeta_h\big(|x| - R - L(\tau - t)\big)\Big)\Big(\delta_i\big(t - \frac{2}{i}\big) - \delta_i(t - \tau)\Big)\delta_j(x - y)\delta_j(t - s)\\
&- \bigg(L\big|u(x,t) - v(y,s)\big| + \operatorname{sign}(x)\operatorname{sign}\big(u(x,t) - v(y,s)\big)\Big(f\big(u(x,t)\big) - f\big(v(y,s)\big)\Big)\bigg)\\
&\quad \cdot \delta_h\big(|x| - R - L(\tau - t)\big)\Big(\zeta_i\big(t - \frac{2}{i}\big) - \zeta_i(t - \tau)\Big)\delta_j(x - y)\delta_j(t - s)\\
&\quad + \operatorname{sign}\big(u(x,t) - v(y,s)\big)\big(g(x,t) - \tilde{g}(y,s)\big)\phi(x,t,y,s)\Bigg)\,dx\,dy\,dt\,ds \geq 0.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\int_0^T\!\!\int_0^T\!\!\int_{\mathbb{R}}\!\int_{\mathbb{R}} \Big(&\big|u(x,t) - v(y,s)\big|\Big(1 - \zeta_h\big(|x| - R - L(\tau - t)\big)\Big)\Big(\delta_i\big(t - \frac{2}{i}\big) - \delta_i(t - \tau)\Big)\delta_j(x - y)\delta_j(t - s)\\
&+ \big|g(x,t) - \tilde{g}(y,s)\big|\phi(x,t,y,s)\Big)\,dxdydtds \geq 0.\end{aligned}$$

Taking the limit $j \to \infty$ in the above inequality, we obtain

$$\begin{aligned}\int_0^T\!\!\int_{\mathbb{R}} \Big(&\big|u(x,t) - v(x,t)\big|\Big(1 - \zeta_h\big(|x| - R - L(\tau - t)\big)\Big)\Big(\delta_i\big(t - \frac{2}{i}\big) - \delta_i(t - \tau)\Big)\\
&+ \big|g(x,t) - \tilde{g}(x,t)\big|\Big(1 - \zeta_h\big(|x| - R - L(\tau - t)\big)\Big)\Big(\zeta_i\big(t - \frac{2}{i}\big) - \zeta_i(t - \tau)\Big)\Big)\,dx\,dt \geq 0. \qquad (2.3)\end{aligned}$$

Inequality (2.3) can be rewritten in the form

$$\begin{aligned}&\int_0^T \mu(t)\,\Big(\delta_i\big(t - \frac{2}{i}\big) - \delta_i(t - \tau)\Big)\,dt\\
&\quad + \int_0^T\!\!\int_{\mathbb{R}} \big|g(x,t) - \tilde{g}(x,t)\big|\Big(1 - \operatorname{sign}^+\big(|x| - R - L(\tau - t)\big)\Big)\Big(\zeta_i\big(t - \frac{2}{i}\big) - \zeta_i(t - \tau)\Big)\,dx\,dt \geq 0, \qquad (2.4)\end{aligned}$$

where

$$\mu(t) := \int_{\mathbb{R}} \big|u(x,t) - v(x,t)\big|\Big(1 - \operatorname{sign}^+\big(|x| - R - L(\tau - t)\big)\Big)\,dx \quad \big(t \in [0, T]\big).$$

Here, $\operatorname{sign}^+(w)$ is defined by

$$\operatorname{sign}^+(w) = \begin{cases} 1 & (w > 0) \\ \frac{1}{2} & (w = 0) \\ 0 & (w < 0) \end{cases}.$$

Since $u, v \in L^\infty\big(\mathbb{R} \times [0, T]\big) \cap C\big([0, T]; L^1_{\mathrm{loc}}(\mathbb{R})\big)$, μ is continuous with respect to t. Noting that $t = 0$ and $t = \tau$ are Lebesgue points of μ, we have

$$\lim_{i\to\infty} \int_0^T \mu(t)\,\delta_i\big(t - \frac{2}{i}\big)\,dt = \mu(0), \quad \lim_{i\to\infty} \int_0^T \mu(t)\,\delta_i(t - \tau)\,dt = \mu(\tau), \qquad (2.5)$$

and, by the dominated convergence theorem together with the definition of ζ_i,

$$\begin{aligned}
&\lim_{i\to\infty}\int_0^T\int_{\mathbb{R}}\left|g(x,t)-\tilde{g}(x,t)\right|\Big(1-\operatorname{sign}^+\big(|x|-R-L(\tau-t)\big)\Big)\Big(\zeta_i(t-\frac{2}{i})-\zeta_i(t-\tau)\Big)\,dx\,dt\\
&=\int_0^T\int_{\mathbb{R}}\left|g(x,t)-\tilde{g}(x,t)\right|\Big(1-\operatorname{sign}^+\big(|x|-R-L(\tau-t)\big)\Big)\big(\operatorname{sign}^+(t)-\operatorname{sign}^+(t-\tau)\big)\,dx\,dt\\
&=\int_0^\tau\int_{|x|\le R+L(\tau-t)}\left|g(x,t)-\tilde{g}(x,t)\right|\,dx\,dt. \qquad (2.6)
\end{aligned}$$

Therefore, combining inequality (2.4) with equations (2.5) and (2.6), we obtain

$$\mu(\tau)\le\mu(0)+\int_0^\tau\int_{|x|\le R+L(\tau-t)}\left|g(x,t)-\tilde{g}(x,t)\right|\,dx\,dt,$$

which is precisely inequality (2.1). □

3 Existence

To prove the existence of entropy solutions to the Cauchy problem (1.1)–(1.2), we begin by considering a regularized setting, temporarily assuming that $g\in C_c^\infty(\mathbb{R}^2)$ and $\bar{u}\in BV(\mathbb{R})$. Under these assumptions, we establish the existence of entropy solutions on interval $[0,T]$ for each $T>0$.

Proposition 1. Let $f\in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$, $g\in C_c^\infty(\mathbb{R}^2)$, and $\bar{u}\in BV(\mathbb{R})$. Then, for each $T>0$, there exists an entropy solution to the Cauchy problem (1.1)–(1.2) on the interval $[0,T]$.

Remark. A function u is said to be an entropy solution to the Cauchy problem (1.1)–(1.2) on the interval $[0,T]$ if it satisfies the following conditions (i)–(iii):

(i) $u(x,0)=\bar{u}(x)$ (a.e. $x\in\mathbb{R}$).

(ii) $u\in L^\infty\big(\mathbb{R}\times[0,T]\big)\cap C\big([0,T];L^1_{\mathrm{loc}}(\mathbb{R})\big)$.

(iii) For any $k\in\mathbb{R}$ and any $\phi\in C_c^\infty(\mathbb{R}\times(0,T))$ with $\phi\ge 0$, inequality (1.3) holds.

Proof sketch of Proposition 1. Consider the Cauchy problem for conservation laws

$$u_t+f(u)_x=0\quad (x\in\mathbb{R},\ t>0), \qquad (3.1)$$

$$u(x,0)=\bar{u}(x)\quad (x\in\mathbb{R}). \qquad (3.2)$$

We denote by $S(t)\bar{u}(x)$ the entropy solution at time t constructed by the wave-front tracking algorithm (see [1, 2]). Moreover, for each $s\ge 0$, consider the Cauchy value problem for partial differential equations

$$u_t(x,t)=g\Big(x,\frac{t+s}{2}\Big)\quad (t>0), \qquad (3.3)$$

$$u(x,0)=\bar{u}(x)\quad (x\in\mathbb{R}). \qquad (3.4)$$

We denote by $R(t,s)\bar{u}(x)$ the solution at time t.

For each $N\in\mathbb{N}$, set

$$\Delta_N=\frac{T}{N}>0.$$

In what follows, we write Δ in place of Δ_N when no confusion is likely to arise. For each $n\in\mathbb{N}\cup\{0\}$ with $n\le N-1$ and each $x\in\mathbb{R}$, define

$$u^0(x):=\bar{u}(x),\quad u^{n+\frac{1}{2}}(x):=R(\Delta,2t_n)u^n(x),\quad u^{n+1}(x):=S(\Delta)u^{n+\frac{1}{2}}(x).$$

We then define $u_\Delta(x,t)$ by

$$u_\Delta(x,0)=u^0(x)\quad (x\in\mathbb{R}),$$

$$u_\Delta(x,t) = \begin{cases} R\big(2(t-t_n), 2t_n\big)u^n(x) & \big((x,t)\in\mathbb{R}\times(t_n, t_{n+\frac{1}{2}}]\big) \\ S\big(2(t-t_{n+\frac{1}{2}})\big)u^{n+\frac{1}{2}}(x) & \big((x,t)\in\mathbb{R}\times(t_{n+\frac{1}{2}}, t_{n+1}]\big) \end{cases} \quad (n=0,\dots,N-1), \tag{3.5}$$

where

$$t_n := n\Delta, \quad t_{n+\frac{1}{2}} := \Big(n+\frac{1}{2}\Big)\Delta, \quad t_N := T.$$

Let $T>0$ be given, and let $\Delta>0$ be arbitrary. There exist constants $M_1, M_2 \geq 0$, independent of Δ and n, such that for all $t\in[0,T]$,

$$\big\|u_\Delta(\cdot,t)\big\|_{L^\infty(\mathbb{R})} \leq M_1, \quad \text{T.V.}\,\big\{u_\Delta(\cdot,t)\big\} \leq M_2.$$

Moreover, for any compact set $K\subset\mathbb{R}$, there exists a constant $M_3\geq 0$, independent of Δ and n, such that for all $s,t\in[0,T]$,

$$\int_K \big|u_\Delta(x,s)-u_\Delta(x,t)\big|\,dx \leq M_3|s-t|.$$

Then, applying Helly's theorem (see [1]), we can construct a subsequence $u_{\Delta_{N(l)}}$ that converges in $L^1_{\text{loc}}\big(\mathbb{R}\times[0,T]\big)$ to a limit function u. This limit satisfies, for all $t\in[0,T]$,

$$\big\|u(\cdot,t)\big\|_{L^\infty(\mathbb{R})} \leq M_1, \quad \text{T.V.}\,\big\{u(\cdot,t)\big\} \leq M_2, \tag{3.6}$$

and, for all $s,t\in[0,T]$,

$$\int_K \big|u(x,s)-u(x,t)\big|\,dx \leq M_3|s-t|. \tag{3.7}$$

We now proceed to show that the limit u is an entropy solution to the Cauchy problem (1.1)–(1.2) on the interval $[0,T]$. Since

$$u_{\Delta_{N(l)}}(x,0) = \bar{u}(x) \quad (x\in\mathbb{R})$$

for all $l\in\mathbb{N}$, it follows that

$$u(x,0)=\bar{u}(x) \quad (\text{a.e. } x\in\mathbb{R}).$$

Moreover, by inequalities (3.6) and (3.7), we have $u\in L^\infty\big(\mathbb{R}\times[0,T]\big)\cap C\big([0,T];L^1_{\text{loc}}(\mathbb{R})\big)$. For each $n\in\mathbb{N}\cup\{0\}$ with $n\leq N(l)-1$, any $k\in\mathbb{R}$ and any $\phi\in C_c^\infty\big(\mathbb{R}\times(0,T)\big)$ with $\phi\geq 0$, set

$$r = 2\big(t-t_{n+\frac{1}{2}}\big) \quad \big(t_{n+\frac{1}{2}}\leq t\leq t_{n+1}\big),$$

and define

$$\varphi(x,r) = \phi\Big(x, \frac{r}{2}+t_{n+\frac{1}{2}}\Big) = \phi(x,t).$$

Then, $\varphi\in C_c^\infty\big(\mathbb{R}\times[0,\Delta]\big)$ with $\varphi\geq 0$, and

$$\phi_t(x,t) = \varphi_r(x,r)\frac{dr}{dt} = 2\varphi_r(x,r).$$

Therefore, we obtain

$$\begin{aligned} &2\int_{t_{n+\frac{1}{2}}}^{t_{n+1}}\int_{\mathbb{R}} \Big(\frac{1}{2}\big|u_\Delta-k\big|\phi_t + \operatorname{sign}\big(u_\Delta-k\big)\big(f(u_\Delta)-f(k)\big)\phi_x\Big)\,dxdt \\ &\qquad + \int_{\mathbb{R}} \big|u_\Delta\big(x,t_{n+\frac{1}{2}}\big)-k\big|\phi\big(x,t_{n+\frac{1}{2}}\big)\,dx - \int_{\mathbb{R}} \big|u_\Delta(x,t_{n+1})-k\big|\phi(x,t_{n+1})\,dx \\ &= \int_0^\Delta\int_{\mathbb{R}} \Big(\big|S(r)u^{n+\frac{1}{2}}(x)-k\big|\varphi_r(x,r) \\ &\qquad + \operatorname{sign}\Big(S(r)u^{n+\frac{1}{2}}(x)-k\Big)\Big(f\big(S(r)u^{n+\frac{1}{2}}(x)\big)-f(k)\Big)\varphi_x(x,r)\Big)\,dxdr \end{aligned}$$

$$+\int_{\mathbb{R}}\left|u^{n+\frac{1}{2}}(x)-k\right|\varphi(x,0)\,dx-\int_{\mathbb{R}}\left|S(\Delta)u^{n+\frac{1}{2}}(x)-k\right|\varphi(x,\Delta)\,dx\geq 0. \tag{3.8}$$

Similarly, set

$$s=2\big(t-t_n\big)\quad\big(t_n\leq t\leq t_{n+\frac{1}{2}}\big),$$

and define

$$\tilde{\varphi}(x,s)=\phi\Big(x,\frac{s}{2}+t_n\Big)=\phi(x,t).$$

Then, $\tilde{\varphi}\in C_c^\infty\big(\mathbb{R}\times[0,\Delta]\big)$ with $\tilde{\varphi}\geq 0$, and

$$\phi_t(x,t)=\tilde{\varphi}_s(x,s)\frac{ds}{dt}=2\tilde{\varphi}_s(x,s).$$

Therefore, we obtain

$$\begin{aligned}
&2\int_{t_n}^{t_{n+\frac{1}{2}}}\int_{\mathbb{R}}\left(\frac{1}{2}\left|u_\Delta-k\right|\phi_t+\operatorname{sign}\big(u_\Delta-k\big)g(x,t)\phi\right)dxdt\\
&\qquad+\int_{\mathbb{R}}\left|u_\Delta(x,t_n)-k\right|\phi(x,t_n)\,dx-\int_{\mathbb{R}}\left|u_\Delta(x,t_{n+\frac{1}{2}})-k\right|\phi(x,t_{n+\frac{1}{2}})\,dx\\
&=\int_0^\Delta\int_{\mathbb{R}}\left(\left|R(s,2t_n)u^n(x)-k\right|\tilde{\varphi}_s(x,s)+\operatorname{sign}\big(R(s,2t_n)u^n(x)-k\big)g\Big(x,\frac{s}{2}+t_n\Big)\tilde{\varphi}(x,s)\right)dxds\\
&\qquad+\int_{\mathbb{R}}\left|u^n(x)-k\right|\tilde{\varphi}(x,0)\,dx-\int_{\mathbb{R}}\left|R(\Delta,2t_n)u^n(x)-k\right|\tilde{\varphi}(x,\Delta)\,dx=0.
\end{aligned}\tag{3.9}$$

For each $n\in\mathbb{N}\cup\{0\}$ with $n\leq N(l)-1$, adding inequality (3.8) and equation (3.9) and using the fact that $\phi\in C_c^\infty\big(\mathbb{R}\times(0,T)\big)$, we obtain

$$2\int_0^T\int_{\mathbb{R}}\left(\frac{1}{2}\left|u_\Delta-k\right|\phi_t+\chi_\Delta\operatorname{sign}\big(u_\Delta-k\big)\big(f(u_\Delta)-f(k)\big)\phi_x+\tilde{\chi}_\Delta\operatorname{sign}\big(u_\Delta-k\big)g(x,t)\phi\right)dxdt\geq 0.$$

Consequently, for any $\psi\in C_c^\infty(\mathbb{R})$ with $\psi\geq 0$, multiplying the above inequality by $\psi(k)$ and integrating over $\mathbb{R}$ with respect to k yields

$$\begin{aligned}
2\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}}\bigg(&\frac{1}{2}\left|u_\Delta-k\right|\phi_t(x,t)\psi(k)+\chi_\Delta(x,t)\operatorname{sign}\big(u_\Delta-k\big)\big(f(u_\Delta)-f(k)\big)\phi_x(x,t)\psi(k)\\
&+\tilde{\chi}_\Delta(x,t)\operatorname{sign}\big(u_\Delta-k\big)g(x,t)\phi(x,t)\psi(k)\bigg)\,dkdxdt\geq 0.
\end{aligned}$$

Letting $\Delta\to 0$, that is, $l\to\infty$, in the above inequality, we obtain

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}}\Big(|u-k|\phi_t(x,t)\psi(k)+\operatorname{sign}(u-k)\big(f(u)-f(k)\big)\phi_x(x,t)\psi(k)\Big)\,dkdxdt\\
&\qquad\geq-\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}}\operatorname{sign}(u-k)g(x,t)\phi(x,t)\psi(k)\,dkdxdt.
\end{aligned}\tag{3.10}$$

Let $\kappa\in\mathbb{R}$ be arbitrary, and set

$$\psi(k)=\delta_h(k-\kappa)\quad(k\in\mathbb{R}).$$

Then, $\psi\in C_c^\infty(\mathbb{R})$ with $\psi\geq 0$. Substituting this choice of ψ into inequality (3.10) and letting $h\to 0$, we have

$$\int_0^T\int_{\mathbb{R}}\Big(|u-\kappa|\phi_t+\operatorname{sign}(u-\kappa)\big(f(u)-f(\kappa)\big)\phi_x\Big)\,dxdt\geq-\int_0^T\int_{\mathbb{R}}\operatorname{sign}(u-\kappa)g(x,t)\phi\,dxdt.$$

This establishes Proposition 1. □

Then, using Proposition 1, we establish the existence of entropy solutions to the original Cauchy problem (1.1)–(1.2).

Theorem 2. *Let $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$, $\bar{u} \in L^\infty(\mathbb{R})$, and let $g : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ be such that $g \in L^\infty\big(\mathbb{R} \times [0,T]\big)$ for each $T > 0$. Then, there exists an entropy solution to the Cauchy problem* (1.1)–(1.2)*.*

Proof sketch of Theorem 2. Let $\bar{u}_n \to \bar{u}$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ as $n \to \infty$, and for each $n \in \mathbb{N}$, let

$$\|\bar{u}_n\|_{L^\infty(\mathbb{R})} \leq \|\bar{u}\|_{L^\infty(\mathbb{R})}$$

with $\bar{u}_n \in C_c^\infty(\mathbb{R}) \subset BV(\mathbb{R})$. Similarly, let $g_n \to g$ in $L^1_{\mathrm{loc}}(\mathbb{R} \times [0,T])$ as $n \to \infty$, and for each $n \in \mathbb{N}$, let

$$\|g_n\|_{L^\infty(\mathbb{R}^2)} \leq \|g\|_{L^\infty(\mathbb{R}\times[0,T])}$$

with $g_n \in C_c^\infty(\mathbb{R}^2)$. Then, by Proposition 1, the Cauchy problem

$$u_t + f(u)_x = g_n(x,t) \quad (x \in \mathbb{R},\ t > 0), \tag{3.11}$$

$$u(x,0) = \bar{u}_n(x) \quad (x \in \mathbb{R}) \tag{3.12}$$

admits a bounded entropy solution u_n on the interval $[0,T]$, for which there exists a constant $M_0 \geq 0$ such that for all $n \in \mathbb{N}$ and all $t \in [0,T]$,

$$\big\|u_n(\cdot,t)\big\|_{L^\infty(\mathbb{R})} \leq M_0.$$

By Theorem 1, for any $n, m \in \mathbb{N}$, any $R > 0$ and all $t \in [0,T]$, we have

$$\int_{|x|\leq R} |u_n(x,t) - u_m(x,t)|\,dx \leq \int_{|x|\leq R+Lt} |\bar{u}_n(x) - \bar{u}_m(x)|\,dx + \int_0^t \int_{|x|\leq R+L(t-s)} |g_n(x,s) - g_m(x,s)|\,dxds.$$

This estimate implies that $\{u_n\}$ is a Cauchy sequence in $C\big([0,T]; L^1_{\mathrm{loc}}(\mathbb{R})\big)$, and hence, there exists a Lebesgue-measurable function u on $\mathbb{R} \times [0,T]$, with $u \in C\big([0,T]; L^1_{\mathrm{loc}}(\mathbb{R})\big)$, such that for any $n \in \mathbb{N}$, any $R > 0$ and all $t \in [0,T]$,

$$\begin{aligned} \int_{|x|\leq R} \big|u_n(x,t) - u(x,t)\big|\,dx &\leq \int_{|x|\leq R+Lt} \big|\bar{u}_n(x) - \bar{u}(x)\big|\,dx + \int_0^t\int_{|x|\leq R+L(t-s)} \big|g_n(x,s) - g(x,s)\big|\,dxds \\ &\to 0 \quad (n \to \infty). \end{aligned} \tag{3.13}$$

Furthermore, we can construct a subsequence $\{u_{n(l)}\}$ of $\{u_n\}$ such that

$$\lim_{l\to\infty} u_{n(l)}(x,t) = u(x,t) \quad \big(\text{a.e. } (x,t) \in \mathbb{R} \times [0,\infty)\big).$$

Consequently, for any $T > 0$, we have $u \in L^\infty\big(\mathbb{R} \times [0,T]\big) \cap C\big([0,T]; L^1_{\mathrm{loc}}(\mathbb{R})\big)$, and moreover, for any $R > 0$ and all $t \in [0,T]$,

$$\lim_{l\to\infty} \int_{|x|\leq R} \big|u_{n(l)}(x,t) - u(x,t)\big|\,dx = 0.$$

Also, by inequality (3.13), for all $R > 0$, we have

$$\int_{|x|\leq R} \big|u(x,0) - \bar{u}(x)\big|\,dx \leq \int_{|x|\leq R} \big|u(x,0) - u_{n(l)}(x,0)\big|\,dx + \int_{|x|\leq R} \big|u_{n(l)}(x,0) - \bar{u}(x)\big|\,dx \to 0 \quad (l \to \infty),$$

which shows that $u(x,0) = \bar{u}(x)$ (a.e. $x \in \mathbb{R}$).

In what follows, to avoid unnecessary notation, we denote a convergent subsequence $\{u_{n(l)}\}$ of $\{u_n\}$ simply by $\{u_n\}$. Let $k \in \mathbb{R}$ and $\phi \in C_c^\infty\big(\mathbb{R} \times (0,\infty)\big)$ with $\phi \geq 0$, such that

$$\bigcup_{x\in\mathbb{R}} \mathrm{supp}\,\big(\phi(x,\cdot)\big) \subset [0,T].$$

Since u_n is an entropy solution to the Cauchy problem (3.11)–(3.12) on the interval $[0,T]$, we obtain

$$\int_0^\infty\int_{\mathbb{R}} \Big(|u_n - k|\phi_t + \mathrm{sign}(u_n - k)\big(f(u_n) - f(k)\big)\phi_x\Big)\,dxdt \geq -\int_0^\infty\int_{\mathbb{R}} \mathrm{sign}(u_n - k) g_n(x,t)\phi\,dxdt.$$

Consequently, for any $\psi \in C_c^\infty(\mathbb{R})$ with $\psi \geq 0$, multiplying the above inequality by $\psi(k)$ and integrating over $\mathbb{R}$ with respect to k yields

$$\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}}\Big(|u_n-k|\phi_t(x,t)\psi(k)+\operatorname{sign}(u_n-k)\big(f(u_n)-f(k)\big)\phi_x(x,t)\psi(k)\Big)\,dkdxdt$$
$$\geq -\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}}\operatorname{sign}(u_n-k)g_n(x,t)\phi(x,t)\psi(k)\,dkdxdt.$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}}\Big(|u-k|\phi_t(x,t)\psi(k)+\operatorname{sign}(u-k)\big(f(u)-f(k)\big)\phi_x(x,t)\psi(k)\Big)\,dkdxdt$$
$$\geq -\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}}\operatorname{sign}(u-k)g(x,t)\phi(x,t)\psi(k)\,dkdxdt. \tag{3.14}$$

Let $\kappa \in \mathbb{R}$ be arbitrary, and set

$$\psi(k) = \delta_h(k-\kappa) \quad (k \in \mathbb{R}).$$

Then, $\psi \in C_c^\infty(\mathbb{R})$ with $\psi \geq 0$. Substituting this choice of ψ into inequality (3.14) and letting $h \to 0$, we have

$$\int_0^\infty\int_{\mathbb{R}}\Big(|u-\kappa|\phi_t+\operatorname{sign}(u-\kappa)\big(f(u)-f(\kappa)\big)\phi_x\Big)\,dxdt \geq -\int_0^\infty\int_{\mathbb{R}}\operatorname{sign}(u-\kappa)g(x,t)\phi\,dxdt.$$

This establishes Theorem 2. □

Conclusion. *As a consequence of Theorems 1 and 2, we obtain the well-posedness of the Cauchy problem* (1.1)–(1.2).

4 Concluding Remarks

The proof technique presented in this paper can also be applied to the case where the source term takes the form $g(u,x,t)$. In fact, if there exists a constant $L_g \geq 0$ such that for all $(x,t) \in \mathbb{R}\times[0,\infty)$ and all $u, v \in \mathbb{R}$,

$$\big|g(u,x,t)-g(v,x,t)\big| \leq L_g|u-v|,$$

and if for each $u \in \mathbb{R}$ and each $T > 0$, the mapping $(x,t) \mapsto g(u,x,t)$ belongs to $L^\infty\big(\mathbb{R}\times[0,T]\big)$, then the well-posedness of entropy solutions to the corresponding Cauchy problem (1.1)–(1.2) can also be established. Moreover, under suitable additional assumptions, a continuous dependence estimate with respect to the flux function can also be derived for the same class of problems. These results will be presented in detail in a forthcoming paper.

Acknowledgments

This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University. The first author was supported by JSPS KAKENHI Grant Numbers JP25KJ1341. The second author was supported by JSPS KAKENHI Grant Numbers JP22K03349.

References

[1] A. Bressan, *Hyperbolic Systems of Conservation Laws*, Oxford University Press, Oxford, 2000.

[2] C. Dafermos, Polygonal approximations of solutions of the initial value for a conservation law, *J. Math. Anal. Appl.*, **38** (1972), 33–41.

[3] T. Ebata, M. Hiraguri and H. Ohwa, Continuous dependence on the initial and flux functions for entropy solutions of non-homogeneous conservation laws, *Commun. Pure Appl. Anal.* **24** (2025), 626–640.

[4] T. Ebata, H. Ohwa and K. Tomita, Uniqueness and existence of solutions for the Cauchy Problem of balance laws, submitted.

[5] H. Holden and N. H. Risebro,*Front Tracking for Hyperbolic Conservation Laws.* Second Edition, Springer, 2015.

[6] S. N. Kružkov, First order quasilinear equations in several space variables, *Math. USSR Sb.* **10** (1970), 217–243.

[7] J. Langseth, A. Tveito and R. Winther, On the convergence of operator splitting applied to conservation laws with source terms, *SIAM J. Numer. Anal.* **33** (1996), 843–863.